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How Thick are the Saturn's Rings?

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Abstract. It has been long thought that the rings of saturn are probably not thicker than a few hundreds of meters. By actually following the motion of particles inside the decaying bending waves of the Saturn's rings and matching the resulting damping length of the bending waves launched by a number of moons with the actual damping lengths observed by Voyagar-II, we show that the thickness of the rings need not be larger than a few meters at the most. We present a few of these results.

Key words: Planetary rings – satellites – solar system – Hydrodynamic waves

1. Introduction

From the point of view of a Keplerian disk in vertical equilibrium the thickness of the saturn's ring should have been a few thousand kilometers. Howevrer, even in the early eighties, it was realized that the vertical height of the Saturn's rings are probably not greater than a few kilometers (Brahic & Sicardy, 1981; Sicardy et al. 1982). After the results of Voyagar-II photo-polarimetric experiment were analyzed, it was found out that the upper limit of the local thickness of the outer edge of the A ring is no more than about 200m. The resolution of the Voyagar camera was 100m and therefore the thickness of the ring, could not be determined with certainty. Presently, it is believed that the thickness of the inner rings could be much less, and probably a few meters (Chakrabarti, 1989; Rosen, Tyler, Marouf & Lissauer 1991) and the early confusions regarding the thickness arose out of the projection effects of the bending waves inside the ring. More recently, Chakrabarti and Bhattacharyya (2001) and Bhattacharyya and Chakrabarti (2002) carried out detailed computation of the bending wave properties on the Saturn's rings and showed that this general result is valid for the A and C rings. In the present article, we show how one can arive at these numbers.

Before we enter into the topic, we like to remind what is going on in a Saturn's ring. Moons of Saturn are in orbits inclined at an angle with the plane of the ring. These moons gravitationally attract ring particles. The force has one component acting along the ring plane and the other component act along the direction perpendicular to the ring plane. There are some orbits, where the angular velocity of the ring particle is commensurate with that of a particular moon and particles in those orbits feel 'resonance' to come out of the plane. The locations of these orbits are known as the vertical resonance orbits. Similarly, there are orbits where the horizontal forcing is resonantly pulling matter out of their usual orbits. These locations are the horizontal resonance orbits. Because of oblateness of the Saturn, these two resonance locations are not the same.

During Voyagar missions these bending waves were detected for various moons, such as Mimas, Titan, Prometheus etc. Although a large number of such locations are expected for each moon, only a few were resolved, since many of them were overlapping. Fig. 1 shows the bending (upper left) and density waves (lower right) at 5:3 resonance of Mimas. The picture is 400km on each side. The Figure is taken from http://ringmaster.arc.nasa.gov. The damping of this bending wave seen to be within 150km which is farely small, considering that the particles do not collide that often. So it was realized that some extra source of shear could be required to explain this rapid damping. Chakrabarti (1989) suggested that this extra source of shear comes from the variation of velocity components in the vertical direction inside the ring.

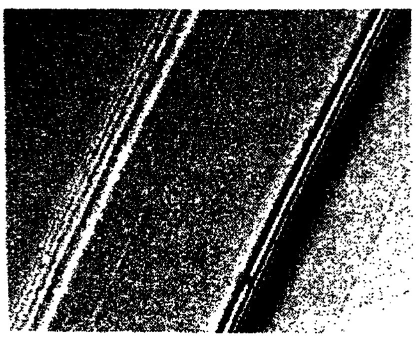

Figure 1. Voyagar-II photograph of the bending (upper left) and density waves (lower right) seen on the Saturn's ring at the 5:3 resonance.

Our procedure is to follow motions of sparingly colliding particles in presence of the planet and the perturbing moon, self-gravity of the ring, Coriolis and centrifugal forces. We integerate the equations of motion of a single particle and calculate the average shear in all directions. Particularly interesting is the fact that the shear which arises out of the variation of the radial velocity components in the vertical direction turns out to be sufficiently significant to damp out the bending waves in a very short distance. In order to see how a wave can be damped, one has to integrate the particle motions inside the ring and estimate the shear and dissipation of energy of the wave. If this problem is studied for a range of the input parameters such as the surface density and the disk height, then after matching the damping length and the wavelength of the wave profile with the observed wavelength the thickness of the ring could be determined. Usually there are independent determination of the surface density from the radio occultation date, so the estimation of the thickness becomes easier.

