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Distribution of Stars Perpendicular to the Plane of the Galaxy

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Abstract. We present here rigorous analytical solutions for the Boltzmann-Poisson equation concerning the distribution of stars above the galactic plane. The number density of stars is considered to follow a behaviour $n(m, 0) \sim H(m - m_0)m^{-x}$, where m is the mass of a star and x an arbitrary exponent greater than 2 and also the velocity dispersion of the stars is assumed to behave as $\langle v^2(m) \rangle \sim m^{-\theta}$ the exponent θ being arbitrary and positive. It is shown that an analytic expression can be found for the gravitational field K_z, in terms of confluent hypergeometric functions, the limiting trends being $K_z \sim z$ for $z \to 0$, while $K_z \to$ constant for $z \to$ infinity. We also study the behaviour of $\langle |z(m)|^2 \rangle$, *i.e.* the dispersion of the distance from the galactic disc for the stars of mass m. It is seen that the quantity $\langle |z(m)|^2 \rangle \sim m^{-\theta}$, for $m \to \infty$, while it departs significantly from this harmonic oscillator behaviour for stars of lighter masses. It is suggested that observation of $\langle |z(m)|^2 \rangle$ can be used as a probe to find x and hence obtain information about the mass spectrum.

Key words: mass spectrum—self consistent field—velocity dispersion

1. Introduction

The vertical distribution of stars (*i.e.* in the z-direction) above the galactic plane requires the solution of the Boltzman equation for the distribution function, self-consistent with the Poisson equation for the gravitational potential. The solution is expected to be quite sensitive to the velocity dispersion spectrum $\langle v^2(m) \rangle$ and the mass dispersion, $n(m)$, – the stars being considered to have different masses. This contention is justified because the ditribution of masses in phase space is decided by the velocity dispersion $\langle v^2(m) \rangle$ and the gravitational potential $\phi(z)$, where the calculation of $\phi(z)$ is again decided by the mass spectrum and the distribution of the masses in phase space. This self consistency condition, as stated above, serves as the motivation for the present study, concerning the effect of mass spectrum and velocity dispersion on the vertical distribution of stars above the galactic plane.

The solutions given below, are consistent under the assumption of Gaussian velocity dispersion, with the quantity $\langle v^2(m) \rangle$ assumed to be independent of z (Bahcall 1984a, 1984b). The important assumptions in our theory are:

(1) The mass spectrum follows the law:

$$n(m, 0) \sim H(m - m_0)(m/m_0)^{-x} \tag{1}$$

with $x > 2$, where $n(m, 0)$ is the number of stars per unit volume per unit mass interval

in a cylindrical volume of infinitesimal thickness around the galactic plane and $H(m - m_0)$ is the Heaviside unit step function.
(2) The velocity dispersion follows:

$$\langle |v(m)|^2 \rangle \sim m^{-\theta}, \tag{2}$$

which we derive from the idea of random walk of the stars in the velocity space, under the influence of rapidly fluctuating forces (Wielen 1977). The m dependence of $\langle |v(m)|^2 \rangle$, as given in (2) arises by incorporating the assumption that the velocity diffusion takes place roughly over a time scale which is a small fraction of the lifetime of the stars, the lifetime being m-dependent.

The solutions given below are exact, in the sense they are non-perturbative. They are seen to have sensitive dependence on x and θ. In the limit $x \to \infty$, *i.e.* stars are of the same mass m_0, we recover the well-known Spitzer formula (Spitzer 1944). The present work elegantly takes into account the uniform variation of the masses. The existing solutions, on the other hand, divide the masses as majority and minority components, such that all the stars in the majority lot have the same mass, while those in the minority lot also have the same mass. Their starting point is to take a Spitzer-type solution with the majority masses alone and then to use the rest of the masses as perturbations. In contrast, our scheme is non-perturbative and the method can be extended to any collection of different species, provided any i-th lot has a mass spectrum $n_i(m, 0) \sim (m/m_i)^{-x_i}$, and the velocity spectrum $\langle |v_i(m)|^2 \rangle \sim m_i^{-\theta}$. Finally, we calculate the dispersion $\langle |z(m)|^2 \rangle$ in the position of the stars. We find that it goes as $m^{-\theta}$, for $(m/m_0) \to \infty$, the exact variation being dependent on θ and x.

