THE ASTROPHYSICAL JOURNAL, 371:673–683, 1991 April 20

ABERRATION AND ADVECTION EFFECTS IN EXPANDING SPHERICALLY SYMMETRIC SHELLS

A. PERAIAH
Indian Institute of Astrophysics, Sarjapur Road, Koramangala, Bangalore 560034, India
Received 1990 April 4; accepted 1990 October 4

ABSTRACT

We treated the aberration and advection effects in a spherical medium scattering radiation isotropically and coherently. We assumed that the spherical shell is moving radially with velocities 1000, 2000, 3000, 4000, and 5000 km s^{-1} ($\beta \approx 0.003$–0.017, where $\beta = v/c$ where v is the velocity of the gases and c is the velocity of light). The transfer equation is solved in the comoving frame. We essentially studied how these high velocities change the radiation field in terms of $\bar{J}$ where $\bar{J} = \{[J(V = 0) - J(V > 0)]/J(V = 0)\}$ where J is the mean intensity. $\bar{J}$ changes from -4% to 2% at $\tau = 1$, $B/A = 2$ (B and A are the outer to inner radii of the spherical shell) and to a maximum of 450% at $\tau = 10$ and $B/A = 2$. We notice that in spherical media unlike in plane parallel layers the changes in J are not proportional to the optical depth τ. When the optical depth is increased to 50, the deviations in $\bar{J}$ vary between -30% and 65%. When the geometrical thickness of the spherical shell is increased, the changes in $\bar{J}$ are not considerable. The amplification factors defined as $J_{max}/100\beta$ show a maximum at about $\tau = 10$ for $B/A = 2$ and 5, and the maxima in the latter case are much smaller than those in the formercase.

Subject headings: radiative transfer — stars: circumstellar shells — stars: supernovae

1. INTRODUCTION

The extreme outer layers of supernovae, novae, AGNs, QSOs, and many supergiant stars are known to be in rapid expansion or contraction. The information regarding internal physical structures, etc., is obtained from the radiation that passes through these outer layers. If the medium is in rapid radial motion, the radiation that passes through these layers will get modified. Therefore it becomes necessary to estimate this modification of large velocities on the emergent radiation. Mihalas, Kunasz, & Hummer (1976) studied the effect of aberration and advection that arises due to large velocities, on the formation of lines. They concluded that the terms that contain the Doppler shifts are more important than those containing aberration and advection with velocities of the order $v/c \approx 0.01$.

In an earlier paper (Peraiah 1987, henceforth paper I) we have shown that the aberration and advection terms produce large changes in the radiation field in a plane-parallel medium which scatters monochromatic radiation isotropically and coherently. We would like to examine whether or not such changes continue to exist in a spherically symmetric medium. We shall assume the same physical characteristics of the medium as in paper I. We shall describe the procedure below. Although it is described in Peraiah (1987), we present the derivation for the sake of completeness.

2. DERIVATION OF THE SOLUTION OF THE EQUATION OF TRANSFER

In this section we shall derive the solution of the equation of transfer by integrating it on the angle-radius mesh. We obtain the reflection and transmission operators, $r(i, i-1)$ and $r(i-1, i)$; $t(i, i-1)$ and $t(i-1, i)$, on this mesh using the principle of interaction (see Peraiah & Grant 1973). The equation of radiative transfer in a spherically symmetric media in the comoving frame of the fluid with relativistic terms of the order of v/c is written as (Castor 1972; Mihalas 1978; Munier & Weaver 1986, Peraiah 1987)

$$(m+\beta)\frac{\partial U(r,m)}{\partial r} + \frac{1-m^2}{r}\left[1 + m\beta\left(1 - \frac{r}{\beta}\frac{d\beta}{dr}\right)\right]\frac{\partial U(r,m)}{\partial m} = K(r)[S(r) - U(r,m)] + \frac{2(m+\beta)U(r,m)}{r} - 3\left[\frac{\beta}{r}(1-m^2) + m^2\frac{d\beta}{dr}\right]U(r,m)\,, \quad (1)$$

where $\beta = v/c$, $m = (\mu - \beta)/(1 - \mu\beta)$, $U(r, m) = 4\pi r^2 I(r, m)$.

The quantity $I(r, m)$ is the specific intensity of the ray whose direction makes an angle of $\cos^{-1}\mu$ with the radius vector r. $K(r)$ is the absorption coefficient and $S(r)$ is the source function. Equation (1) can be integrated on the angle-radius mesh defined on $[m_{j-1}, m_j][r_{i-1}, r_i]$ (see Peraiah & Varghese 1985). The integration of equation (1) is done by expressing the specific intensity in terms of certain interpolation coefficients and weight factors:

$$U(r, m) = U_0 + U_r\xi + U_m\eta + U_{rm}\xi\eta\,, \quad (2)$$

where U_0, U_r, U_m, U_{rm} are the interpolation coefficients. The source function S is similarly defined. In equation (2), the quantities ξ and η are defined as

$$\xi = \frac{r - \bar{r}}{\Delta r/2} \quad \text{and} \quad \eta = \frac{m - \bar{m}}{\Delta m/2}\,, \quad (3)$$

where

$$\bar{r} = (r_{i-1} + r_i)/2\ , \qquad \Delta r = (r_i - r_{i-1})\ , \tag{4}$$

$$\bar{m} = (m_{j-1} + m_j)/2\ , \qquad \Delta m = (m_j - m_{j-1})\ . \tag{5}$$

Substituting equation (2) in (1) and simplifying, we obtain,

$$\frac{2}{\Delta r}(m+\beta)(U_r + \eta U_{rm}) + \frac{2}{\Delta m}\frac{1-m^2}{r}\left[1 + m\beta\left(1 - \frac{r}{\beta}\frac{d\beta}{dr}\right)\right]\left[U_m + \xi U_{rm}\right]$$
$$= K(r)\left[(S_0 + S_r\xi + S_m\eta + S_{rm}\xi\eta) - (U_0 + U_r\xi + U_m\eta + U_{rm}\xi\eta)\right]$$
$$+ \left[\frac{2(m+\beta)}{r} - 3\left\{\frac{\beta}{r}(1-m^2) + m^2\frac{d\beta}{dr}\right\}\right]\left\{U_0 + U_r\xi + U_m\eta + U_{rm}\xi\eta\right\}\ , \tag{6}$$

where the source function S is expanded in a similar manner as the specific intensity given in question (2). We shall now apply the operator

$$X_m = \frac{1}{\Delta m}\int_{\Delta m} \dots\, dm \tag{7}$$

on equation (6). In this process we use the following identities:

$$X_m[Z] = Z \qquad \text{where } Z = \text{constant}\ , \tag{8}$$

$$X_m[\eta] = 0\ , \tag{9}$$

$$X_m[m] = \bar{m} \tag{10}$$

$$X_m[\eta m] = \tfrac{1}{6}\Delta m\ , \tag{11}$$

$$X_m[m^2] = \bar{m}^2\ , \tag{12}$$

$$X_m[\eta m^2] = \tfrac{1}{3}\Delta m \cdot \bar{m}\ , \tag{13}$$

$$X_m[m^3] = \bar{m}\langle m^2\rangle\ , \tag{14}$$

where

$$\bar{m}^2 = \tfrac{1}{3}(m_j^2 + m_j m_{j-1} + m_{j-1}^2)\ , \tag{15}$$

and

$$\langle m^2\rangle = \tfrac{1}{2}(m_j^2 + m_{j-1}^2)\ . \tag{16}$$

Applying the operator (7) on equation (6) and using the identities given in (8) to (16) will give us

