SCALE TRANSFORMATIONS AND EVOLUTION OF THE EARLY UNIVERSE

(Letter to the Editor)

C. SIVARAM
Indian Institute of Astrophysics, Bangalore, India

(Received 27 July, 1993)

Abstract. The notion of discrete scale transformations is invoked to suggest strong links between fundamental interactions and cosmology giving rise to a hierarchy of cosmic scales.

In recent papers (Sivaram, 1982a, b, 1986a, b) it was pointed out that the cosmological parameters characterising the universe can be arrived at from microphysical considerations involving the fundamental interactions of elementary particle physics. For instance (Sivaram, 1982b, 1986a), the total baryon number (N_b) of the universe was related to the dimensionless coupling constants, α_g and α respectively as:

$$N_b \approx \alpha_g^{-2}\alpha^{-1} \tag{1}$$

where $\alpha_g^{-1} \approx (\hbar c/Gm_p^2) = 1.694 \times 10^{38}$ and

$$\alpha^{-1} = \frac{\hbar c}{e^2} = 137.036 \tag{2}$$

$\hbar$, c, m_p, G and e are respectively the Plank's constant, the velocity of light, the proton mass, the Newtonian gravitational constant and the electron charge. As noted in earlier works $\alpha_s \approx \frac{1}{2}\ln\alpha^{-1}/\pi \approx 14$ (the pion-nucleon dimensionless constant) and $\ln\alpha_g^{-1} = 2/3\alpha^{-1}$. Again α and α_s are both logarithmically varying functions of the energy E, which would be of significance in the early universe. For instance α^{-1} at the Z^0 scale (≈ 100 GeV) is $\alpha_{Z^0}^{-1} \approx 128$ described by an empirical Gamow–Teller type of relation as:

$$\frac{1}{\varepsilon}(\alpha^{-1} - \alpha_{Z^0}^{-1}) \approx \left(\frac{2\alpha^{-1}}{3} - \ln\alpha_g^{-1}\right)$$

where ε is the Naperian logarithmic base.

Combined with the early universe nucleosynthesis the above results were extrapolated to give the present temperature of the microwave background as: (Sivaram, 1992, 1993)

Astrophysics and Space Science **207**: 317–324, 1993.

$$T_0 = \left(\frac{m_p c^2}{K_B}\right)\left(\frac{8}{3}\frac{Gm_p^2}{\hbar c}\right)^{1/3} = 2.73\ \text{K}. \quad (3)$$

K_B is the Boltzmann constant.

Also in another recent paper (Sivaram, 1990, 1992) the role of Weyl gravity which is scale invariant was explored for sub-Planckian as well as macroscopic domains and the existence of a hierarchy of scales was deduced. A fixed discrete scale factor was used to connect the sub-Planckian and cosmological domains giving a series of self-similar structures.

We will now elaborate a little on this approach giving some specific examples. For continuous scale transformations we have:

$$\chi_\mu \rightarrow x'\hat{\mu} \equiv \varepsilon x_\mu \quad (4)$$

where ε is continuous. Considering discrete scale transformations we have generally.

$$x_\mu \rightarrow x'_\mu = \beta^n x_\mu \quad (n = 3, 2, 1, 0, -1, -2, -3, \ldots) \quad (5)$$

where β can take some fixed value, say $\alpha^{-1} = 137$, or $m_p/m_e = 1838$, etc. For invariance under continuous scale transformations, given by Eq. (4), the wave function, ψ, in the Dirac equation given by the Lagrangian density:

$$\mathcal{L} = -\overline{\psi}\gamma_\mu \partial_\mu \psi - m\overline{\psi}\psi$$

transforms as:

$$\psi(x) \rightarrow \psi'(x') = \varepsilon^{-3/2}\psi(\varepsilon x) \quad (6)$$