2. Theory

2.1 *The Collisionless Boltzmann Equation*

The distribution function $f(z, v, t)$ is known to follow the collisionless Boltzmann equation in one dimension (here the z-direction):

$$[\partial/\partial t + v\partial/\partial z + (F_s + F_f)\partial/\partial v]f = 0, \tag{3}$$

where F_s denotes a slowly varying acceleration and F_f a fast varying acceleration on the particle.
Let us consider F_f to have a correlation:

$$\langle F_f(t) F_f(t') \rangle = |F_f|^2 \exp(-|t - t'|/\tau_0). \tag{4}$$

Defining $\tau = t/\tau_0$, Equation (3) can be written as

$$[\partial/\partial\tau + \tau_0 F_f(\tau)\partial/\partial v]f + \tau_0(v\partial/\partial z + F_s\partial/\partial v)f = 0. \tag{5}$$

Considering $\tau_0 \to 0$, $|F_f \tau_0| \to$ constant, but for the term in the parentheses in Equation (5) to be finite, we can identify the powers of τ_0 on both sides of Equation (5) and equate each of the terms in (5) to zero (Van Kampen 1985). Thus

$$[\partial/\partial t + F_f(t)\partial/\partial v]f = 0, \tag{5.1}$$

$$(v\partial/\partial z + F_s\partial/\partial v]f = 0. \tag{5.2}$$

This enables us to write:

$$f(t,z,v)=f_1(t,v)f_2(z,v). \tag{6}$$

From (6) and (5.1), we get,

$$\partial f_1/\partial t=-iF_f(t)\hat{p}f_1, \tag{7}$$

where

$$\hat{p}=-i\partial/\partial v.$$

We now expand f_1 in terms of the eigenfunctions of p and write,

$$f_1=\sum a(p,t)\exp(ipv), \tag{8}$$

so that (7) yields,

$$(\partial/\partial t)\,a(p,t)=-i\,F_f(t)\hat{p}\,a(p,t). \tag{9}$$

The quantity $F_f(t)$ being a random variable, we have on an average (Van Kampen 1973), for $(t/\tau_0)\gg 1$,

$$(\partial/\partial t)\langle a(p,t)\rangle=-p^2\left[\int_0^\infty dt'\langle F(t)F(t')\rangle\right]\langle a(p,t)\rangle. \tag{10}$$

Considering the correlation in $F_f(t)$ to follow the form given in (4), we have the solution of (10) to be,

$$\langle a(p,t)\rangle=a(p,0)\exp(-|F_f|^2\tau_0 tp^2). \tag{11}$$

The coefficient $a(p,0)$ has to be found from the initial condition,

$$f_1(v,0)=\delta(v-v_0), \tag{12}$$

giving

$$a(p,0)=\exp(-ipv_0). \tag{13}$$

Hence

$$f_1(v,t)=[\sqrt{\pi}\,\delta v(t)]^{-1}\exp[-(v-v_0)^2/[\delta v(t)]^2], \tag{14}$$

where

$$\delta v(t)=[|F_f|^2\tau_0 t]^{1/2}.$$

This shows,

$$\langle v\rangle=v_0$$

and

$$\langle(v-v_0)^2\rangle=2(\delta v(t))^2. \tag{15}$$

Further the initial velocity v_0 is assumed to be a Gaussian:

$$f(v_0)=[\pi\sigma_0]^{-1/2}\exp(-v_0^2/\sigma_0^2), \tag{16}$$

so that averaging $f(v,t)$ over all possible v_0's we get,

$$f_1(v,t)=[\pi\langle v^2(t)\rangle]^{-1/2}\exp(-v^2/2\langle v^2(t)\rangle), \tag{17}$$

where

$$\langle v^2(t)\rangle=\sigma_0^2+2(\delta v(t))^2, \tag{18}$$

which is Wielen's formula (Wielen 1977), obtained using the Central Limit Theorem. Similar problems have also been tried by Spitzer & Schwarszchild (1951, 1953) and

Lacey (1984), by considering the velocity relaxation to be dependent on the kinetic energy of the particle.

Equation (17) and (18) describe the fast varying part of the distribution function. The slowly varying part follows equation (5.2), which is written as,

$$(v\partial/\delta z)f = (\partial\phi/\partial z)(\partial/\partial v)f, \tag{19}$$

$\phi(z)$ being the slowly varying part of the potential *i.e.* the part arising due to the distribution of stellar masses ($F_s = -\partial\phi/\partial z$). Thus putting $f = f_1(v,t)f_2(z,v)$, as given in (6) and substituting $f_1(v,t)$ from (17) in (19), we have,

$$f_2(z) = \exp(-\phi/\langle v^2(t)\rangle).$$

Hence

$$f(z,v,t) \sim [\pi\langle v^2(t)\rangle]^{-1/2}\exp(-v^2/2\langle v^2(t)\rangle)\cdot\exp(-\phi/\langle v^2(t)\rangle) \tag{20}$$

so that averaging over all velocities v we obtain,

$$f(z,t) \sim \exp(-\phi/\langle v^2(t)\rangle) \tag{21}$$

which is also the starting point of Bahcall's calculation (Bahcall 1984a, 1984b).

