Astron. Astrophys. 298, 438–444 (1995)

ASTRONOMY AND ASTROPHYSICS

Distribution of stars perpendicular to the plane of the Galaxy. II

S. Chatterjee

Indian Institute of Astrophysics, Bangalore 560 034, India

Received 8 February 1994 / Accepted 4 October 1994

Abstract. We present here rigourous analytical solutions of the Boltzmann–Poisson equations concerning the distribution of stars perpendicular to the galactic plane. The number density of stars at the galactic disk is assumed to follow $n(m,0) \sim H(m-m_0)m^{-x}$, where m is the mass of the star and x is an arbitrary exponent greater than 2, while $H(m-m_0)$ is the unit Heaviside step function. The velocity dispersion of the stars is assumed to arise from the stellar motion in a random force field – leading to $\langle v^2(m)\rangle \sim$ constant for $m \le m_*$, $\langle v^2(m)\rangle \sim m^{-\theta}$, for $m > m_*$, where m_* is the stellar mass for which the stellar life-time equals the galactic age. It is seen that the height distribution of stars is very sensitive to the values of x and θ. Finally we have derived an expression connecting the surface density volume density and velocity dispersion of stars, and show that this relation is a sensitive function of θ and x and use them to obtain certain plausible numbers for our Galaxy in the limits of the present day data.

Key words: stars: rotation – Galaxy: kinematics and dynamics – Galaxy: stellar content

1. Introduction

Distribution of stars, perpendicular to the galactic plane (xy plane) requires the solution of the Boltzmann equation for the distribution of masses in the phase space, self-consistent with the Poisson equation for the gravitational potential. The solution exhibits an interplay between the "randomizing effect" due to the velocity dispersion $\langle v^2(m)\rangle$ and the "ordering effect" of the gravitational potential $\phi(z)$, of which the latter depends upon the mass spectrum and the velocity dispersion via the Boltzmann–Poisson (BP) eqations. The self-consistency condition, stated above – serves as the motivation of the present calculations – concerning the effect of the mass spectrum and velocity dispersion on the vertical distribution of stars above the galactic plane.

The solutions given below are consistent under the assumption of Gaussian velocity dispersion, with the quantity $\langle v^2(m)\rangle$ assumed to be independent of z (Bahcall 1984a, 1984b). The important conditions in our theory are as follows

(1) The mass spectrum follows the law:

$$n(m,0) \sim H(m-m_0)(m/m_0)^{-x}$$

with $x > 2$, where $n(m,0)$ is the number of stars per unit volume per unit mass interval in a cylindrical volume of infinitesimal thickness around the galactic plane and $H(m-m_0)$ is the Heaviside unit step function.

(2) The velocity dispersion follows:

$$\begin{aligned}\langle v^2(m)\rangle &= \text{constant}\,, \quad \text{for} \quad m \le m_* \\ &\sim m^{-\theta}\,, \quad \text{for} \quad m \ge m_*\end{aligned}$$

where m_* is the mass for which the stellar lifetime is equal to the age of the galaxy. The above velocity dispersion is derived by considering random walk of stars in velocity space under rapidly fluctuating random force fields, $\langle v^2(m)\rangle$ being directly proportional to the time t that the star has lived in the random force field (Wielen 1977).

The solutions given below are exact, in the sense, that they are non perturbative. They are seen to have sensitive dependence on x and θ. In the limit $x \to \infty$ and (or) $\theta \to 0$, i.e. either for all masses equal and (or) for all the velocities equal, we recover the well known Spitzer formula (Spitzer 1942). The present work elegantly takes into account the variation of the masses and the dependence of the velocity dispersion on mass. The paper is divided into five sections. Section 2 gives the rigourous method of self consistent solution of the BP equations. In Sect. 3, we use the above solutions to obtain a relation between the mass, stellar velocity dispersion, scale height and the mass dispersion at the galactic disk. We demonstrate in Sect. 4 that this relation enables us to ascertain the relative merits of the different sets of data on the galactic parameters while Sect. 5 covers concluding remarks with comments on the need to reexamine the existing data on the basis of the present self consistent solutions.

2. Theory

2.1. The collisionless Boltzmann equation

The distribution function $f(z,v,t)$ for stars in the space perpendicular to the galactic plane (xy plane) is known to follow the collision-less Boltzmann equation in one dimension (here z-direction):

$$\left[\partial/\partial t + v\partial/\partial z + (F_{\rm s}(t) + F_{\rm f}(t))\,\partial/\partial v\right] f(z,v,t) = 0 \quad (1)$$

where $F_{\rm s}$ denotes the slowly varying and $F_{\rm f}$ denotes the fast varying acceleration on the stars.

We consider $F_{\rm f}(t)$ to be zero mean Gaussian random variable, with a correlation function:

$$\langle F_{\rm f}(t)F_{\rm f}(t')\rangle = |F_{\rm f}|^2 \exp(-|t-t'|/\tau_0)\,. \quad (2)$$

In what follows, we determine the distribution function $f(z,v,t)$ at times $t \gg \tau_0$. The method of solution of the stochastic differential equation (1) has been presented by the author (Chatterjee 1991), which may be referred to for mathematical details, the techniques being elaborated in the two classic review articles by Van Kampen on stochastic differential equations (Van Kampen 1973, 1985). The essential results are as follows.