In the next Section, we present the governing equations and the assumptions made to solve them. We solve these equations perturbatively. In §3, we present results for perturbations by Mimas and Titan. Finally in §4, the conclusions are drawn.

2. Model Equations

We choose a right handed Cartesian co-ordinate system $(X,\ Y,\ Z)$ at a radial distance r from the center of the planet with the origin located on the equatorial plane of the planet with $X - Y$ plane coinciding with this plane (Fig. 2). The X-axis points radially outward and Y-axis points toward the azimuthal direction. The frame is rotating around the planet with local Keplerian frequency $\Omega(r)$. Let the amplitude of the bending wave

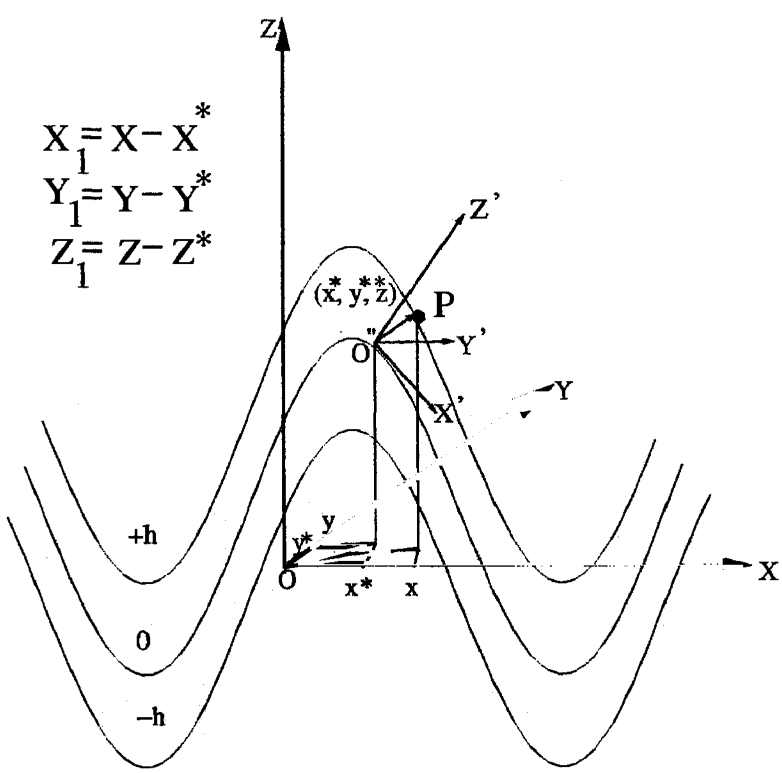

Figure 2: The co-ordinate systems used in computing the shear components.

be ϵ. The mid-plane itself oscillates up and down with this amplitude. Sitting on the mid-plane, the particle oscillates up and down with amplitude h, the half-thickness of the ring. Let the co-ordinate of the origin of a Cartesian frame (X', Y', Z') which is oscillating with the mid-plane of the disk be (x^*, y^*, z^*). In the absence of oscillations of the mid-plane, these two co-ordinate systems merge. A particle moving within the ring having coordinate (x, y, z) has a coordinate of $x_1 = x - x^*$, $y_1 = y - y^*$ and $z_1 = z - z^*$ with respect to the origin of the oscillating frame. If ω be the angular frequency of the propagating wave, then the phase of the mid-plane is $\phi^* = k_x x^* + k_y y^* - \omega t$ and that of the particle located at a point A (x, y, z) is $\phi = k_x x + k_y y - \omega t$. By definition, $\omega = (\Omega_p - \Omega)$ where, Ω_p is the pattern frequency. Here, k_x and k_y are the $x-$ and $y-$ components of the wave vector $\vec{k}$. Let (x', y', z') be the co-ordinate of the particle in the (X', Y', Z') frame. Thus,

$$z' = h + \epsilon \cos\phi^*, \tag{1}$$

where, h is the half thickness of the ring. One derives the following important relations,