2.2 *Mass Dependence of the Velocity Dispersion*

The velocity dispersion $\langle(v^2(t))\rangle$ given in (18) depends upon the time t spent by the masses in the field of the random forces. This time t depends upon the lifetime of the stars, which thus yields a mass dependence in the velocity dispersion $\langle v^2(t)\rangle$. We know, the mass consumption is related to the luminosity $L(m)$ as,

$$c^2(\mathrm{d}m/\mathrm{d}t) = -L(m) = -L_\odot(m/m_\odot)^{\theta+1}. \tag{22}$$

The exponent θ is known to be nearly 2.0 for main sequence stars. Following Wielen (1977), we consider, the stars to have spent nearly half their lifetime on the main sequence. This yields.

$$t(m) = (c^2/2L_\odot)\int_m^{0.9m} \mathrm{d}m/(m/m_\odot)^{\theta+1}$$

$$= (m_\odot c^2/L_\odot\theta)(m_\odot/m)^\theta. \tag{23}$$

Thus

$$\langle v^2(m)\rangle = \sigma_0^2 + C_v(m_\odot c^2/L_\odot{}^\theta)(m_\odot/m)^\theta \tag{24}$$

giving a mass dependence of the velocity dispersion, with $C_v = 2|F_f|^2\tau_0$. Substituting $m_\odot = 1.9892 \times 10^{33}$ gms, $L_\odot = 3.8268 \times 10^{33}\,\mathrm{erg\,sec^{-1}}$ and values from Wielen (1977) $C_v = 6 \times 10^{-7}(\mathrm{km\,s^{-1}})^2/\mathrm{year}$ and $\sigma_0^2 = 10^{12}(\mathrm{cm/sec})^2$, we find that the first term can be neglected for $(m/m_\odot) < 500$, if $\theta = 2$. Thus for all practical purposes, we can use,

$$\langle v^2(m)\rangle = (\alpha m^\theta)^{-1} \tag{25}$$

where

$$1/\alpha = (C_v m_\odot c^2)/(L_\odot\theta).$$

The mass dependence of the velocity dispersion as given in (24) and (25) is an indirect one, being decided by the time $\tau(m)$ spent by the star in the random force field, where $\tau(m)$ is decided by the mass-consumption rate. The mass-luminosity formula $L(m) \sim m^\theta$, makes $\tau(m) \sim m^{-\theta}$ and hence Equation (24) results. For higher $\tau(m)$, hence for lower masses, (24) approximates (25) but deviations from (25) would result if $(m/m_{\odot}) > 500$ *i.e.* for very massive stars. However, these stars being very few in number, contribute little to the total mass density and hence the discrepancy between (24) and (25) would not show up in the calculations that follow. Also (18) has good observational support for old stars (Wielen 1977; Wielen & Fuchs 1989), from which we indirectly conclude that light stars which live longer will follow (24) more closely. It should also be kept in mind that the exponent θ is mass dependent, but on an average has a value ~ 2 for low mass stars. However, for the problem at hand, the variation of $\theta(m)$ with m does not have severe effects as is explained in section 2.3.

2.3 *Solution of the Poisson Equation with Mass Spectrum and Velocity Dispersion*

We use the above input to solve the Poisson equation. Here we introduce the mass spectrum as given in (1) as another important input. We consider the system to consist of ν different "lots" such that the mass spectrum for any i-th lot is:

$$n_i(m,0) = H(m - m_i) n_i m_i^{x_i - 1}(x_i - 1) m^{-x_i} \tag{26}$$

and each has a velocity dispersion

$$\langle v_i^2(m) \rangle = (\alpha_i m)^{-\theta_i} \tag{27}$$

where

$$i = 0, 1, 2 \ldots . (\nu - 1).$$

The choice of the exponent x_i depends upon the present-day mass function of the stars and other matter. Further, we consider that at any subsequent epoch, the total birthrate of stars contributes weakly to the total mass density. One would then expect the mid-plane number density to follow a scaling law as is given by (26). If the time elapsed after the birth of the stars be much smaller than the main-sequence lifetime of the stars, then x_i can be found from the initial mass function and the mass-luminosity relation for the main sequence.