$$\frac{2}{\Delta r}\left\{(\bar{m}+\beta)U_r + \frac{1}{6}\Delta m U_{rm}\right\} + \frac{2}{r}\left(\frac{1}{\Delta m}\right)\left\{(1-\bar{m}^2) + \bar{m}(1 - \langle m^2\rangle)\beta R\right\}(U_m + U_{rm}\xi)$$
$$= K(r)\left[(S_0 + S_r\xi) - (U_0 + U_r\xi)\right] + \left\{\frac{2\bar{m} + \beta(3\bar{m}^2 - 1)}{r} - 3\bar{m}^2\frac{d\beta}{dr}\right\}\left\{U_0 + U_r\xi\right\}$$
$$+ \Delta m \cdot \bar{m}\left\{\frac{1}{r}\left(\beta + \frac{1}{3\bar{m}}\right) - \frac{d\beta}{dr}\right\}\left\{U_m + U_{rm}\xi\right\}\ , \tag{17}$$

where

$$R = 1 - \frac{r}{\beta}\frac{d\beta}{dr}\ . \tag{18}$$

Let us assume that

$$\frac{d\beta}{dr} \approx \frac{\Delta\beta}{\Delta r} = S = \text{constant}\ . \tag{19}$$

Then, we have

$$\Delta\beta = S \cdot \Delta r\ , \tag{20}$$

where

$$\Delta r = (r_i - r_{i-1})\ . \tag{21}$$

Now, we shall perform the integration on the radial mesh. For this purpose, we apply the operator

$$Y_v = \frac{1}{V}\int_{\Delta r} \dots 4\pi r^2\,dr\,, \tag{22}$$

where

$$V = \frac{4\pi}{3}(r_i^3 - r_{i-1}^3)\,, \tag{23}$$

on equation (17). This gives us

$$\begin{aligned}
&\frac{2}{\Delta r}\left\{(\bar{m}+\beta)U_r + \frac{1}{6}\Delta m \cdot U_{rm}\right\} + \left\{\frac{1}{2}\frac{\Delta A}{V}G - \frac{G}{\beta}\left(\frac{d\beta}{dr}\right) + \left(\frac{\Delta A}{V}\right)\frac{1-\bar{m}^2}{\Delta m}\right\}U_m + \left\{\frac{G}{\Delta r}\left(2 - \frac{r\,\Delta A}{V}\right)\right.\\
&\qquad \left. - \frac{1}{6}\frac{\Delta A}{\bar{A}}\frac{G}{\beta}\frac{d\beta}{dr} + \frac{2}{\Delta r}\frac{1-\bar{m}^2}{\Delta m}\left(2 - \frac{\bar{r}\,\Delta A}{V}\right)\right\}U_{rm} = K\left\{\left(S_0 + \frac{1}{6}\frac{\Delta A}{\bar{A}}S_r\right) - \left(U_0 + \frac{1}{6}\frac{\Delta A}{\bar{A}}U_r\right)\right\}\\
&\qquad + \left\{2\bar{m} + \beta(3\bar{m}^2 - 1)\right\}\left\{\frac{1}{2}\frac{\Delta A}{V}U_0 + \frac{1}{\Delta r}\left(2 - \frac{\bar{r}\,\Delta A}{V}\right)U_r\right\} - 3\bar{m}^2\frac{d\beta}{dr}\left(U_0 + \frac{1}{6}\frac{\Delta A}{\bar{A}}U_r\right)\\
&\qquad + \bar{m}\cdot\Delta\bar{m}\left[\left(\beta + \frac{1}{3\bar{m}}\right)\left\{\frac{1}{2}\frac{\Delta A}{V}U_m + \frac{1}{\Delta r}\left(2 - \frac{\bar{r}\,\Delta A}{V}\right)U_{rm}\right\} - \frac{d\beta}{dr}\left(U_m + \frac{1}{6}\frac{\Delta A}{\bar{A}}U_{rm}\right)\right],
\end{aligned} \tag{24}$$

where

$$G = \frac{2\bar{m}}{\Delta m}(1 - \langle m^2\rangle)\beta\,, \tag{25}$$

$$\Delta A = 4\pi(r_i^2 - r_{i-1}^2)\,, \tag{26}$$

$$\bar{A} = V/\Delta r\,. \tag{27}$$

Collecting the coefficients of the interpolation coefficients U_0, U_r, etc., we can rewrite equation (24)

$$\begin{aligned}
&U_r\left[\frac{2}{\Delta r}(\bar{m}+\beta) + \frac{1}{6}\frac{\Delta A}{\bar{A}}K - p\left\{2\bar{m} + \beta(3\bar{m}^2 - 1)\right\} + \frac{1}{6}\frac{\Delta A}{\bar{A}}3\bar{m}^2\frac{d\beta}{dr}\right]\\
&\qquad + U_m\left[G\left(\frac{1}{2}\frac{\Delta A}{V} - \frac{1}{\beta}\frac{d\beta}{dr}\right) + \frac{\Delta A}{V}\left(\frac{1-\bar{m}^2}{\Delta m}\right) - \Delta m\cdot\bar{m}\left\{\left(\beta + \frac{1}{3\bar{m}}\right)\frac{1}{2}\frac{\Delta A}{V} - \frac{d\beta}{dr}\right\}\right]\\
&\qquad + U_{rm}\left[\frac{1}{3}\frac{\Delta m}{\Delta r} + Gp - \frac{1}{6}\frac{\Delta A}{\bar{A}}\frac{G}{\beta}\frac{d\beta}{dr} + 2p\frac{1-\bar{m}^2}{\Delta m} - \Delta m\cdot\bar{m}\left\{\left(\beta + \frac{1}{3\bar{m}}\right)p - \frac{1}{6}\frac{\Delta A}{\bar{A}}\frac{d\beta}{dr}\right\}\right]\\
&\qquad + U_0\left[K - \frac{1}{2}\frac{\Delta A}{V}\left\{2\bar{m} + \beta(3\bar{m}^2 - 1)\right\} + 3\bar{m}^2\frac{d\beta}{dr}\right] = K\left[S_0 + \frac{1}{6}\frac{\Delta A}{\bar{A}}S_r\right].
\end{aligned} \tag{28}$$

For $-m$, the transfer equation is written as

$$\begin{aligned}
&(-m+\beta)\frac{\partial U(r,-m)}{\partial r} - \frac{1-m^2}{r}\left[1 - m\beta\left(1 - \frac{r}{\beta}\frac{d\beta}{dr}\right)\right]\frac{\partial U(r,-m)}{\partial m}\\
&\qquad = K[S(r) - U(r,-m)] + 2\frac{(-m+\beta)}{r}U(r,-m) - 3\left[\frac{\beta}{r}(1-m^2) + m^2\frac{d\beta}{dr}\right]U(r,-m)\,.
\end{aligned} \tag{29}$$

Substituting equation (2) in equation (29), we obtain

$$\begin{aligned}
&\frac{2}{\Delta r}(-m+\beta)(U_r + U_{rm}\eta) - \frac{2}{\Delta m}\frac{1-m^2}{r}\left[1 - m\beta\left(1 - \frac{r}{\beta}\frac{d\beta}{dr}\right)\right]\left(U_m + U_{rm}\xi\right) = K(r)S(r)\\
&\qquad + \left[K(r) + \frac{1}{r}\left\{2(-m+\beta) - 3\beta(1-m^2)\right\} - 3m^2\frac{d\beta}{dr}\right]\left(U_0 + U_r\xi + U_m\eta + U_{rm}\xi\eta\right).
\end{aligned} \tag{30}$$