At any $t \gg \tau$, stars possess a velocity dispersion

$$\left\langle v^2(t)\right\rangle = \sigma_0^2 + 2\left\langle(\delta v(t))^2\right\rangle \quad (3)$$

where σ_0^2 is the velocity dispersion of the stars at the time of their formation while

$$\left\langle(\delta v(t))^2\right\rangle = |F_{\rm f}|^2\tau_0 t = C_{\rm V} t \quad (4)$$

with

$$C_{\rm V} = |F_{\rm f}|^2\tau_0$$

gives the velocity dispersion due to the action of the fast varying random forces.

Also considering the initial velocity at the time of formation to be a zero mean Gaussian random variable, one obtains

$$f(z,v) \sim \exp(-\phi(z)/\left\langle v^2(t)\right\rangle) \quad (5)$$

where $\left\langle v^2(t)\right\rangle$ is given in (3) and (4). Equation (5) is also the starting point of Bahcall's calculation (Bahcall 1984a,1984b).

2.2. Mass dependence of the velocity dispersion

The velocity dispersion $\left\langle v^2(t)\right\rangle$ which appears in the distribution function is seen from (4) to be directly proportional to the time t, that the star has spent in the random force field.

Considering that the rate of star formation follows

$$n(t)\mathrm{d}t = n_0 \exp(-t/\tau)\mathrm{d}t \quad (6)$$

– being the number of stars created within the time t and $t+\mathrm{d}t$ – then after a time T, the average age of the stars is:

$$\begin{aligned}\langle t_{\rm age}\rangle &= \int_0^T (T-t)n(t)\mathrm{d}t \Big/ \int_0^T n(t)\mathrm{d}t \\ &= T + \tau[\{(T/\tau)+1\}\exp(-T/\tau) - 1] \\ &\quad /[1-\exp(-T/\tau)]\,. \end{aligned} \quad (7)$$

If the star generation rate be uniform with time, then we can put $(T/\tau) \to 0$, so that we have, from (7),

$$\langle t_{\rm age}\rangle \sim T/2 \quad (8)$$

For stars with life-time $\tau(m)$ much longer than the galactic age, ($T_{\rm gal}$) we must put $T = T_{\rm gal}$, and hence

$$\langle t_{\rm age}\rangle \sim T_{\rm gal}/2\,. \quad (9)$$

Similarly, for stars with life-time $\tau(m)$ less than $T_{\rm gal}/2$ we must have $T = \tau(m)$ and hence

$$\langle t_{\rm age}\rangle \sim \tau(m)/2\,. \quad (10)$$

Hence Wielen's assertion, that on an average the stars have lived half their main sequence life time, is equivalent to the case of constant rate of stellar birth.

To estimate the main-sequence life-time, we note that this is the time needed to burn 10% of the hydrogen. In this process since only 0.007 fraction of the hydrogen mass is converted into energy, at the end of the main-sequence life-time, the stellar mass is 0.9993 of its initial mass.

From the mass-burning formula,

$$c^2\mathrm{d}m/\mathrm{d}t = -\mathrm{L}_\odot\left(m/\mathrm{m}_\odot\right)^{(\theta+1)} \quad (11)$$

the main-sequence life-time can be calculated as,

$$\begin{aligned}\tau(m) &= \left(\mathrm{m}_\odot c^2/\mathrm{L}_\odot\theta\right)\left(m/\mathrm{m}_\odot\right)^{-\theta}\int_{0.9993}^{1} y^{-(\theta+1)}\mathrm{d}y \\ &= \alpha\left(m/\mathrm{m}_\odot\right)^{-\theta}\end{aligned} \quad (12)$$

where

$$\alpha = 0.001752\left(\mathrm{m}_\odot c^2/\theta\mathrm{L}_\odot\right) \quad (13)$$

Putting $\mathrm{m}_\odot = 1.9892\,10^{33}$ g, $\mathrm{L}_\odot = 3.8268$ erg s^{-1}, $c = 3\,10^{10}$ cm s^{-1}, $\theta = 2.5$ and $T_{\rm gal} = 10^{10}$ yr $\sim 3\,10^{17}$ s, one finds that for $m < m_* \sim 0.85\mathrm{m}_\odot$, $\tau(m) > T_{\rm gal}$.

Thus substituting T from Eqs. (9) and (10) the velocity dispersion follows,

$$\left\langle v^2(m)\right\rangle = \sigma_0^2 + v_0^2 \qquad \text{for} \quad m \le m_* \quad (14.a)$$

$$\left\langle v^2(m)\right\rangle = \sigma_0^2 + v_0^2\left(m/\mathrm{m}_\odot\right) \qquad \text{for} \quad m \ge m_* \quad (14.b)$$

where

$$\begin{aligned}\left\langle v_0^2\right\rangle &= \left(0.001752 C_{\rm V}\mathrm{m}_\odot c^2/\mathrm{L}_\odot\theta\right)\left(\mathrm{m}_\odot/m_*\right)^\theta \\ &= C_{\rm V}T_{\rm gal}/2 = 3000\,(\mathrm{km\,s^{-1}})^2\end{aligned} \quad (15)$$

– as we find from Wielen's estimate that $C_{\rm V}$ $6\,10^{-7}\,(\mathrm{km\,s^{-1}})^2\,\mathrm{yr}^{-1}$.