$$x_1 = z_1 \epsilon\ k_x \sin\phi^*, \tag{2a}$$

and

$$y_1 = z_1 \epsilon\ k_y \sin\phi^*. \tag{2b}$$

When the oblateness of the planet is ignored, the vertical frequency μ of the particle defined by,

$$\mu^2 = \frac{\partial^2 \phi_p}{\partial z^2}|_z = 0, \tag{3a}$$

and the epicyclic frequency κ of the particle defined by,

$$\kappa^2 = \frac{1}{r^3}\frac{d}{dr}\left[(r^2\Omega)^2\right]_z = 0, \tag{3b}$$

are identical to the local Keplerian frequency Ω.

The components of the equation of motion of the test particle are given by (Chakrabarti, 1989),

$$\frac{d^2x}{dt^2} = -2\Omega\frac{dy}{dt} + 3\Omega^2 x - \nu^2 x_1 - (\Omega^2 - \omega^2)\epsilon^2 \kappa_x cos(\phi) sin(\phi^*)/W, \tag{5a}$$

$$\frac{d^2y}{dt^2} = 2\Omega\frac{dx}{dt} - \nu^2 y_1 - (\Omega^2 - \omega^2)\epsilon^2 \kappa_y cos(\phi) sin(\phi^*)/W, \tag{5b}$$

$$\frac{d^2z}{dt^2} = -\Omega^2 z - \nu^2 z_1 + (\Omega^2 - \omega^2)\epsilon cos(\phi)/W. \tag{5c}$$

Here, $\nu^2 = 4\pi G\rho$ is the vertical oscillation frequency due to the self-gravity and ρ is the mass density of the ring matter. The first term of the R.H.S. of Eq. (5a) comes from the well known Coriolis acceleration. The second term of that equation comes from the difference between the centrifugal acceleration of the particle and the centrifugal acceleration of the observer in the rotating frame. We consider motions with $k << 1$, i.e., the wavelengths which are large compared to the amplitude of the bending wave. In this limit, $W \approx 1$ and $\phi = \phi^* + z_1 \epsilon k^2 sin\phi^*$ and $\cos\phi \approx \cos\phi^* - z_1 \epsilon k^2 sin^2\phi^*$. Using this, the $z-$component of the equation is re-written as,

$$\frac{dz_1}{dt^2} = -\alpha^2 z_1 - \epsilon^2(\Omega^2 - \omega^2) z_1 k^2 \sin^2\phi^*, \tag{6}$$

where, $\alpha^2 = \Omega^2 + \nu^2$. Re-writing in terms of phase,

$$\frac{\partial^2 z_1}{\partial \phi^{*2}} = -\frac{z_1}{\omega^2}\left[\alpha^2 + \frac{\epsilon^2(\Omega^2 - \omega^2)k^2}{2}\right] + \frac{\epsilon^2(\Omega^2 - \omega^2) z_1 k^2 \cos 2\phi^*}{2\omega^2}. \tag{7}$$

Close to the resonance orbit, $\Omega \sim \omega$ and the parameter,

$$\gamma^2 = \frac{\epsilon^2 k^2 (\Omega^2 - \omega^2)}{\omega^2} \tag{8}$$

could be treated as a perturbation parameter (Chakrabarti, 1989). Defining,

$$\eta^2 = \frac{\alpha^2}{\omega^2} + \frac{\gamma^2}{2}, \tag{9}$$

Eq. (7) becomes,

$$\frac{\partial^2 z_1}{\partial \phi^{*2}} = -\eta^2 z_1 + \frac{\gamma^2 \cos 2\phi^*}{2} z_1. \tag{10}$$

To obtain a complete solution, we expand z_1 in powers of γ^2 as,

$$z_1 = z_1^{(0)} + \sum_{i=0}^{N} \gamma^{2i} z_1^{(i)}. \tag{11}$$

Upon substitution in Eq. (10) and equating coefficients of the powers of γ^2 the solution of z is obtained, up to powers of γ^4. This then gives rise to x and y.