Application of X_m on equation (30) gives

$$\frac{2}{\Delta r}\left\{(\beta-\bar{m})U_r-\frac{1}{6}\Delta m U_{rm}\right\}-\frac{2}{\Delta m}\frac{1}{r}\left\{(1-\bar{m}^2)-\bar{m}(1-\langle\bar{m}^2\rangle)\beta\left(1-\frac{r}{\beta}\frac{d\beta}{dr}\right)\right\}(U_m+U_{rm}\xi)$$
$$=K\left[(S_0+S_r\xi)-(U_0+U_r\xi)\right]-\left\{\frac{2m+\beta(1-3\bar{m}^2)}{r}+3\bar{m}^2\frac{d\beta}{dr}\right\}(U_0+U_r\xi)$$
$$-\frac{\Delta m\cdot m}{r}\left(r\frac{d\beta}{dr}-\beta+\frac{1}{3\bar{m}}\right)(U_m+U_{rm}\xi)\,. \quad (31)$$

Applying the operator Y_v on equation (31), we obtain

$$\frac{2}{\Delta r}\left\{(\beta-\bar{m})U_r-\frac{1}{6}\Delta m U_{rm}\right\}+\left(\frac{1}{2}\frac{\Delta A}{V}G-\frac{G}{\beta}\frac{d\beta}{dr}-\frac{\Delta A}{V}\frac{1-\bar{m}^2}{\Delta m}\right)U_m+\left\{G_p-\frac{1}{6}\frac{\Delta A}{\bar{A}}\frac{G}{\beta}\frac{d\beta}{dr}-\frac{2}{\Delta m}(1-\bar{m}^2)p\right\}U_{rm}$$
$$=K\left\{\left(S_0+\frac{1}{6}\frac{\Delta A}{\bar{A}}S_r\right)-\left(U_0+\frac{1}{6}\frac{\Delta A}{\bar{A}}U_r\right)\right\}-\left\{\frac{1}{2}\frac{\Delta A}{V}(2\bar{m}+\beta-3\beta\bar{m}^2)+3\bar{m}^2\frac{d\beta}{dr}\right\}U_0$$
$$-\left\{p(2\bar{m}+\beta-3\beta\bar{m}^2)+3\bar{m}^2\frac{d\beta}{dr}\frac{1}{6}\frac{\Delta A}{\bar{A}}\right\}U_r-\left\{\Delta m\cdot\bar{m}\left[\frac{d\beta}{dr}+\frac{1}{2}\frac{\Delta A}{V}\left(\frac{1}{3\bar{m}}-\beta\right)\right]\right\}U_m$$
$$-\left\{\Delta m\cdot\bar{m}\left[\frac{1}{6}\frac{\Delta A}{\bar{A}}\frac{d\beta}{dr}+p\left(\frac{1}{3\bar{m}}-\beta\right)\right]\right\}U_m\,, \quad (32)$$

where

$$p=\frac{1}{\Delta r}\left(2-\frac{\bar{r}\cdot\Delta A}{V}\right). \quad (33)$$

Equation (32) is rewritten after rearranging the interpolation coefficients:

$$U_r\left[\frac{2}{\Delta r}(\beta-\bar{m})+\frac{1}{6}\frac{\Delta A}{\bar{A}}K+p(2\bar{m}+\beta-3\beta\bar{m}^2)+3\bar{m}^2\frac{d\beta}{dr}\frac{1}{6}\frac{\Delta A}{\bar{A}}\right]$$
$$+U_m\left[\frac{1}{2}\frac{\Delta A}{V}G-\frac{G}{\beta}\frac{d\beta}{dr}-\frac{\Delta A}{V}\frac{1-\bar{m}^2}{\Delta m}+\Delta m\bar{m}\left\{\frac{d\beta}{dr}+\frac{1}{2}\frac{\Delta A}{V}\left(\frac{1}{3\bar{m}}-\beta\right)\right\}\right]$$
$$+U_{rm}\left[-\frac{1}{3}\frac{\Delta m}{\Delta r}+Gp-\frac{1}{6}\frac{\Delta A}{\bar{A}}\frac{G}{\beta}\frac{d\beta}{dr}-\frac{2}{\Delta m}(1-\bar{m}^2)p+\Delta m\cdot\bar{m}\left\{\frac{1}{6}\frac{\Delta A}{\bar{A}}\frac{d\beta}{dr}+p\left(\frac{1}{3\bar{m}}-\beta\right)\right\}\right]$$
$$+U_0\left[K+\frac{1}{2}\frac{\Delta A}{V}(2\bar{m}+\beta-3\beta\bar{m}^2+3\bar{m}^2\frac{d\beta}{dr}\right]=K\left(S_0+\frac{1}{6}\frac{\Delta A}{\bar{A}}S_r\right). \quad (34)$$

Multiplying equations (24) and (34) by Δr, we get

$$\alpha U_r+\beta_1 U_m+\gamma U_{rm}+\delta U_0=\Delta r\cdot K\left[S_0+\frac{1}{6}\frac{\Delta A}{\bar{A}}S_r\right], \quad (35)$$

and

$$\alpha' U_r+\beta_1' U_m+\gamma' U_{rm}+\delta' U_0=\Delta r\cdot K\left[S_0+\frac{1}{6}\frac{\Delta A}{\bar{A}}S_r\right], \quad (36)$$

where

$$\alpha=\Delta r\left\{\frac{2}{\Delta r}(\bar{m}+g)+\frac{1}{6}\frac{\Delta A}{\bar{A}}K-p(2\bar{m}+\beta-3\bar{m}^2-1)+\frac{1}{6}\frac{\Delta A}{\bar{A}}3\bar{m}^2\frac{d\beta}{dr}\right\}, \quad (37)$$

$$\beta_1=\Delta r\left\{G\left(\frac{1}{2}\frac{\Delta A}{V}-\frac{1}{\beta}\frac{d\beta}{dr}\right)+\frac{\Delta A}{V}\frac{1-\bar{m}^2}{\Delta m}-\Delta m\bar{m}\left[\left(\beta+\frac{1}{3\bar{m}}\right)\frac{1}{2}\frac{\Delta A}{V}-\frac{d\beta}{dr}\right]\right\}, \quad (38)$$

$$\gamma=\Delta r\left\{\frac{1}{3}\frac{\Delta\bar{m}}{\Delta r}+Gp-\frac{1}{6}\frac{\Delta A}{\bar{A}}\frac{G}{\beta}\frac{d\beta}{dr}+2p\frac{1-\bar{m}^2}{\Delta m}-\Delta m\bar{m}\left[\left(\beta+\frac{1}{3\bar{m}}\right)p-\frac{1}{6}\frac{\Delta A}{\bar{A}}\frac{d\beta}{dr}\right]\right\}, \quad (39)$$

$$\delta=\Delta r\left\{K-\frac{1}{2}\frac{\Delta A}{V}[2\bar{m}+\beta(3\bar{m}^2-1)]+3\bar{m}^2\frac{d\beta}{dr}\right\}. \quad (40)$$