Once again using $\sigma_0^2 \sim 30\,(\mathrm{km\,s^{-1}})^2$ one finds that the σ_0^2 term in (14.a) and (14.b) can be neglected if $(m/m_*) < 7.0$. From the mass spectrum given below in (16) we find that for $m_0 < 0.1\mathrm{m}_\odot$ and $x = 2.3$, only a fraction $4.6\,10^{-3}$ of the stars in the galaxy have masses greater than $7\mathrm{m}_\odot$. The neglect of σ

thus gives a small error. Hence we shall consider the velocity dispersion to be as given in (17) below, i.e. the σ_0^2 term being neglected.

As is well known, the gravitational force on a body being propotional to its mass, any gravitational field imparts the same acceleration on all masses. Thus different masses under identical gravitational field would, after a time t, gain the same velocity $\Delta v(t)$ over the initial value. This would make $\langle v^2(m)\rangle$ the same for all masses had all the stars lived for equal lengths of time in the random force field. However, since different masses have different average life times, their average velocity dispersions also acquire a mass dependence at any given time. These manifestations have been derived in this section, to be used in the subsequent calculations.

2.3. Solution of the Poisson equation with mass spectrum and velocity spectrum

We use the above input to solve the Poisson equation. Here we introduce the mass spectrum of the stars to be

$$n_i(m,0) = H(m-m_i)n_i m_i^{x_i-1}(x_i-1)m^{-x_i} \tag{16}$$

with all $x_i > 2.0$, where the subscript i denotes the species under consideration (e.g. may mean the thin disk or the thick disk cases), according to Salpeter's estimate, $x = 2.35$, for the initial mass spectrum (Salpeter 1955).

From the foregoing discussions, we take the velocity dispersion to be,

$$\begin{aligned} \langle v^2(m)\rangle &= v_0^2 && \text{for} \quad m \le m_* \\ &= v_0^2(m_*/m)^\theta && \text{for} \quad m > m_* \end{aligned} \tag{17}$$

From (5) the Poisson equation is given by,

$$(\mathrm{d}^2/\mathrm{d}z^2)\phi(z) = 4\pi G\sum_i \int n_i(m) m \exp(-\phi(z)/\langle v_i^2(m)\rangle)\mathrm{d}m \tag{18}$$

Multiplying both sides of (18) by $2\mathrm{d}\phi/\mathrm{d}z$ and integrating, with the boundary condition: $\phi(0) = 0$, $\phi'(0) = 0$, we get

$$\begin{aligned} (\mathrm{d}\eta/\mathrm{d}\chi)^2 = \sum_i \Big\{ &\Big[1-(m_i/m_*)^{x_i-2}\,\theta_i/(x_i+\theta_i-2)\Big] \\ &\times \left[1-\mathrm{e}^{-\eta_i}\right] + \left[(x_i-2)/(x_i+\theta_i-2)\right] \\ &\times (m_i/m_*)^{x_i-2}\,\eta^{(x_i-\theta_i-2)/\theta_i}Q_i(\eta)\Big\}\,\varepsilon_i \end{aligned} \tag{19}$$

where we have introduced the dimensionless variables,

$$\eta = \phi/v_0^2 \tag{19.1}$$
$$\chi = z/z_0 \tag{19.2}$$
$$\varepsilon_i = \rho_i(0)/\rho(0) \tag{19.3}$$

with

$$z_i^2 = v_i^2/(8\pi G\rho(0)) \tag{19.4}$$
$$\rho(0) = \sum_i \rho_i(0) \tag{19.5}$$

where

$$\rho_i(0) = n_i m_i (x_i-1)/(x_i-2) \tag{19.6}$$

denotes the midplane mass density contributed by the i-th lot so that $\rho(0)$ gives the total midplane mass density, and $Q_1(\eta)$ is the integral defined as,

$$Q_i(\eta) = \int_\eta^\infty \mathrm{e}^{-\xi}\xi^{-(2-x_i-\theta_i)/\theta_i}\mathrm{d}\xi \tag{20}$$

We can also write the above result as,

$$\begin{aligned} (\mathrm{d}\eta/\mathrm{d}\chi)^2 = \sum_i &\left[1-(m_i/m_*)^{x_i-2}\right][1-\exp(-\eta)] \\ &+(m_i/m_*)^{x_i-2}[(x_i-2)/(x_i+\theta_i-2)] \\ &\times[1-((x_i+\theta_i-2)/\theta_i)\exp(-\eta)U(1,(2-x_i)/\theta_i,\eta)]\varepsilon_i \end{aligned} \tag{21}$$

$U(a,b,z)$ being a confluent hypergeometric series (Gradstey'n & Ryzhik 1980; Abramowicz & Stegun 1965).