Thus the mass density at the galactic plane for any i-th lot is given by:

$$\rho_i(0) = n_i m_i (x_i - 1)/(x_i - 2). \tag{28}$$

This shows that the above choice of the mass-spectrum is valid for $x_i > 2$, lest there be divergence in the mass density in the galactic plane. This divergence problem can be eliminated by putting an upper cut-off for the mass. This is a trivial exercise as far as the method of our solution is concerned and is important only if $x_i \leqslant 2$.

The Poisson equation is then given by:

$$(\mathrm{d}^2\phi/\mathrm{d}z^2) = 4\pi G \sum_{i=0}^{(\nu-1)} n_i m_i [(x_i - 1)/(x_i - 2)] \cdot \int_{m_i}^{\infty} m^{-x_i} m \, \mathrm{d}m \exp(-\phi \alpha m_i^{\theta_i}), \tag{29}$$

where we have used,

$$n_i(m,z) = n_i(m,0) \exp(-\phi/\langle v_i^2(m) \rangle). \tag{30}$$

It is to be noted, however, that the exponent θ varies with mass (Lang 1974). However, being a Laplace type integral, the method of steepest descent shows that the integral in (29) will have much larger contributions for low m values and hence the error involved in neglecting the variation in $\theta(m)$ will be insignificant (Jeffereys 1962; Olver 1974). The other alternative would be to treat θ_i as a free parameter. However, this exercise would still make θ_i have a large weightage close to that for low value of m. Hence, in the following, we proceed with our calculations by putting θ_i as a constant, to be chosen as that for the low mass stars.

Multiplying both sides of (29) by $2(\mathrm{d}\phi/\mathrm{d}z)$ and integrating, we have

$$(\mathrm{d}\phi/\mathrm{d}z)^2 = 8\pi G \sum_{i=0}^{(\nu-1)} \rho_i(0) m_i^{(x_i-1)}[(x_i-2)/\alpha_i] \times \int_{m_i}^{\infty} [1-\exp(-\phi\alpha_i m^{\theta_i})]\cdot m^{1-x_i}\mathrm{d}m \tag{31}$$

with the boundary conditions $(\mathrm{d}\phi/\mathrm{d}z) = 0$ and $\phi = 0$ at $z = 0$.
Defining:

$$z_i^2 = [4\pi G\,\rho_i(0)(\alpha_i m_i^{\theta_i})]^{-1} \tag{32.1}$$

$$\eta = \phi(\alpha_0 m_0)^{\theta_0}, \tag{32.2}$$

$$\varepsilon_i = (m_i^{\theta_i}\alpha_i)/(m_0^{\theta_0}\alpha_0) \tag{32.3}$$

$$\beta_i = z_0^2/(z_i^2\varepsilon_i^2) \tag{32.4}$$

$$\chi = z/z_0 \tag{32.5}$$

we can easily integrate (31) (Gradsteyn & Ryzhik 1980) as

$$(\mathrm{d}\eta/\mathrm{d}\chi)^2 = 2\sum_{i=0}^{(\nu-1)} \beta_i[(x_i-2)/(x_i+\theta_i-2)][1-[(x_i+\theta_i-2)/\theta_i] \times \exp(-\varepsilon_i\eta)U(1,s_i,\varepsilon_i\eta)] \tag{33}$$

where

$$s_i = (2-x_i)/\theta_i.$$

$U(a,b,z)$ is a confluent hypergeometric series (Abramowicz & Stegun 1965).

3. Special cases

(1) *Low* χ limits.

In this limit, the series expansion of (33) in terms of $\varepsilon_i\eta \to 0$ gives,

$$(\mathrm{d}\eta/\mathrm{d}\chi)^2 = 2\sum_{i=0}^{(\nu-1)} \beta_i\varepsilon_i\eta$$

i.e.

$$(\mathrm{d}\phi/\mathrm{d}z)^2 = 8\pi G\rho(0)\phi \tag{34}$$

where

$$\rho(0) = \sum_{i=0}^{(\nu-1)} \rho_i(0)$$

is the total density at the galactic plane.
Hence,

$$\phi \sim 2\pi G\rho(0)z^2 \tag{35}$$

giving a quadratic dependence on height, in this limit.