Furthermore, we have,

$$\alpha' = \Delta r\left[\frac{2}{\Delta r}(\beta - \bar{m}) + \frac{1}{6}\frac{\Delta A}{\bar{A}}K + A_2\right], \tag{41}$$

$$\beta_1' = \Delta r(B_1 + A_3), \tag{42}$$

$$\gamma' = \Delta r\left(A_4 + B_2 - \frac{1}{3}\frac{\Delta m}{\Delta r}\right), \tag{43}$$

where

$$\delta' = \Delta r(K + A_1), \tag{44}$$

$$A_1 = \frac{1}{2}\frac{\Delta A}{V}(2\bar{m} + \beta - 3\beta\bar{m}^2) + 3\bar{m}^2\frac{d\beta}{dr}, \tag{45}$$

$$A_2 = p(2\bar{m} + \beta - 3\beta\bar{m}^2) + 3\bar{m}^2\frac{d\beta}{dr}\frac{1}{6}\frac{\Delta A}{\bar{A}}, \tag{46}$$

$$A_3 = \Delta\bar{m}\bar{m}\left[\frac{d\beta}{dr} + \frac{1}{2}\frac{\Delta A}{V}\left(\frac{1}{3\bar{m}} - \beta\right)\right], \tag{47}$$

$$A_4 = \Delta m\bar{m}\left[\frac{1}{6}\frac{\Delta A}{\bar{A}}\frac{d\beta}{dr} + p\left(\frac{1}{3\bar{m}} - \beta\right)\right], \tag{48}$$

$$B_1 = \frac{1}{2}\frac{\Delta A}{V}G - \frac{G}{\beta}\frac{d\beta}{dr} - \frac{\Delta A}{V}\frac{1 - \bar{m}^2}{\Delta m}, \tag{49}$$

$$B_2 = Gp - \frac{1}{6}\frac{\Delta A}{\bar{A}}\frac{G}{\beta}\frac{d\beta}{dr} - \frac{2}{\Delta m}(1 - \bar{m}^2)p. \tag{50}$$

We now replace U_0, U_r, U_m, U_{rm} in equations (35) and (36), by the nodal values U_a, U_b, etc. (see Peraiah & Varghese 1985):

$$\alpha(-U_a - U_b - U_c - U_d) + \beta_1(-U_a + U_b + U_c - U_d) + \gamma(U_a - U_b - U_c + U_d) + \delta(U_a + U_b + U_c + U_d) = \tau\left[(S_a + S_b + S_c + S_d)\left(1 - \frac{1}{6}\frac{\Delta A}{\bar{A}}\right)\right], \tag{51}$$

and

$$\alpha'(-U_a - U_b - U_c - U_d) + \beta_1'(-U_a + U_b + U_c - U_d) + \gamma(U_a - U_b - U_c + U_d) + \delta(U_a + U_b + U_c + U_d) = \tau\left(1 - \frac{1}{6}\frac{\Delta A}{\bar{A}}\right)(S_a + S_b + S_c + S_d). \tag{52}$$

On rearranging,

$$U_a(-\alpha - \beta_1 + \gamma + \delta) + U_b(-\alpha + \beta_1 - \gamma + \delta) + U_c(-\alpha + \beta_1 - \gamma + \delta) + U_d(-\alpha - \beta_1 + \gamma + \delta) = \tau\left(1 - \frac{1}{6}\frac{\Delta A}{\bar{A}}\right)(S_a + S_b + S_c + S_d), \tag{53}$$

and

$$U_a(-\alpha' - \beta_1' + \gamma' + \delta') + U_b(-\alpha' + \beta_1' - \gamma' + \delta') + U_c(-\alpha' + \beta_1' - \gamma' + \delta') + U_d(-\alpha' - \beta_1' + \gamma' + \delta') = \tau\left(1 - \frac{1}{6}\frac{\Delta A}{\bar{A}}\right)(S_a + S_b + S_c + S_d), \tag{54}$$

where

$$\tau = K \cdot \Delta r. \tag{55}$$

Letting

$$A_a = -\alpha - \beta_1 + \gamma + \delta, \tag{56}$$

$$A_b = -\alpha + \beta_1 - \gamma + \delta, \tag{57}$$

$$A_c = -\alpha + \beta_1 - \gamma + \delta, \tag{58}$$

$$A_d = -\alpha - \beta_1 + \gamma + \delta, \tag{59}$$

and

$$A'_a = -\alpha' - \beta'_1 + \gamma' + \delta' , \tag{60}$$

$$A'_b = -\alpha' + \beta'_1 - \gamma' + \delta' , \tag{61}$$

$$A'_c = -\alpha' + \beta'_1 - \gamma' + \delta' , \tag{62}$$

$$A'_d = -\alpha' - \beta'_1 + \gamma' + \delta' , \tag{63}$$

we can write equations (53) and (54) as

$$A_a U_a^+ + A_b U_b^+ + A_c U_c^+ + A_d U_d^+ = \left(1 - \frac{1}{6}\frac{\Delta A}{\bar{A}}\right)(S_a^+ + S_b^+ + S_c^+ + S_d^+) , \tag{64}$$

and

$$A'_a U_a^- + A'_b U_b^- + A'_c U_c^- + A'_d U_d^- = \left(1 - \frac{1}{6}\frac{\Delta A}{\bar{A}}\right)(S_a^- + S_b^- + S_c^- + S_d^-) . \tag{65}$$

By writing the nodal values in their full form, such as

$$U_a^+ = U_{j-1}^{i-1,+} , \qquad U_b^+ = U_j^{i-1,+} , \text{ etc.} , \tag{66}$$

we obtain

$$A_a U_{j-1}^{i-1,+} + A_b U_j^{i-1,+} + A_c U_{j-1}^{i,+} + A_d U_j^{i,+} = \tau\left(1 - \frac{1}{6}\frac{\Delta A}{\bar{A}}\right)(S_{j-1}^{i-1,+} + S_j^{i-1,+}) + \tau\left(1 + \frac{1}{6}\frac{\Delta A}{\bar{A}}\right)(S_{j-1}^{i,+} + S_j^{i,+}) , \tag{67}$$

and

$$A'_a U_{j-1}^{i-1,-} + A'_b U_j^{i-1,-} + A'_c U_{j-1}^{i,-} + A'_d U_j^{i,-} = \tau\left(1 - \frac{1}{6}\frac{\Delta A}{\bar{A}}\right)(S_{j-1}^{i-1,-} + S_j^{i-1,-}) + \tau\left(1 + \frac{1}{6}\frac{\Delta A}{\bar{A}}\right)(S_{j-1}^{i,-} + S_j^{i,-}) , \tag{68}$$

where

$$S_{j-1}^{i-1,+} = \sum_{j=1}^{J} \frac{1}{2}\,\omega(P^{++}CU^{+,i-1} + P^{-+}CU^{-,i-1})_{j-1} + (1-\omega)B_j^{i-1,+} , \tag{69}$$

$$S_j^{i-1,+} = \sum_{j=1}^{J} \frac{1}{2}\,\omega(P^{++}CU^{+,i-1} + P^{+-}CU^{-,i-1})_j + (1-\omega)B_j^{i-1,+} , \tag{70}$$

where ω is the albedo for single scattering, the quantities P^{++}, P^{+-} etc., represent the phase function for different directions, C is the quadrature weight for angle integration, and the B's are the internal sources. Defining

$$\mathbf{A}^{ab} = \begin{bmatrix} A_a^j & A_b^{j+1} & & \\ & A_a^{j+1} & A_b^{j+2} & \\ & & A_a^{J-1} & A_b^J \\ & & & A_a^J \end{bmatrix} , \tag{71}$$

and similarly for A^{dc}, we can rewrite equations (67) and (68) as

$$[\mathbf{A}^{dc} - \tau^+\mathbf{Q}\gamma^{++}]\mathbf{U}_i^+ + [\mathbf{A}^{ab} - \tau^-\mathbf{Q}\gamma^{++}]\mathbf{U}_{i-1}^+ = (1-\omega)\tau\mathbf{Q}\mathbf{B}^+ + \tau^+\mathbf{Q}\gamma^{+-}\mathbf{U}_i^- + \tau^-\mathbf{Q}\gamma^{+-}\mathbf{U}_{i-1}^- , \tag{72}$$