Equations (19–21) give exact relationships for the gravitational field when x_i and θ_i are any set of arbitrary constants, with $x_i > 2$. The exponent θ_i given above is consldered to be a constant but in reality is itself m dependent, – though the dependence on m is not very rapid. However, since the integral given in (20) is a Laplace type integral, its magnitude is dominated by the values where $\exp(-\eta)$ is maximum (see, for example, Jeffereys 1962; Olver 1964). Hence, we shall take $\theta = 2.5$, which is the value around $m = m_* \sim \mathrm{m}_\odot$ (Lang 1984).

2.4. Special cases

For the moment we consider a single species case, the extension to a multiple species case is straightforward and a case with two species will be presented in the last part of the paper.

(1) Low χ limits

In the limit $\eta \to 0$ one can evaluate (20) as

$$\mathop{\mathrm{Lt}}_{\eta\to 0} Q(\eta) \sim [\theta/(x-2)]\eta^{(2-x)/\theta} \tag{22}$$

and hence (19) reduces to

$$(\mathrm{d}\eta/\mathrm{d}\chi)^2 = \eta \tag{23}$$

$$\text{i.e.} \quad \eta \sim \chi^2/4 \tag{24}$$

so that the gravitational potential $\phi(z)$ goes as,

$$\phi(z) \sim 2\pi G\rho(0)z^2 \tag{25}$$

i.e. has a quadratic dependence on the height, z and the fleld K_z goes linearly as,

$$K_Z \equiv \phi' \sim 4\pi G\rho(0)z\,. \tag{26}$$

(2) High χ limit

For $\eta \to \infty$, as given in (20) can be asymptotically expanded by Laplace's method (Jeffereys 1962; Olver 1964) as:

$$\mathop{\mathrm{Lt}}_{\eta\to 0} Q(\eta) \sim \mathrm{e}^{-\eta}/\eta^{(x+\theta-2)/\theta} \tag{27}$$

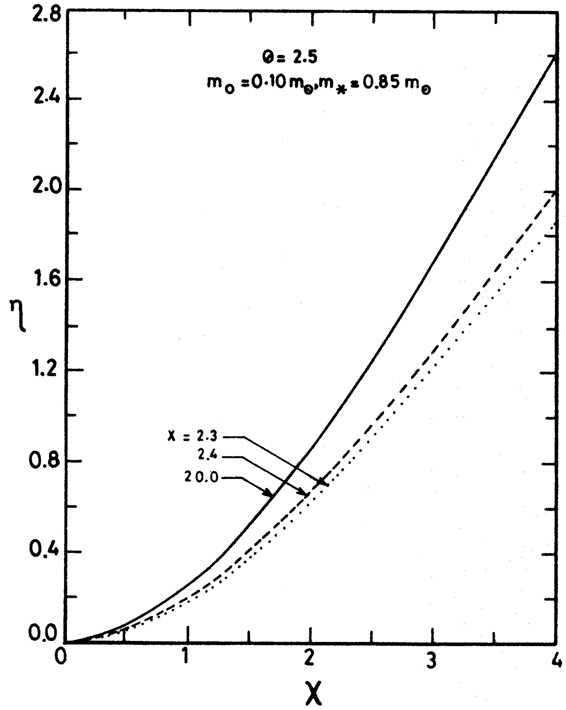

Fig. 1. η versus χ plot for $\theta = 2.5$ with $x = 2.3, 2.4, 20.0$

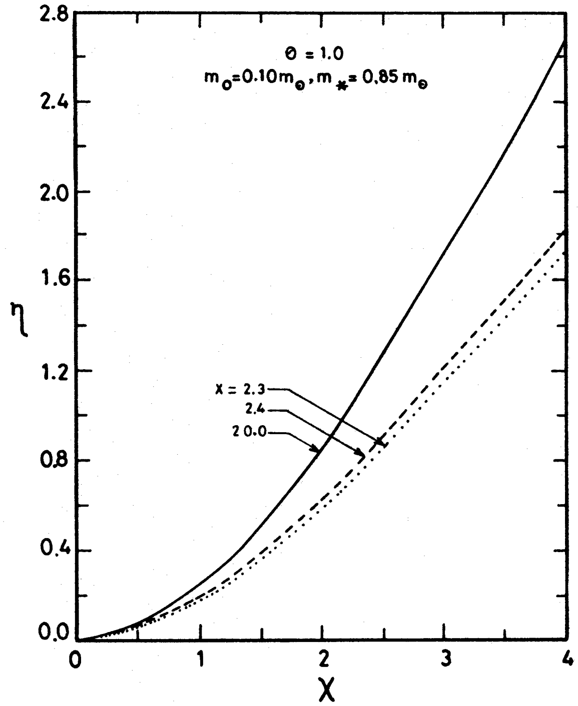

Fig. 2. η versus χ plot for $\theta = 1.0$ with $x = 2.3, 2.4, 20.0$

Equation (19) then reads,

$$(\mathrm{d}\eta/\mathrm{d}\chi)^2 = 1 - (m_0/m_*)^{x-2}[\theta/(\theta + x - 2)] \quad (28)$$

so that

$$\eta \sim \left[1 - (m_0/m_*)^{x-2}[\theta/(\theta + x - 2)]\right]^{1/2} \chi \quad (29)$$

i.e. there is a uniform gravitational field pointed towards the galactic mid-plane. As seen from (29), the value of the field depends crucially on the values of x and θ and hence on the mass spectrum and the velocity spectrum. These asymptotic power law trends are independent of the mass and velocity spectra but are characteristic of the fact that we are dealing here with a one dimensional problem.