(since $\langle\psi\psi\rangle$ is a density of dimensions l^{-3}, ψ has dimensions $l^{-3/2}$). under the discrete scale transformations (Eq. (5)), $\mathcal{L}$ transforms as:

$$\mathcal{L} \rightarrow \mathcal{L}' = \beta^{-4n}[\overline{\psi}'\gamma_\mu \partial'_\mu \psi' - (m\beta^n)\overline{\psi'} \cdot \psi] \quad (7)$$

i.e. has the same form but with m replaced by $m\beta^n$, i.e. a Dirac field with quanta of mass m also implies existence of quanta with a mass spectrum.

$$m_n = m_{n0}\beta^n\ . \quad (8)$$

As an example we have a particle mass formula of the type:

$$M = nk\left[\frac{1}{2}\frac{\sqrt{\hbar c}}{e}\right]^n m_e \quad (9)$$

(m_e is the electron mass, n, k are integers and here $\beta = \sqrt{\hbar c}/e = \alpha^{-1/2}$, for correspondence with Eq. (8)). Now in Sivaram (1982a) and earlier references therein, it was shown that Eq. (9) does give the masses of a large number of known elementary particles, for e.g. $n = 2$, $k = 3$, gives the muon mass, $n = 2$, $k = 4$ gives the pion mass, etc. The transformations given by Eqs. (6) and (7) also apply to a self interacting spinor field coupling to gravity for which the appropriate equation is (Sivaram, 1979):

$$\beta^{-4n}\left[\gamma^{\mu}\psi_{;\mu} + (L_0^2\beta^{2n})\gamma^{\mu}(\overline{\psi}\gamma_{\mu}\gamma_5)\gamma^5\psi\right] = 0. \tag{10}$$

Here L_0 is a basis length scale arising from the correlation of the spinor field in a background gravitational field, so that here we have a discrete scale invariant length spectrum given by:

$$L_n = L_{0n}\beta^n. \tag{11}$$

As an example of a possible significance of this for a hierarchy of cosmic length scales, consider a gravitational 'Bohr radius' given by (for the significance, see Sivaram (1990)):

$$R_{\beta\varepsilon} = \frac{\hbar^2}{Gm_p^3} = \frac{e^2\,\hbar}{\alpha G m_p^3 c} = 3\times 10^{24}\ \mathrm{cm} \approx 1\ \mathrm{Mpc} \tag{12}$$

we use this for L_{0n} in Eq. (11). As in the earlier example use $\beta = \alpha$ for the discrete scale parameter. ($\alpha = e^2/\hbar c$). For $n = 1$, we have

$$L_1 = \frac{\alpha\,\hbar^2}{Gm_p^3} = \frac{e^2\,\hbar}{Gm_p^3 c} \simeq 7.5\ \mathrm{kpc}. \tag{13}$$

For $n = 2$, we have:

$$L_2 = \frac{\alpha^2\,\hbar^2}{Gm_p^3} \simeq 50\ \mathrm{pc}. \tag{14}$$

For $n = 3$

$$L_3 = \frac{\alpha^3\,\hbar^2}{Gm_p^3} \simeq 10^{18}\ \mathrm{cm} \approx 1\ \mathrm{light-year}. \tag{15}$$

Since there is no restriction on n being positive (see Eq. (5)) for discrete scale invariance:

$$L_{-1} = \frac{\hbar^2}{\alpha G m_p^3} = \frac{\hbar^3\,c}{Ge^2 m_p^3} = 137\ \mathrm{Mpc}. \tag{16}$$

It is very interesting that Eqs. (12)–(16) do give the length scales of observed ordered large scale structures in the universe! For instance Eq. (12), i.e. $R = L_0$, gives the average distance between galaxies, i.e. between Andromeda and the