and

$$[\mathbf{A}'^{f\beta} - \tau^+\mathbf{Q}\gamma^{--}]\mathbf{U}_i^- + [\mathbf{A}'^{ab} - \tau^-\mathbf{Q}\gamma^{--}]\mathbf{U}_{i-1}^- = (1-\omega)\tau\mathbf{Q}\mathbf{B}^- + \tau^+\mathbf{Q}\gamma^{-+}\mathbf{U}_i^+ + \tau^-\mathbf{Q}\gamma^{-+}\mathbf{U}_{i-1}^+ , \tag{73}$$

where

$$\mathbf{Q} = \begin{pmatrix} 1 & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \\ & & & 1 \end{pmatrix} , \tag{74}$$

$$\tau^+ = \tau\left(1 + \frac{1}{6}\frac{\Delta A}{\bar{A}}\right) , \tag{75}$$

$$\tau^- = \tau\left(1 - \frac{1}{6}\frac{\Delta A}{\bar{A}}\right) , \tag{76}$$

$$\gamma^{++} = \tfrac{1}{2}\omega\mathbf{P}^{++}\mathbf{C} , \tag{77}$$

$$\gamma^{+-} = \tfrac{1}{2}\omega\mathbf{P}^{+-}\mathbf{C} . \tag{78}$$

The quantities γ^{--}, γ^{-+} are defined similarly. With

$$\bar{\mathbf{A}}_{ab} = \mathbf{Q}^{-1}\mathbf{A}^{ab} , \tag{79}$$

$$\bar{\mathbf{A}}_{dc} = \mathbf{Q}^{-1}\mathbf{A}^{dc} , \tag{80}$$

$$\bar{\mathbf{A}}'_{ab} = \mathbf{Q}^{-1}\mathbf{A}'^{ab} , \tag{81}$$

$$\bar{\mathbf{A}}'_{dc} = \mathbf{Q}^{-1}\mathbf{A}'^{dc} , \tag{82}$$

equations (72) and (73) become

$$[\bar{\mathbf{A}}_{dc} - \tau^{+}\gamma^{++}]\mathbf{U}_i^{+} + [\bar{\mathbf{A}}_{ab} - \tau^{-}\gamma^{++}]\mathbf{U}_{i-1}^{+} = (1-\omega)\tau\mathbf{B}^{+} + \tau^{+}\gamma^{+-}\mathbf{U}_i^{-} + \tau^{-}\gamma^{+-}\mathbf{U}_{i-1}^{-} , \tag{83}$$

$$[\bar{\mathbf{A}}'_{dc} - \tau^{+}\gamma^{--}]\mathbf{U}_i^{-} + [\bar{\mathbf{A}}'_{dc} - \tau^{-}\gamma^{--}]\mathbf{U}_{i-1}^{-} = (1-\omega)\tau\mathbf{B}^{-} + \tau^{+}\gamma^{-+}\mathbf{U}_i^{-} + \tau^{-}\gamma^{-+}\mathbf{U}_{i-1}^{+} . \tag{84}$$

Equations (83) and (88) are compared using the interaction principle (Peraiah & Grant 1973), and the two pairs of transmission and reflection operators are written as

$$\mathbf{t}(i, i-1) = \mathbf{R}^{+-}[\Delta^{+}\mathbf{A} + \mathbf{r}^{+-}\Delta^{-}\mathbf{C}] , \tag{85}$$

$$\mathbf{t}(i-1, i) = \mathbf{R}^{-+}[\Delta^{-}\mathbf{D} + \mathbf{r}^{-+}\Delta^{+}\mathbf{B}] , \tag{86}$$

$$\mathbf{r}(i, i-1) = \mathbf{R}^{-+}[\Delta^{-}\mathbf{C} + \mathbf{r}^{-+}\Delta^{+}\mathbf{A}] , \tag{87}$$

$$\mathbf{r}(i-1, i) = \mathbf{R}^{+-}[\Delta^{+}\mathbf{B} + \mathbf{r}^{+-}\Delta^{-}\mathbf{D}] , \tag{88}$$

where

$$\Delta^{+} = [\bar{\mathbf{A}}_{cd} - \tau^{+}\gamma^{++}]^{-1} , \tag{89}$$

$$\Delta^{-} = [\bar{\mathbf{A}}_{ab} - \tau^{+}\gamma^{--}]^{-1} , \tag{90}$$

$$A = \tau\gamma^{++} + A_{ab} , \tag{91}$$

$$B = \tau\gamma^{+-} , \tag{92}$$

$$C = \tau\gamma^{-+} , \tag{93}$$

$$D = \tau\gamma^{++} + A_{dc} , \tag{94}$$

$$\mathbf{r}^{+-} = \tau\Delta^{+}\gamma^{+-} , \tag{95}$$

$$\mathbf{R}^{+-} = [\mathbf{I} - \mathbf{r}^{+-}\mathbf{r}^{-+}]^{-1} . \tag{96}$$

The transmission and reflection operators given in equation (85) to (88) are made use of in calculating the internal radiation field as described in Peraiah & Grant (1973).

3. RESULTS AND DISCUSSION

We have considered a spherically symmetric shell bounded by $\tau = \tau_{\max}$ at the inner radius A and $\tau = 0$ at the outer radius B, where τ is the optical depth of the shell. We assumed that the medium is non-emitting (see paper I) with isotropic and coherent scattering. The boundary conditions on the radiation field are

$$U^{-}(T, \mu_j) = 1 , \tag{97}$$

and

$$U^{+}(\tau = 0, \mu_j) = 0 . \tag{98}$$

Regarding the boundary condition of the expansion velocity, we set

$$V(T, r = A) = 0 , \tag{99}$$

and

$$V(\tau = 0, r = B) = V , \tag{100}$$

where

$$T = \tau_{\max} ,$$
$$V = 0 \text{ km s}^{-1}(\beta = 0),\ 1000 \text{ km s}^{-1}(\beta = 0.0033) ,$$
$$2000 \text{ km s}^{-1}(\beta = 0.0067),\ 3000 \text{ km s}^{-1}(\beta = 0.01) ,$$
$$4000 \text{ km s}^{-1}(\beta = 0.013),\ 5000 \text{ km s}^{-1}(\beta = 0.0167) . \tag{101}$$

The velocity gradient is positive and constant. The total optical depths we have considered are

$$T = 1, 5, 10, 30, 50 . \tag{102}$$

Further, the aspect ratio values B/A are

$$\frac{B}{A} = 2 \text{ and } 5 . \tag{103}$$

The mean intensities are computed by using the relation

$$J = \frac{1}{2} \int_{-1}^{+1} U(\mu) d\mu . \tag{104}$$

Here we are not calculating the outward fluxes as was done in paper I, because the fluxes show similar behavior as the mean intensities. The changes in J due to velocities compared to those calculated in the static shell are calculated by using the relation

$$\bar{J} = \frac{\Delta J}{J(V = 0)} \times 100 , \tag{105}$$

where

$$\Delta J = J(V = 0) - J(V > 0) . \tag{106}$$

We divided the spherical shell into several smaller shells so that we obtain a stable solution; we calculated the **r** and **t** matrices in these smaller shells and add them by the star algorithm (see Peraiah & Grant 1973). If the condition of flux conservation is not satisfied, we subdivide the shell both geometrically and optically and repeat the procedure of the star algorithm. When velocities are introduced the shell thickness is reduced further.