Numerical simulations (Chakrabarti, 1989; Chakrabarti and Bhattacharyya, 2001) tend to indicate that the shear due to variation of radial velocity component along vertical direction is significant. However, for the sake of completeness, the variation of vertical velocity component along the radial direction is also considered recently and the entire shear computation was done analytically (Bhattacharyya and Chakrabarti 2002). Analytically, the magnitude of this shear could be computed from (Bhattacharyya and Chakrabarti 2002),

$$\sigma^2 \sim \left(\frac{<\ddot{x}>}{<\dot{z}>})^2 + (\frac{<\ddot{z}>}{<\dot{x}>}\right)^2, \tag{12}$$

where, we used only the significant components in a co-rotating frame and <> indicates that the averaging is being done over a complete cycle ϕ^*. During excursion of the particle in epicyclic orbits inside the ring, it collides with a neighboring particle a few times in each orbit and in this process momentum is transferred from one radial distance to the next. The distance between two colliding particles (measured in the co-rotating frame) can be thought of as either the epicyclic radius or the size of the particle, whichever is bigger. For simplicity, we assume that the epicyclic radius is of the order of the half-height of the ring.

Our analytical solution may be used for the calculation of the wave profile as well as the damping length of the bending wave. For a wave profile, we first compute the amplitude of the wave at the site of the launching from (e.g., Rosen, Tyler and Marouf 1991),

$$\epsilon(r_v) = \frac{\mathcal{A}}{\sigma^{1/2}} \ \mathrm{m}, \tag{13}$$

where, the surface density σ is to be computed in c.g.s. unit. r_v is the location of the vertical resonance. Subsequent amplitude is self-consistently computed from energy dissipation. The radial wavelength of the launched wave is computed from (Rosen, Tyler and Marouf, 1991),

$$\lambda_z(r) = \frac{\mathcal{B}\sigma}{(r - r_v)} \ \mathrm{km}. \tag{14}$$

$\mathcal{A}$ and $\mathcal{B}$ are constants to be obtained from Rosen, Tyler Maraouf & Lissaur (1991) for a given resonance. The procedure is the following: we advance the wave by δr, and compute the reduced energy density at $r_2 = r_1 + \delta r$ from,

$$E_w(r_2) = E_w(r_1) - \frac{\dot{E_w}(r_1)}{c_g(r_1)}\delta r. \tag{15}$$

where, the energy density of the wave is calculated using,

$$E_w(r) = \frac{1}{2}\epsilon(r)^2\Omega(r)^2 \tag{16}$$

and the rate of dissipation of the wave is obtained from,

$$\dot{E}_w(r_1) = \nu_k(r_1)\sigma^2. \tag{17}$$

The instantaneous kinematic viscosity $\nu_k(r)$ is given by,

$$\nu_k(r) = \Omega(r) r \max\left(R^2, R_{epi}^2\right). \tag{18}$$

R is the size of the largest particle and R_{epi} is the amplitude of the epicyclic motion which is taken to be the half thickness of the ring i.e., h. The group velocity $c_g(r)$ is given by (Shu, Cuzzi & Lissauer, 1983),

$$c_g(r) = -\frac{s_k \pi G\sigma}{\omega(r) - m\Omega(r)}, \tag{19}$$

where, $\omega(r) = \Omega_p - \Omega(r)$, $\Omega_p = -2\pi/T_{titan}$ is the pattern frequency, same as the angular velocity of Titan. For -1:0 resonance, $m = 1$ was chosen. s_k is -1 for trailing bending wave propagating inward from inner vertical resonance and outward from the outer vertical resonance. For leading bending waves ($s_k = +1$) this is the opposite. Similarly, when perturbation due to Mimas is considered, $\Omega_p = -2\pi/T_{mimas}$ is used. We repeat the above procedure of integration, till the energy of the wave vanishes. The traversed length gives the damping length of the bending wave. Those cases which give similar to observed length and wavelengths are considered to be important.