Milky Way or the size scale corresponding to a cluster of galaxies. Eq. (13), i.e. L_1 corresponds to the size scale of a galactic disk, i.e. for the Milky Way, it is 8 kpc; Eq. (14) i.e. L_2 corresponds to the size of a globular cluster or a typical star forming region. In Eq. (15), L_3 corresponds to a typical stellar separation distance while L_{-1} in Eq. (16) corresponds to the length scale of the largest structures so far observed (the Great wall of Galaxies). Thus while Eqs. (8) and (9) gave a hierarchy of particle masses, Eqs. (11)–(15) give a hierarchy of length scales in the cosmic structure. In both cases, invariance under discrete scale transformations being the basis and in both cases is the scale factor. Apart from α, Eqs. (12)–(15), also involve m_p, while Eq. (9) involves m_e, the electron mass. Eqs. (1)–(3) as well as earlier works suggest that m_p/m_e can be expressed in terms of α_g and α_e (or alternately in terms of the total baryon number N_b. In Sivaram (1982a), we had the relation:

$$\frac{m_p}{m_e} = \frac{\hbar / m_{\hat{\pi}} c}{(G_{\mathrm{F}} / \hbar c)^{1/2}} \tag{17}$$

where G_{F} is the universal Fermi weak interaction constant ($\simeq 1.5 \times 10^{-49}$ erg cm^3) and $m_{\hat{\pi}}$ is the pion mass. We also had the following interesting relation:

$$G_{\mathrm{F}} = \alpha G M_{\mathrm{pl}}^2 \left(\frac{g_s^2}{2 m_p c^2} \right)^2 \tag{18}$$

where M_{pl} is the planck mass $\sim (\hbar c/G)^{1/2}$ [Note that: $(\alpha_g^{-1} = (M_{\mathrm{pl}}/m_p)^2)$] and g_s is the strong interaction charge. Eq. (18) embodies an interrelationship between the coupling constants of all the four fundamental interactions since as remarked earlier $\frac{1}{2} \ln \alpha_g^{-1}/\pi \approx \alpha_s = 14$ gives the strong pion–nucleon constant and since $m_p/m_e \approx \alpha_s/\alpha$, (Sivaram, 1982a), we would obtain an expression for m_p/m_e as:

$$\alpha^2 \left(\frac{m_p}{m_e} \right)^2 = 2 \ln \left[\frac{\alpha_g^{-1} \alpha^{-1}}{24} \right] \tag{19}$$

(since $N_b = \alpha_g^{-2} \alpha^{-1}$, we have a direct connection between m_p/m_e and cosmological parameters explored in Sivaram (1986a). Eq. (19) gives for m_p/m_e the numerical value of 1836.1527014, to be compared with the experimental value 1836.152701 (Cohen and Taylor, 1990) thus giving remarkable agreement to ten significant figures! Also we have the remarkable relation:

$$\frac{1}{2} \ln \alpha^{-1}/\pi = \exp(2\alpha^{-1}/3)(2\alpha_g)^{-1} = 14 = \alpha_s \tag{20}$$

(the pion–nucleon dimensionless constant). Also from lepton-baryon unification, we had the remarkable identity:

$$m_e + m_\mu + m_\tau = m_p(2 + 2\alpha + \alpha^2) \tag{21}$$

giving for the τ-lepton the mass $m_\tau = 1.784119$ GeV, and $m_p/m_\tau = 0.526$ and $m_p/m_\mu = 8.88$. Again, one must consider the variations with the energy scale, i.e. the relation between α_{Z^0} and α, ($\alpha_{Z^0} = 127.98$). Thus we can write: (as G_F has dimension (mass)$^{-2}$):

$$G_F m_p^2 = \frac{m_e/(m_p \alpha_{Z^0}) M_Z}{[M_W (\ln \alpha_g^{-1})^2]} \tag{22}$$

giving another remarkable relation between electro weak parameters and the gravitational and cosmological parameter. Here M_Z/M_W is the ratio of the W and Z-boson masses related to the Weinberg angle by $\sin^2\theta_W = (1 - M_W^2/M_Z^2)$. $M_Z/M_W = 1.1415$. Eq. (22) gives $G_F = 1.166371 \times 10^{-5}\ \mathrm{GeV}^{-2}/(\hbar c)^3$ agreeing with the value obtained from the muon life time to 6 significant figures (Cohen and Taylor, 1990).