In Figure 1, we plotted $\bar{J}$ for $B/A = 2$ and $T = 1$. Here $\tau = 0(r = B)$ corresponds to the emergent side and $T = 1(r = A)$ corresponds to the incident side of the shell. We notice here that, unlike in the plane-parallel case, (see paper I, Fig. 5) $\bar{J}$ has maximum and minimum values for all the velocities, the greatest variation (for $V = 5000$ km s^{-1}) being -4% to $+2\%$. The range of change is about 6% which is the same as that shown in Figure 5 of paper I.

In Figure 2, we have increased the optical depth to $T = 5$ with the same aspect ratio of 2 as in Figure 1. Here the changes are shown to be enormous, and if we compare these results with those presented in Figure 7 of paper I, the changes introduced by sphericity are spectacular. The $\bar{J}$ values at $V = 5000$ km s^{-1} approach as much as 50%, while in the plane-parallel case only 12%. Figure 3 gives the $\bar{J}$ for $T = 10$ with $B/A = 2$. The changes in $\bar{J}$ are still larger than those given in Figure 2. However, a further increase in optical depth to 30 and 50 brings down the changes considerably, (see Fig. 4 and 5, respectively) although $\bar{J}$ changes between positive and negative values. We have increased the aspect ratio to $B/A = 5$ and plotted $\bar{J}$ in Figure 6 for $T = 1$. Here $\bar{J}$ corresponding to $V = 5000$ km s^{-1} has the maximum value of 2.8% at $\tau = 0.85$ and lowest value at $\tau = .1$. However for $T = 5$, $\bar{J}$ changed considerably (Fig. 7). Figures 8, 9, and 10 show $\bar{J}$'s for $T = 10$, 30, and 50.

In Figures 11 and 12 we plotted the amplification factor $\bar{J}_{max}/100\beta$. These factors again show maxima depending upon the aspect ratio B/A. For $B/A = 2$, we have the amplification factor at about 270 (Fig. 11) while for $B/A = 5$ this is at about 80 (Fig. 12) for $V = 5000$ km s^{-1}.

In a few cases the changes in $\bar{J}$ for the spherical case appear to be rather lower than those in the plane-parallel case. The changes are not due only to B/A but also due to other factors such as the optical depth in each shell. We have to consider the changes due to the optical depth per unit geometrical extension, that is we have to see how $\bar{J}$ at the emergent side of the shell varies with respect to the quantity $T/(B/A)$. We presented these results in Figures 13, 14, and 15 for $T = 1$, 5, and 30, respectively, to show only the character of variation. For $T = 1$, we see that between $B/A = 1$ and $B/A = 2$ larger velocities will have larger $\bar{J}$'s while at $B/A \approx 2$