(3) All masses are equal

This corresponds to the case $x \to \infty$. Since $(\mathrm{m}_\odot/m) < 1$, we have $(m_0/m)^{x-2} \to 0$. Equation (19) then yields,

$$(\mathrm{d}\eta/\mathrm{d}\chi)^2 = \left[1 - \mathrm{e}^{-\eta}\right] \quad (30)$$

Hence one finds the well-known Spitzer result (Spitzer 1942),

$$\eta = 2\ln[\mathrm{ch}(\chi/2)] \quad (31)$$

$$\begin{aligned}\rho(z) &= \rho(0)\mathrm{e}^{-\eta} \\ &= \rho(0)\mathrm{sech}^2(\chi/2)\,. \end{aligned} \quad (32)$$

Similar results (25–32) could be derived from equation (21) also, as was done by using the limiting expressloh for $U(a, b, z)$ in our earlier paper (Chatterjee 1991).

(4) Dispersion in the position of the stars

The quantity $\langle z^2(m)\rangle$ is calculated from:

$$\langle z^2(m)\rangle = I(2)/I(0) \quad (33)$$

where

$$I(\alpha) = \int_0^\infty z^\alpha \exp(-\phi(z)/\langle v^2(m)\rangle)\mathrm{d}z\,. \quad (34)$$

For $m \le m_*$, $\langle v^2(m)\rangle = v_0^2$ and hence the integrals in (34) are mass independent, so that using (33) one gets,

$$\begin{aligned}\langle z^2(m)\rangle &= \text{constant} \quad \text{for} \quad m \le m_* \quad (35)\\ &\sim (m_*/m)^\theta \quad \text{for} \quad m \gg m_* \quad (36)\end{aligned}$$

Thus measurements of $\langle z^2(m)\rangle$ enable us to find θ.

3. Mass $M(z)$ contained within $z = -z$ and $z = z$: relation with the velocity dispersion and the mass density in the galactic midplane

By definition,

$$\begin{aligned}M(z) &= 2\int_0^z \rho(z)\mathrm{d}z \\ &= 2(4\pi G)^{-1}K_z \end{aligned} \quad (37)$$

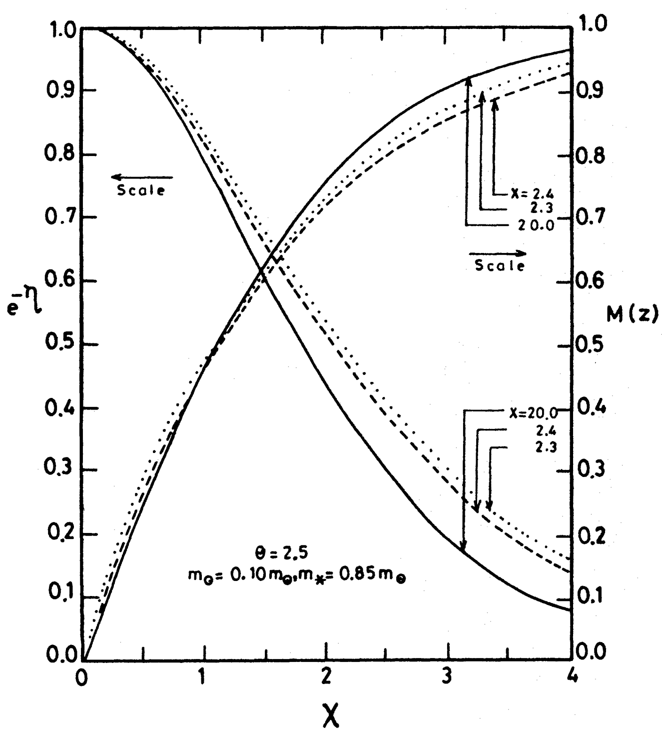

Fig. 3. Plot of $e^{-\eta}$ and $M(z)$ versus χ for θ = 2.5 and x = 2.3, 2.4, 20.0