3. Results for various ring conditions

We present two results, one from Mimas 5:3 resonance and the other for the Titan -1.0 resonance. Before we proceed, we wish to present the past results of wave profiles of the bending waves of Mimas 5:3 and Mimas 7:4 resonances (Fig. 3). The distance unit $\xi = -s_\epsilon x/\sqrt{2|\epsilon|}$, where s_ϵ is the sign of $\epsilon = 2\pi G\sigma/[r(rdD/dr)|_{r_v}]$, $x = (r - r_v)/r_v$, $D = \mu^2 - (\omega - m\Omega)^2$, m being a non-negative integer ($m = 4$ for 5:3 resonance and $m = 1$ for -1:0 resonance). For $\xi > 0$, the wave will propagate and for $\xi < 0$ the wave will be evanescent. The profiles clearly do not agree with the damping, because the normal shear was used.

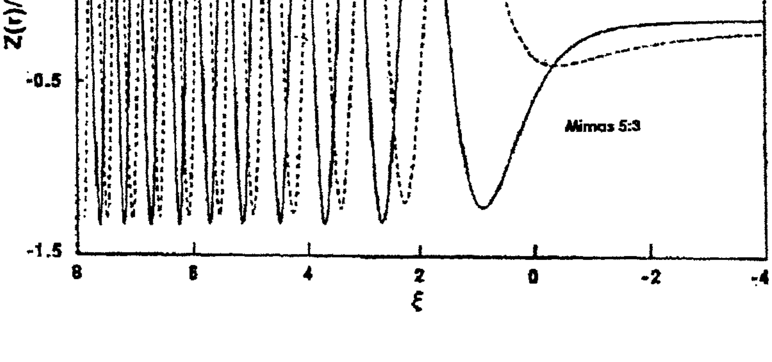

Figure 3. The computation of profiles of bending waves without considering vertical shear (from Shu, Cuzzi and Lissauer, 1983). The damping is negligible.

In Fig. 4, we show the nature of the wave profile that arise from our computation near the 5:3 resonance of Mimas for four sets of parameters. The profile of the lower-left panel drawn for a ring of height 4.2 meters and $\sigma = 46$gm cm^{-2} is preferred, as the damping length and the wavelength roughly agree with observation. For other parameters, the

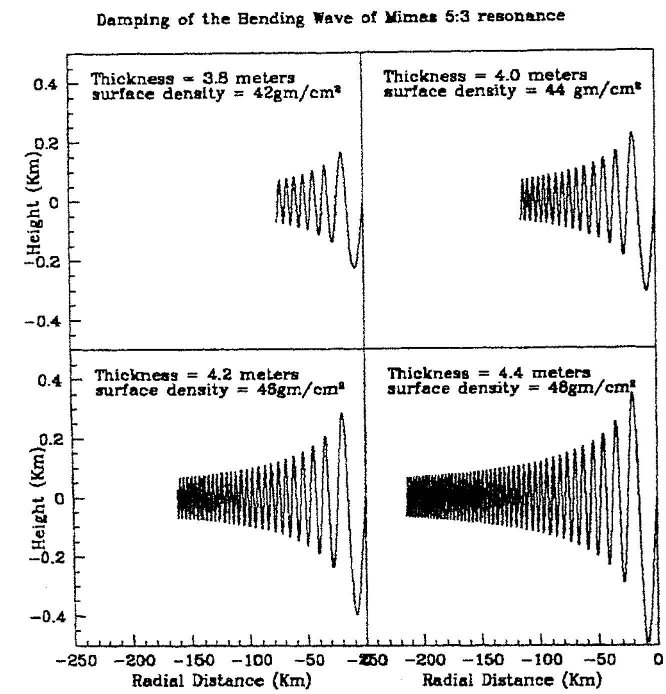

Figure 4. Shape of our bending wave profile at the 5:3 resonance of Mimas as a function of the ring thickness and the surface density. The profile of the lower-left panel drawn for a ring of height 4.2 meters and $\sigma = 46$gm cm^{-2} is preferred as the damping length and wavelength match with observations.

damping length is either too high or too low, and the wavelengths also are either too high or too low. In Fig. 5, we show results for Titan -1:0 resonance with two cases superimposed. Here we chose σ=0.31 gm cm^{-2} and 1.57gm cm^{-2} for ring-height of 2.3m for which the damping length is $\sim$ 85 km, same as the observed value. Clearly, the case with σ=0.31 gm cm^{-2} (dashed curve) is invalid because the total number of oscillations is 148, far too high compared to the observed wave profile. On the other hand, the profile for $\sigma = 1.57$ gm cm^{-2} has 28 waves (solid curve). This is equal to the observed number of oscillations (Rosen and Lissauer, 1988).