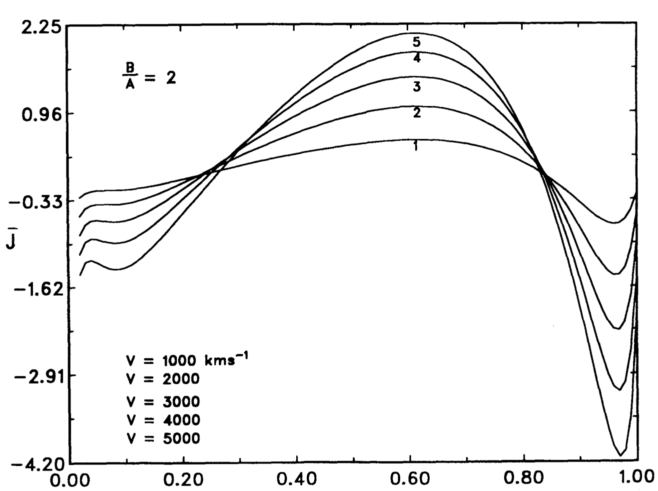

FIG. 1.—$\bar{J}$ for $T = 1$, $B/A = 2$

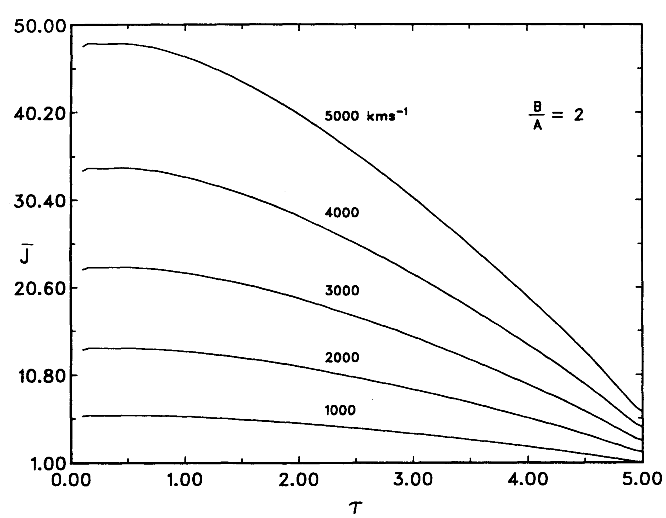

FIG. 2.—$\bar{J}$ for $T = 5$, $B/A = 2$

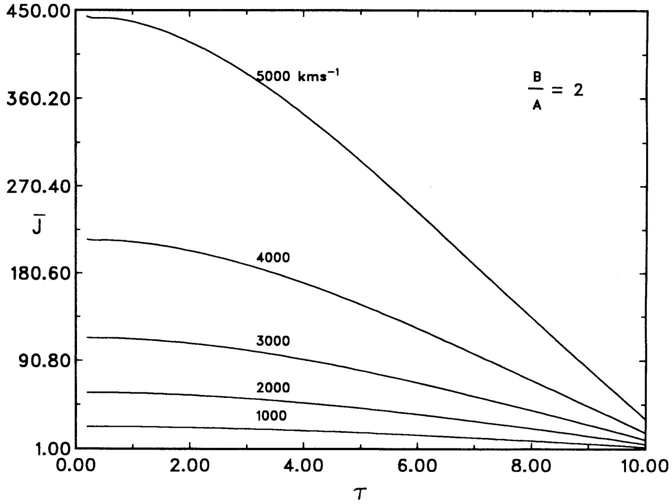

FIG. 3.—$\bar{J}$ for $T = 10$, $B/A = 2$

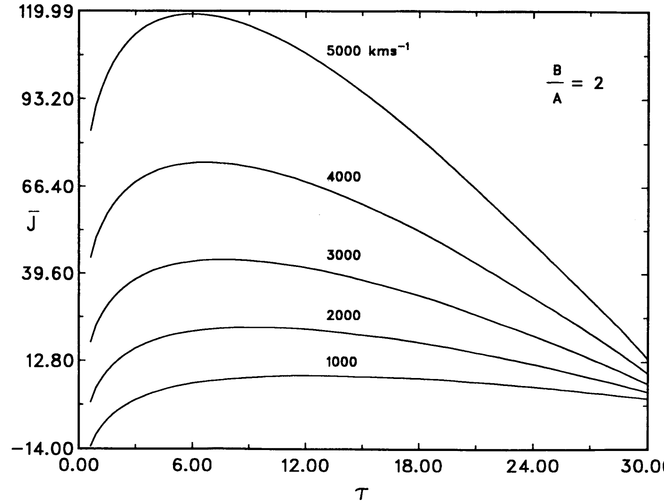

FIG. 4.—$\bar{J}$ for $T = 30$, $B/A = 2$

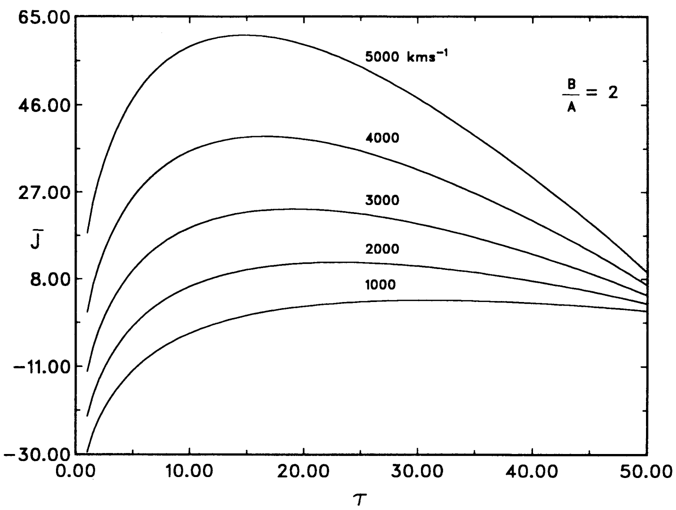

FIG. 5.—$\bar{J}$ for $T = 50$, $B/A = 2$

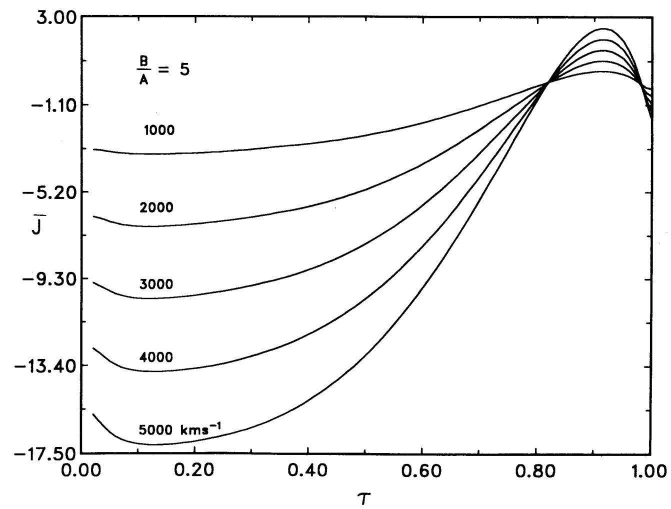

FIG. 6.—$\bar{J}$ for $T = 1$, $B/A = 5$

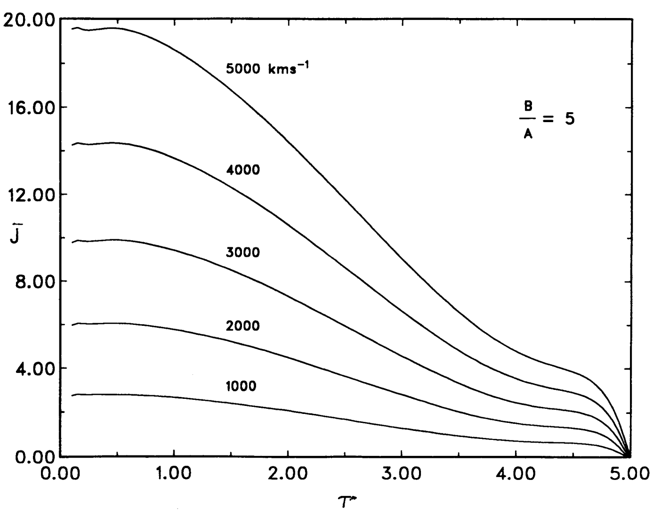

FIG. 7.—$\bar{J}$ for $T = 5$, $B/A = 5$

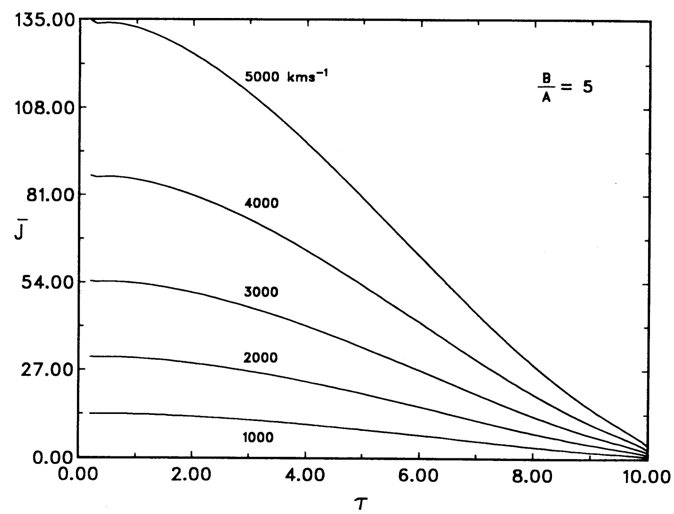

FIG. 8.—$\bar{J}$ for $T = 10$, $B/A = 5$

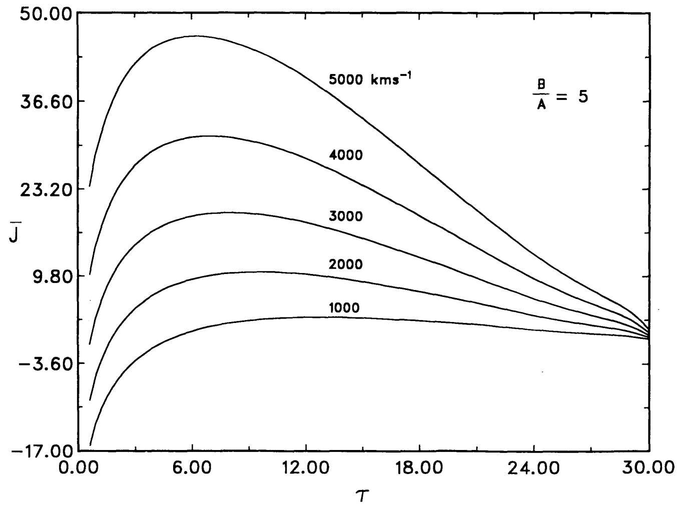

FIG. 9.—$\bar{J}$ for $T = 30$, $B/A = 5$

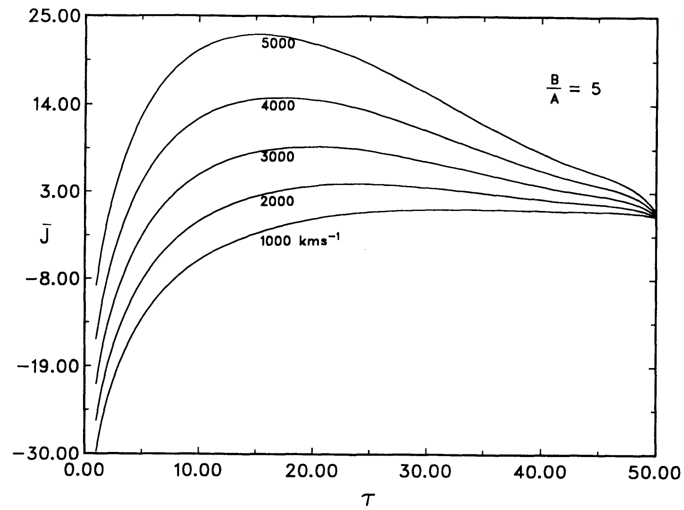

FIG. 10.—$\bar{J}$ for $T = 50$, $B/A = 5$

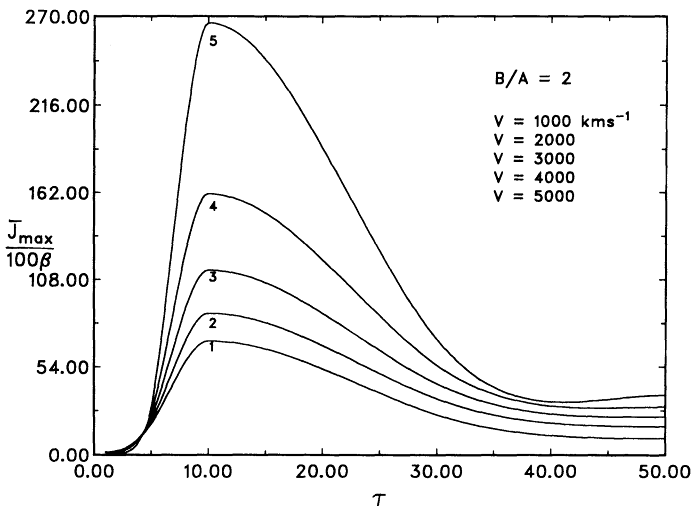

FIG. 11.—Amplification factor $\bar{J}_{max}/100\beta$ for $B/A = 2$

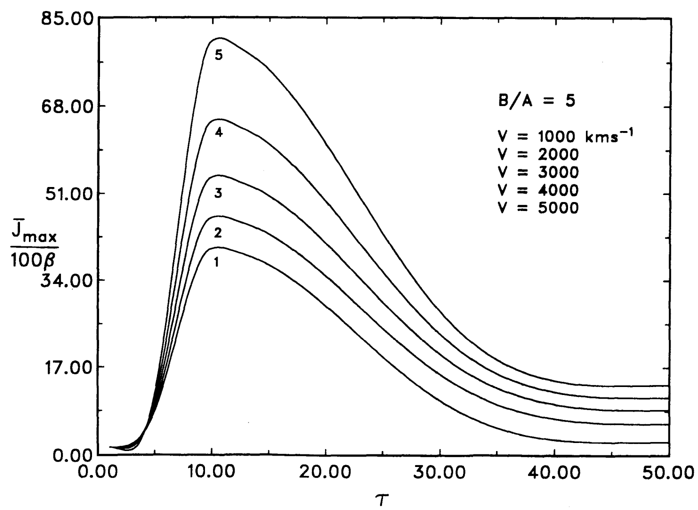

FIG. 12.—Amplification factor $\bar{J}_{max}/100\beta$ for $B/A = 5$

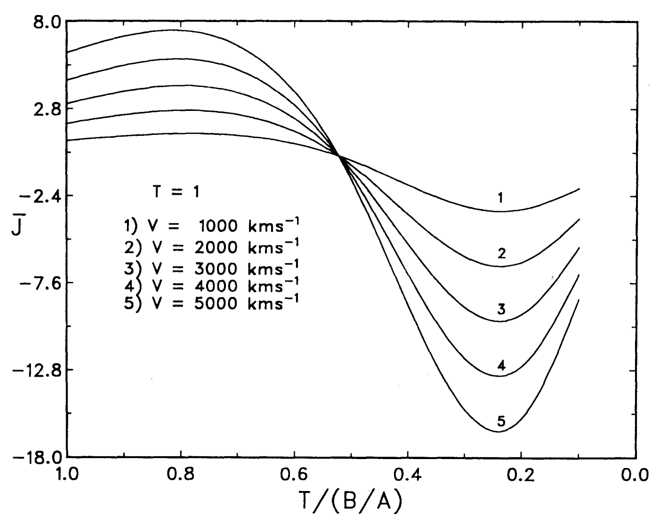

FIG. 13.—Emergent $\bar{J}$'s ($\tau = 0$) vs. $T/(B/A)$ for the total optical depth shown

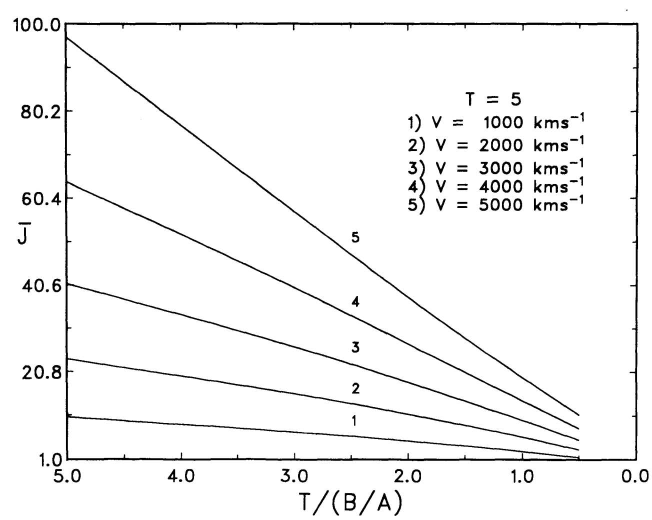

FIG. 14.—Same as those given in Fig. 13 for $T = 5$

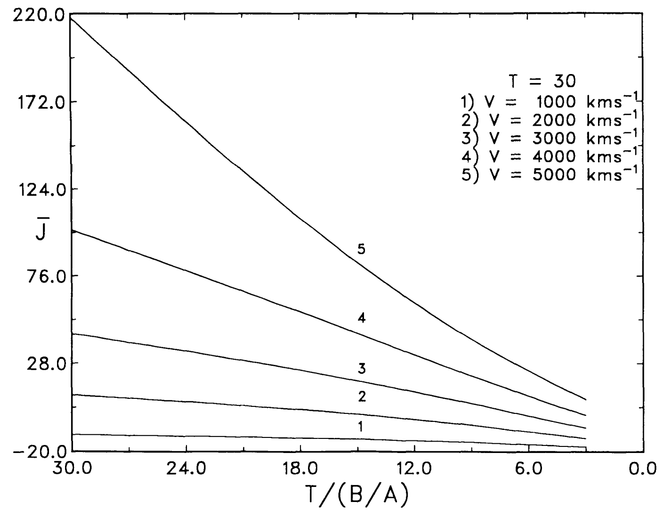

FIG. 15.—Same as those given in Fig. 13 for $T = 30$