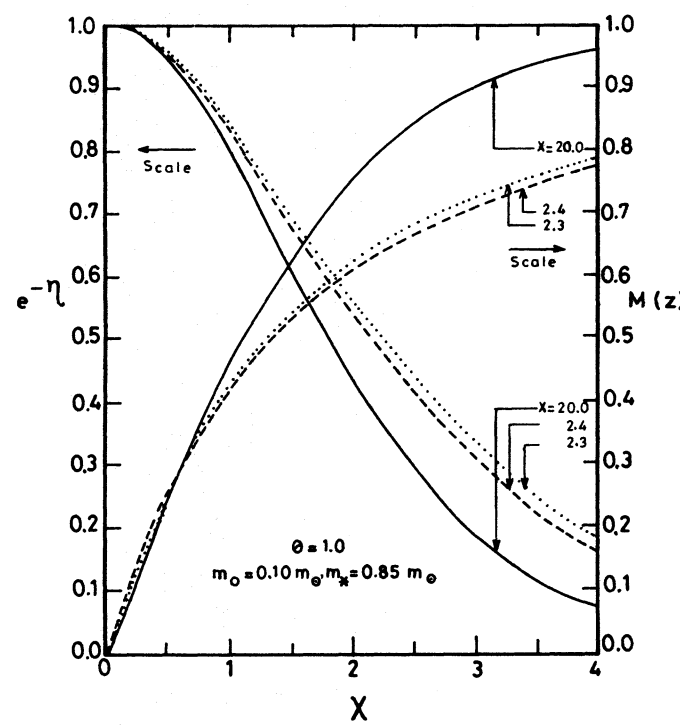

Fig. 4. Plot of $e^{-\eta}$ and $M(z)$ versus χ for θ = 1.0 and x = 2.3, 2.4, 20.0

where $K_Z = |\mathrm{d}\phi/\mathrm{d}z|$ is the magnitude of the gravitational field at $z = z$. Some manifestations of the above result are shown, for a single species case, in Figs. (1–4). Since $K_Z = (v_0^2/z_0)$. $(\mathrm{d}\eta/\mathrm{d}\chi)$, $M(z)$ can be easily calculated from (19) and (21). It is seen that $M(z) \sim z$ for $z \ll z_0$ while $M(z)$ tends to a constant for $z \gg z_0$.

The value of $M(z)$ in the limit $z/z_0 \to \infty$, gives us the surface density of matter (Σ) in the galaxy to be

$$\Sigma = 2(4\pi G)^{-1} K_z(\infty) \tag{38}$$

so that

$$\pi G \Sigma^2 = 2V^2 \rho(0) \tag{39}$$

where V^2 is defined as

$$V^2 = \sum_i \rho_i(0) v_i^2 f_i(\theta_i, x_i)/\rho(0) \tag{40}$$

with

$$f_i(\theta_i, x_i) = \left[1 - (m_i/m_*)^{x_i - 2}[\theta_i/(\theta_i + x_i - 2)]\right] \tag{41}$$

In Eq. (39) V^2 refers to an average velocity dispersion of the system, the factors $f_i(\theta_i, x_i)$ being the weightage factors for the respective lots of the masses given in the system.

Equation (39) is an interesting formula, which connects the surface density of mass with the velocity dispersion and the mid-plane mass density. The general form of $f_i(\theta_i, x_i)$, as given in (41) shows it to be a sensitive function of x_i. We have plotted the variation of $f_i(\theta_i, x_i)$ with x_i for a case $m_i = 0.1\mathrm{m}_\odot$ and $m_* = \mathrm{m}_\odot$ with θ_i being chosen as 2.5 which is a typical value. The graphical plot is shown in Fig. 5. It shows that for x_i greater than 2.5, $f_i(\theta_i, x_i)$ saturates very rapidly.

Equation (40) is in fact, a manifestation of the Virial theorem, connecting the kinetic energy of the system with the potential energy when the particles undergo random motion. It can be used as a rule of thumb to ascertain many of the galactic parameters. This is easily explained on examination of Eqs. (39–41). What needs to be explained is that, for any $x_i < 2.5$ any small error in the knowledge of the exponent x_i may lead to large discrepancies in the evaluation of the right hand side of Eq. (39). This point should be taken note of in future analyses of data regarding galactic mass density. In what follow, we use Eq. (39) to judge the relative merits of the different sets of data. While doing this we recognize that the solutions of the Boltzmann–Poisson equations introduce different scale lengths in the system, which we can use in determining the different galactic properties. The importance of these scale lengths we proceed to discuss next. Possibilities of direct measurements of some of these scale lengths lead us directly to ascertain some of the parameters introduced in this paper.

4. Different scale lengths in the system

These can be understood by investigating in gross terms the asymptotic dependence of $\phi(z)$ on z. Equations (26) and (28) show how the gravitational field has a linear dependence on z for low z and becomes a constant for higher z. The height z_c at

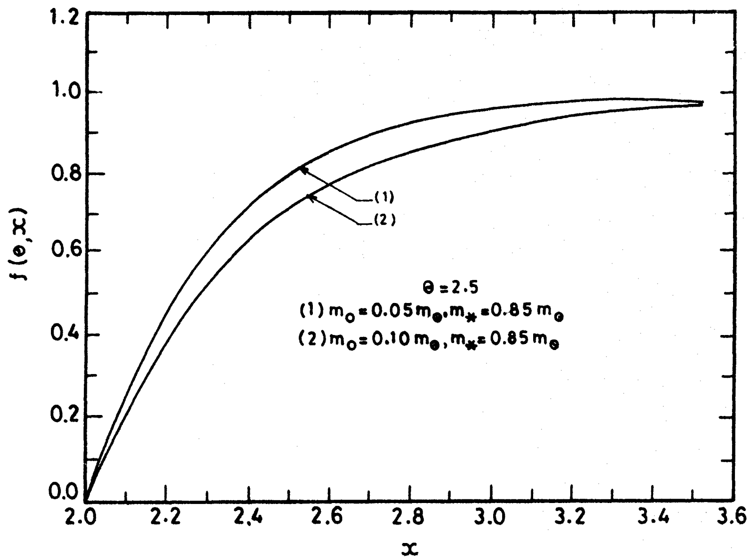

Fig. 5. Dependence of $f(\theta, x)$ on θ and x

which this transition takes place, can be estimated as,

$$z_c = \left[V^2/2\pi G\rho(0)\right]^{1/2} = \Sigma/2\rho(0) \quad (42)$$

In the region $z > z_c$ i.e. where the potential goes linearly with height, it is possible to define a scale length $z_{sc}(m)$, such that the number distribution of star with mass m follows a decrement,