Let us now return to the cosmic length scales given by Eqs. (11)–(15). It may be argued that in these expressions instead of the quantum unit of angular momentum $\hbar$, the angular momenta of the large scale structures (e.g. galaxies) must occur. Thus in the expression for the gravitational Bohr radius, $\hbar$ must be replaced by J_G (a typical galactic angular momentum) and m_p by a typical galactic mass, M_G. As suggested in Sivaram (1990), $J_G \approx 10^{100}\,\hbar \approx 10^{73}$ erg s. (Compare with the so called galactic Planck's constant of (Cocke and Tifft, 1989) to explain several velocity and red shift features!). Then for $M_G \sim 10^{67} m_p$, we find R_{GB} in Eqs. (10) and (11) to be unchanged, i.e. the typical intergalactic distance remains ~ 1 Mpc. Since this fundamental length scale, i.e. L_0 is unchanged, the other scales L_1, L_2, etc. obtained from discrete scale invariance, remain the same. In an earlier paper Sivaram (1990), an energy dependent string tension was invoked to connect the fundamental interaction strengths, the string tension scaling as $T \sim M^2 \sim G^{-1} \sim 1/R^2$. Thus at the Planck scale, $T = T_{\rm pl} = c^2/G$; at other mass scales, M, $T = T_{\rm pl}(M/M_{\rm pl})^2$ or $T = c^2/G_{\rm eff}$, where $G_{\rm eff} = G(M_{\rm pl}/M)^2$, G being the Newtonian gravitational constant, corres- ponding to $M = M_{\rm pl}$. At $M = M_W$, the intermediate boson mass the effective G is the Fermi constant $G_F = G(M_{\rm pl}/M_W)\,\hbar^2/c^2$, at hadron mass scale, ~ 1 GeV $\sim m_h$, $G = G(M_{\rm pl}/M_h)^2 \approx G_f \approx 10^{38}$ G, the strong gravity constant and so on, the interactions becoming effectively stronger as the mass scales decrease. The idea that for masses below the Planck mass ($= \hbar c/G)^{1/2} \approx 2 \times 10^{-5}$ g), which acts as a borderline between macro and micro worlds, the effective universal gravitational constant increases (going as $\sim M^2$, i.e. for $M = M_W$, it is as strong as β-decay and at $M \approx M_p$ it is as strong as strong interactions, i.e. $G = G_f$, etc.) is testable in future space micro-gravity experiments. This change in G is only exclusively for the gravitational interaction between these two masses only. i.e. when analysing the attractive force between two such particles (say in the form of metal or dust grain), we expect the mutual falling time (proportional to $d^{3/2}/\sqrt{G\mu}$, where d is the separation and μ the reduced mass) to decrease respective to the Newtonian prediction, in the case when both these particles have masses smaller than the

Planck mass. For two particles of radii r and equal masses m, the Newtonian falling time is

$$t_N = \frac{1}{3}(d^3/Gm)^{1/2} - \frac{2}{3}(2r^3/Gm)^{1/2},$$

Compressed metallic grains of planck mass have a radius of 0.07 mm. If separated by 0.5 mm their Newtonian falling time is $\sim$ 30 min in a microgravity environment, i.e. exclusively due to their mutual gravitational attraction. For grains with a hundred times lower mass, the effective G should increase to 10^4 the Newtonian value (from the scaling relations used above). So for the same separation, their mutual falling time is now $\sim$ 3 min, ten times smaller. Such an experiment in a microgravity space environment, well shielded and controlled with a high precision from electromagnetic perturbations or earth gravity gradients may be within present or near future feasibility limits. If the mutual gravitational interaction of such sub-planck masses is found to be very different in such experiments from the conventional Newtonian behaviour, there would be a need to have drastic revision of our ideas of uniting gravity with other interactions and of the nature of gravity itself.