the changes in $\bar{J}$ corresponding to all velocities pass through a point, the $\bar{J}$'s corresponding to larger velocities having negative and steeper slopes. The changes continue with the same slope until $B/A \approx 5$, at which the slope becomes positive. From this point the changes in $\bar{J}$ continue to increase with positive slopes which converge as B/A increases.

When T is increased to 5 (see Fig. 14), we see that the values of $T/(B/A)$ increase, which means that the optical depth per unit geometrical extension also increases. At $B/A = 1$ we see large changes in $\bar{J}$ and gradually reduce as B/A increases. These $\bar{J}$'s converge as B/A increases. For large B/A's these become negative. We plotted $\bar{J}$ versus $T/(B/A)$ in Figure 15 for $T = 30$. These results show similar characteristics as those given in Figure 14.

4. CONCLUSIONS

We presented a solution of the radiative transfer equation in spherical symmetry including the aberration and advection terms. The changes in mean intensities are dependent on the geometrical and optical thickness and in particular depend on the ratio $T/(B/A)$.

I thank the referee for useful suggestions and criticisms. I also thank B. A. Varghese in helping me in calculation and drawings of the figures.

REFERENCES

Castor, J. H. 1972, ApJ, 178, 779
Munier, A., & Weaver, R. 1986, preprint
Mihalas, D. 1978, Stellar Atmospheres (San Francisco: Freeman)
Mihalas, D., Kunasz, P. B., & Hummer, D. G. 1976, ApJ, 206, 515
Peraiah, A. 1987, ApJ, 317, 271 (paper I)
———. 1987, Bull. Astr. Soc. India, 15, 1
Peraiah, A., & Grant, I. P. 1973, J. Inst. Math. Appl., 12, 75
Peraiah, A., & Varghese, B. A. 1985, ApJ, 290, 411