$$\nu(m, z) \sim \exp(-z/z_{sc}(m)). \quad (43)$$

It is easy to see

$$z_{sc}(m) = \langle v^2(m)\rangle/(2\pi G\Sigma). \quad (44)$$

The mass dependence of $z_{sc}(m)$ can be gathered from the velocity spectrum (35,36). It also allows us to introduce another scale height

$$z_{sc} = \langle v_i^2\rangle/(2\pi G\Sigma) \quad (45)$$

which is the scale height for stars with mass less than m_*. Thus a study of the scale heights $\langle z(m)\rangle$ can help us to determine the velocity spectrum and hence the exponent θ.

It is seen from the above that z_c depends only on the gross features like Σ and $\rho(0)$ and is insensitive to the mass spectrum exponent x_i or the exponent of mass consumption θ_i. The other scale length z_{sc} on the other hand depends on both these exponents and for a one component case we also get a simple relation, $z_0^2/z_{sc} = \Sigma\rho(0)$.

1. Single-component case

This is the case where $i = 0$. To test the efficacy of the formula we have tried to use it to compare the different sets of data, – giving widely different values for Σ and $\rho(0)$. In Table 1, we exhibit some of the values calculated from the known data. The quantities selected from the observations are the ones for Σ and $\rho(0)$ while all other quantities are calculated from the formulae, given in the present paper. In these calculations, we have used the values $x = 2.35$ (i.e. the Salpeter value) and $\theta = 2.5$. The calculated values show that the values obtained for $x = 2.35$ show closer agreement with observation than those given for $x \to \infty$, i.e. the Spitzer case which considers all masses to be equal. The value of θ given above is indeed an average one. In order to ascertain it accurately, one has to study $\langle z^2(m)\rangle$ for different masses of stars and extract the value of θ.

2. Two-component case

Recent data analyses by Kuijken & Gilmore (Kuijken & Gilmore 1989, 1989a) have shown that the stellar population in our galaxy is composed of two parts – a thin disk with a low velocity dispersion (suffix $i = 0$) and a thick disk with a high velocity dispersion (suffix $i = 1$). It is seen from their data that the K dwarfs in the thin disk have scale height 249 pc while for those in the thick disk lt is 1000 pc, which gives us $(v_1^2/v_0^2) \sim (1000/249) \sim 4.016$. Further using the thick disk data in the range $1500\,\text{pc} \leq z \leq 2500\,\text{pc}$, we have $v_1^2 \sim 871\,(\text{km s}^{-1})^2$ so that $v_0^2 \sim 217.11\,(\text{km s}^{-1})^2$. For the thin disk we take $x_0 \sim 2.35$ and for the thick disc $x_1 \to \infty$. Thus we have $f_1^2(\theta_1, x_1) = 1.0.$ and $f_0^2(\theta_0, x_0) = 0.588$, which when substituted in (39–41) give us $\rho_1(0)/\rho(0) = 0.098$ and $\rho_0(0)/\rho(0) = 0.902$. A different set of values for the θ and x parameters in these disks would give us different values for these parameters, thus justifying the importance of the mass and velocity dispersion spectra.

5. Conclusion

The present paper gives a detailed calculation of the distribution of mass above the galactic plane, for any arbitrary mass spectrum given in (16) and the velocity spectrum given in (17). In Figs. 1–4 we present the manifestations of these results, for the single species case in which we have chosen $x = 2.3$, 2.4, 20 and $\theta = 2.5$, 1.0. The values $x = 2.3$, 2. 4 are chosen in the spirit of Salpeter's suggestion that for the initial mass function, one has in the solar neighbourhood, $x = 2.35$, while $x = 20.0$

Table 1. Comparison of galactic parameters, calculated from different observational sets

Ref.	Σ	$\rho(0)$	θ	v_0 $x \to \infty$	v_0 $x = 2.35$	z_{sc} $x \to \infty$	z_{sc} $x = 2.35$
1.	78 ± 5	0.18± 0.02	2.5	14.34 ±0.92	17.92 ±1.15	137.88 ±8.83	202.15 ±40.95
2.	67 ± 5	0.185± 0.02	2.5	12.30 ±0.90	15.37 ±1.13	119.32 ±9.94	149.15 ±9.95
3.	52	0.0985	2.5	13.80	17.25	191.90	239.80
4.	46 ± 9	0.1	2.5	11.96 ±2.34	14.96 ±2.90	162.63 ±31.82	203.24 ±39.76