In Sivaram (1990), the scaling law $G \sim M^{-2} \sim R^2$, for all energy scales, $M < M_{\text{Planck}}$ was understood in terms of an energy dependent string tension. The constancy of $\hbar$ for such scales (in $MCR \sim \hbar$) would imply that as the string is stretched (R is increased) we have a $M \sim 1/R$ relation, and since this M is now distributed over a larger R, the tension T scales as M^2 (or equivalently as $T \to c^2/G_{\text{eff}}$, where $G_{\text{eff}} \sim 1/M^2$). At the Fermi β-decay scale, the effective strong gravitational constant is $(G_{\text{eff}})_{\text{weak}} \to G_{F'}c^2/\hbar^2$ giving an electro-weak string tension of $T_W = c^2/(G_{\text{eff}})_{\text{weak}} = \hbar^2/G_{\text{F}}$ or in energy units, i.e. erg cm^{-1} of

$$T_W = \frac{\hbar^2 c^2}{G_{\text{F}}} \tag{23}$$

(G_{F} has dimensions of erg cm^3). At the Planck scale, $G_{\text{eff}} = G$ the usual Newtonian constant, the corresponding 'gravitational' Fermi constant being $(G_{\text{F}})_g = G\hbar^2/C^2 = 6 \times 10^{-83}$ erg cm^3, giving the relative strengths of gravitational and weak interactions as

$$\sim (G_{\text{F}})_g/G_{\text{F}} \sim 4 \times 10^{-34}\,. \tag{24}$$

The corresponding string tension is

$$T_{\text{pl}} = \frac{c^2}{G} = \frac{\hbar^2}{(G_{\text{F}})_g}. \tag{25}$$

The above approach suggests a novel way of looking at the these results. The different strengths of the various interactions as measured by the 'equivalent G'

arises from the distribution of the same universal quantum coupling constant $\hbar c$ distributed over regions of space time of different surface areas.

Thus the product ($\hbar c\times$ area) gives the effective 'Fermi constant' $G_{\mathrm{F(eff)}}$. Thus if the area is the square of the beta decay length, $(G_{\mathrm{F}}/hc)^{1/2}$, then $\hbar c\times$ area $= G_{\mathrm{F}}$, the Fermi beta decay constant. If the area is the square of the Planck length, then $\hbar c\times$ area $= (G_{\mathrm{F}})_g$, giving the Newtonian constant as expressed in Eq. (24). so the interaction gets effectively stronger as $(G_{\mathrm{F}})_{\mathrm{eff}}$, increases as the area over which the universal charge $\hbar c$ is distributed gets larger. Thus for strong interactions, the area is the (proton Compton length)^2 giving $(G_{\mathrm{F}})_{\mathrm{strong}} \approx 10^5 G_{\mathrm{F}}$, giving the typical strong interaction $g^2/\hbar c \sim 1$. (10^{-5} for weak, 10^{-38} for gravitation and so on (Eq. (25)).

Thus in general

$$\hbar c \times \text{ area} = (G_{\mathrm{F}})_{\mathrm{eff}} \tag{26}$$

Now using Eq. (23) for the string tension in Eq. (26) we have

$$\frac{\hbar c}{\text{area}} = T,$$

or since curvature $\sim 1/$area (of a surface) we have

$$\hbar c \times \text{ curvature} = T \tag{27}$$

or

$$\frac{T}{\text{curvature}} = \text{constant } \hbar c.$$

Thus, the highest curvature ($c^3/\hbar G = 10^{66}$ cm^{-2}, Sivaram, 1986b) is associated with the maximum string tension of $T_{\mathrm{pl}} = c^2/G$ from Eq. (28). Thus the universality of $\hbar c$ combined with just the geometry of space time through the curvature, gives rises to different string tensions and consequently to interactions of different strength! (Sivaram, 1990). We also note that in Klein–Kaluza theories, a magnetic moment due to the extra dimension is introduced as:

$$\mu = \frac{\hbar}{c}\sqrt{G_{\mathrm{eff}}} = \sqrt{(G_{\mathrm{F}})_{\mathrm{eff}}} \tag{28}$$