In Fig. 6, we show variation of damping length λ_D (km) for Titan -1:0 resonance as a function of the surface mass density σ for various thicknesses ($2h$, marked on each curve). The horizontal line is drawn to mark the observed damping length of 85km (Rosen and Lissauer 1988). This line clearly shows that a particular damping length may be observed

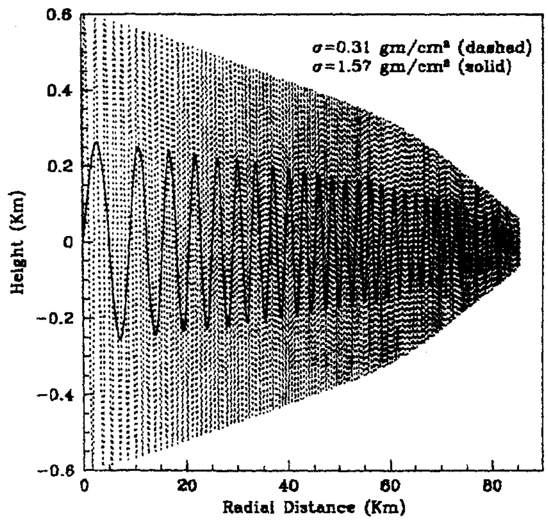

Figure 5. Shape of our bending wave profile at the -1:0 resonance of Titan as a function of the ring thickness and the surface density. The solid curve is drawn for $\sigma = 1.57$ gm cm^{-2} and $h = 2.3$m and has 28 waves as is observed.

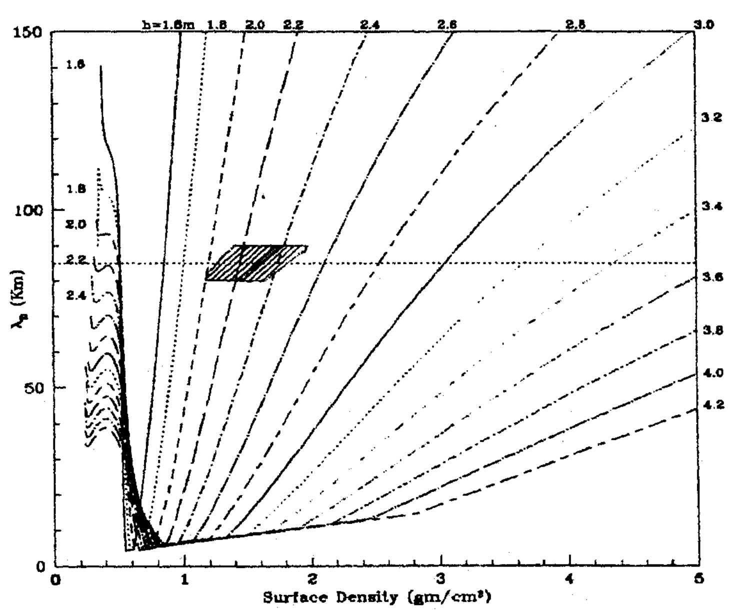

Figure 6. The parameter space spanned by surface density σ and the damping length λ_d for a range of disk height $2h$. The shaded region shows the plusible parameters for the C ring.

for two different values of surface mass density. It also shows that the ring height cannot be more that 3.6m. The shaded region around $\sigma \sim 1.3\mathrm{gm\ cm}^{-2}$ includes those parameters for which the number of oscillations lie between 25 and 35 and the damping length between 80 and 90 km and therefore our best guess for the C ring parameters lie in this shaded area.

4. Concluding Remarks

Calculation with the inclusion of various shear components indicates that the thickness of the ring (especially rings A and C for which carried out our analysis) is no more than $2-5$ meters. These numbers are comfortably below the estimated observed value of 100 meters, because of poor resolution of the camera. In future we wish to carry out this work for other resonances.

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