Units: Σ in $m_\odot$ pc^{-2}; $\rho(0)$ in $m_\odot$ pc^{-3}; v_0 in km s^{-1}; z_{sc} in pc.
References: 1. Hill, Hilditch & Barnes 1979. 2. Bahcall & Soneira 1980; Bahcall, Schmidt, Soneira 1983. 3. Hill, Hilditch & Barnes 1979; Bahcall 1984. 4. Kuijken & Gilmore 1989, 1989a.

corresponds to the Spitzer case of $x \gg 1$. In choosing the velocity spectrum, one has been led by the concept of random walk in velocity space under rapidly fluctuating forces. In this respect, the interstellar collisions have been neglected. These interstellar collisions are expected to lead to an equipartition result in thermal equilibrium, $\langle v^2(m) \rangle \sim 1/m$. i.e. $\theta = 1$. At the present epoch, when the age of the galaxy is much smaller than the relaxation time, the relaxation processes can be neglected, as has been done here. The $\theta = 1$ cases are considered to show the dependence of the results on θ. It is to be noted that since the velocity spectrum exponent θ appears as an exponential term $\exp(-\eta)$, as compared to the mass spectrum exponent x, which appears as a power law term $(m_0/m)^x$, in expressions (19,21), the results are more sensitive to the choice of the velocity spectrum exponent θ. Hence, for any fit with observations, it is natural to demand that the value of θ should be ascertained. Most of the theories, as also observational fits take $\theta = 0$, i.e. the velocity dispersion is independent of the mass of the star, a hypothesis accepted more on account of paucity of data than due to compelling physical grounds. To extract this parameter observationally, one must extend the observations to A and B type stars, while the present day observations are limited to K dwarfs and F type stars. The situation may improve when the Hipparcos data on the stellar positions and stellar motions are available.

The present paper has employed a relation between the velocity dispersion and the mass of the star by estimating the time spent by the stars in a stochastic field – the time being related to the life-time of the stars, which is calculated from the mass burning formula. In the computations, we have used the average value $\theta = 2.5$, $m_0 = 0.1 m_\odot$ and $m_* = 0.85 m_\odot$.

It follows from this work that observational data on the distribution of stars above the galactic plane are required to be fitted to a more general theory as given here than to empirical fits that have been done till now (Van der Kruit 1988; Hill et al. 1979; Kuijken & Gilmore 1989). In the Part 3 of the present paper we have obtained a relation between the surface density of matter (Σ), mass density ($\rho(0)$) and the velocity dispersion (V^2) in a disk like galaxy – this relation being a manifestation of the virial theorem. The sensitivity of the calculated parameters to the choice of x is shown in Table 1. These have been done for several sets of data and it is seen that for the same set, the case $x \to 2.35$ i.e. the Salpeter case gives better fit than the Spitzer case $x \to \infty$. This is seen from the fact that the scale height of old stars, like K giants is given by $\sqrt{2}\, z_{sc}$ which in the Salpeter case works out to lie between 200–300 pc, while that in the Spitzer case lies between 150–200 pc as can be calculated from the last two columns of Table 1. The observed data, though not definitive, give the scale height to be 250 pc. This, we believe, is an indication that in order to determine the midplane mass density and the surface density, the proper mass spectrum is to be considered.

Acknowledgements. The author would like to thank his colleague Dr. D.C.V. Mallik for various discussions on velocity dispersion spectrum. He also thanks his colleague Dr. H.C. Bhatt for some useful comments on the manuscript. The referee of the paper is thanked for some interesting comments.

References

Abramowicz M., Stegun I.A., 1965, Handbook of Mathematical Functions, Chapter 13. Dover, New York
Bahcall J.N., 1984a, ApJ 276, 156
Bahcall J.N., 1984b, ApJ 276, 169
Bahcall J.N., Schmidt M., Soneira R.M., 1983, ApJ 265, 730
Bahcall J.N., Soneira R.M., 1980, ApJS 44, 73
Chatterjee S., 1991, J. Astrophys. Astron. 12, 269
Gradsteyn I.S., Ryzhik I.M., 1980, Table of integrals, Series and Products, Equation 3.381.6. Academic Press, New York
Hill G., Hildich R.W., Barnes J.V., 1979, MNRAS 186, 813
Jeffereys H., 1962, Asymptotic Expansions, Sect. 2.3. Clarendon Press, Oxford
Kuijken K., Gilmore G., 1989a, MNRAS 239, 571
Kuijken K., Gllmore G., 1989b, MNRAS 239, 651
Lang K.R., 1974, Astrophysical Formulae, Sect. 5.4.3. Springer, Berlin
Olver F.W.J., 1974, Asymptotics and Special Functions, Sect. 7. Academic Press, New York
Salpeter E.E., 1955, ApJ 121, 161
Spitzer L., 1942, ApJ 95, 329
Van der Krult P.C., 1988, A&A 192, 117
Van Kampen N.G., 1973, Phys. Rep. 21, 171 (see Sect. 12 therein)
Van Kampen N.G., 1985, Phys. Rep. 124, 263
Wielen R., 1977, A&A 60, 263