Thus $(G_{\mathrm{F}})_{\mathrm{eff}}^{1/2}$ has the dimensions of magnetic moment! Thus for the distribution of the universal $\hbar c$ over a proton Compton wavelength, the induced magnetic moment is

$$\mu_s \approx (hc \times \text{ area})^{1/2} \doteq \frac{\hbar}{c}\sqrt{G_f} = (G_{\mathrm{F}})^{1/2}{}_{\mathrm{strong}} =$$
$$= 9 \times 10^{-23} \text{ esu} = \text{ proton magnetic moment!} \tag{29}$$

The smallest magnetic moment in this picture would be when $hc\times$ area is the smallest, i.e. when the area is smallest say of (Planck length)^2 Then the magnetic moment is $\mu_g = \hbar/c\sqrt{G} = 8 \times 10^{-42} \simeq 10^{-22}\mu_B$ (μ_B is the Bohr magneton). Eqs. (27), (28) and (29) now enable us to understand what happens to the relation $\hbar c \times$ area $= (G_F)_{eff}$ for macroscopic objects $\gg M_{Planck}$. Here we would replace ($\hbar c$) in Eqs. (29) and (30) by Jv, where J is the angular momentum and $v < c$, a typical velocity (e.g. intrinsic rotational speed) of the object. This then would give the object of area $\sim (4\pi R^2)$, an effective magnetic moment as given by Eq. (30) of:

$$\mu = (Jv\, 4\pi R^2)^{1/2} \tag{30}$$

and since $\mu = BR^3$ where B is the magnetic field we have a typical magnetic field associated with the body as:

$$B = (4\mu JV)^{1/2}/R^2. \tag{31}$$

For the a typical galaxy where as seen above $J \sim 10^{100}\,\hbar$, $v \sim 10^{-3}c$, $R \approx 8$ kpc, we have $B \approx 10^{-6}$ Gauss and so on for other celestial objects. Eq. (31) embodies the old Blackett type relation. Thus the progressive increase of G_{eff} or $(G_F)_{eff}$ scaling as R^2 or as $\sim 1/M^2$ for all $R > L_{pl}$ or $M > M_{pl}$ is now interpreted as resulting in a magnetic moment for macroscopic objects, scaling as $J^{1/2}R$. For sub-Planckian scales as noted in Sivaram (1990), G_{eff} is constant, so the magnetic moment remains at its minimal value $\sim \hbar/c\sqrt{G_{eff}} \sim 6 \times 10^{-42}$. We have thus a unified frame work for describing scales from sub-Planckian to super galactic clusters.

References

Cocke, W.J. and Tifft, W.: 1989, *Astrophys. J.* **346**, 613.
Sivaram, C.: 1979, *Physics Reports* **51C**, 111.
Sivaram, C.: 1982, *Astrophys. Space Sci.* **88**, 507.
Sivaram, C.: 1982, *Amer. J. Phys.* **50**, 279; **51**, 277.
Sivaram, C.: 1986a, *Astrophys. Space Sci.* **124**, 195.
Sivaram, C.: 1986b, *Astrophys. Space Sci.* **125**, 189.
Sivaram, C.: 1987b, *Nature* **327**, 108.
Sivaram, C.: 1987a, *Int. J. Theor. Phys.* **26**, 1127.
Sivaram, C.: 1992, *Astrophys. Space Sci.* (in press).
Sivaram, C.: 1990, *Astrophys. Space Sci.* **167**, 335.