(2) *High χ limit.*

Expansion of (33) for $\varepsilon_i \eta \to \infty$ yields

$$(\mathrm{d}\eta/\mathrm{d}\chi)^2 = 2 \sum_{i=0}^{(\nu-1)} \beta_i (x_i - 2)/(x_i + \theta_i - 2)$$

or

$$(\mathrm{d}\phi/\mathrm{d}z)^2 = 8\pi G \sum_{i=0}^{(\nu-1)} \rho_i(0) \langle v_i^2(m_i) \rangle (x_i - 2)/(x_i + \theta_i - 1). \tag{36}$$

This shows an interplay between the mass density $\rho_i(0)$ and the velocity dispersion $\langle v_i^2(m_i) \rangle$ of the different species. The right hand side of Equation (36) is however, independent of η. Hence, in this limit, we get, $\phi \propto z$, signifying a uniform gravitational field directed towards the mid-plane.

(3) *All masses equal.*

This corresponds to the case $\varepsilon_i = \delta_{i,0}, \rho_i = \rho_0 \delta_{i,0}$, so that $\beta_i = \delta_{i,0}$ and $x_0 \to \infty$. In this limit (see Abramowicz & Stegun 1965) we have

$$U(1, s_i, \varepsilon_0 \eta) \to \theta_0/(x_0 + \theta_0 - 2) \tag{37}$$

so that Equation (33) reduces to

$$(\mathrm{d}\eta/\mathrm{d}\chi)^2 = 2(1 - e^{-\eta}) \tag{38}$$

whose solution is the well known result obtained by Spitzer (1942), *i.e.*

$$\eta = 2 \ln \mathrm{ch}\, (\chi/\sqrt{2})$$

and hence

$$\rho(z) = \rho(0) e^{-\eta} = \rho(0)\, \mathrm{sech}^2(\chi/\sqrt{2}).$$

(4) *Dispersion in the position of the particles.*

This quantity $\langle z_i^2(m) \rangle$ is calculated from

$$\langle z_i^2(m) \rangle = I(2)/I(0) \tag{39}$$

where

$$I(2) = \int_0^\infty z^2 \exp(-\phi/\langle v_i^2(m) \rangle) \mathrm{d}z$$

and

$$I(0) = \int_0^\infty \exp(-\phi/\langle v_i^2(m) \rangle) \mathrm{d}z,$$

i denoting the lot referred to.

In the limit $(m/m_i) \to \infty$, the term $\exp(-\phi/\langle v_i^2(m) \rangle)$ becomes very small at all points other than the points very close to the origin. Here $\phi \sim z^2$, as given in (35) so that the integrals in (39) yield,

$$\tilde{f}_i(m) \equiv (m/m_i)^{\theta_i} \cdot \langle z_i^2(m) \rangle / \langle z_s^2 \rangle \tag{40}$$

which tends to 1 as $(m/m_i) \to \infty$, but goes to higher values as $(m/m_i) \to 1$, z_s being a scale height.

(5) *Mass $M(z)$ contained between $z = 0$ and $z = z$.*

By definition,

$$M(z) = \int_0^z \rho(z)\mathrm{d}z$$
$$= (4\pi G)^{-1} K_z \qquad (41)$$

where $K_z = (\mathrm{d}\phi/\mathrm{d}z)$ is the magnitude of the gravitational field. Obviously, $K_z \sim (\partial\eta/\partial\chi)$ which is calculated from (33). It is easily seen from (34), (36) that $M(z) \sim z$ for z much smaller than the scale height z_s and $M(z)$ tending to a constant for z far beyond the scale height.

4. Conclusions

The present paper gives a detailed calculation of the distribution of mass above the galactic plane for any arbitrary mass distribution as given in (1) and velocity distribution as given in (25). The solutions are exact and valid for the presence of any arbitrary number of species, $i = 0, 1, 2, \ldots (\nu - 1)$, some of which may be referred to as halo. Another hypothesis that has appeared is that the Gaussian velocity distribution is valid and the collisionless Boltzmann equation, in presence of fast varying random external forces gives the velocity dispersion as given by Wielen (1977). In Figs 1–6 we have investigated the manifestation of these results for a single species system *i.e.* $\nu = 1$, for the values $x_0 = 2.3$, 2.4 and 20, while $\theta_0 = 2$ (*i.e.* main sequence case) and 1. The values $x_0 = 2.3$, 2.4 are chosen in accordance with Salpeter's initial mass function (Salpeter 1955) and the case $x_0 = 20$ corresponds to a case with a very weak mass dispersion spectrum. The case $\theta_0 = 1$ corresponds to the one in which the stars

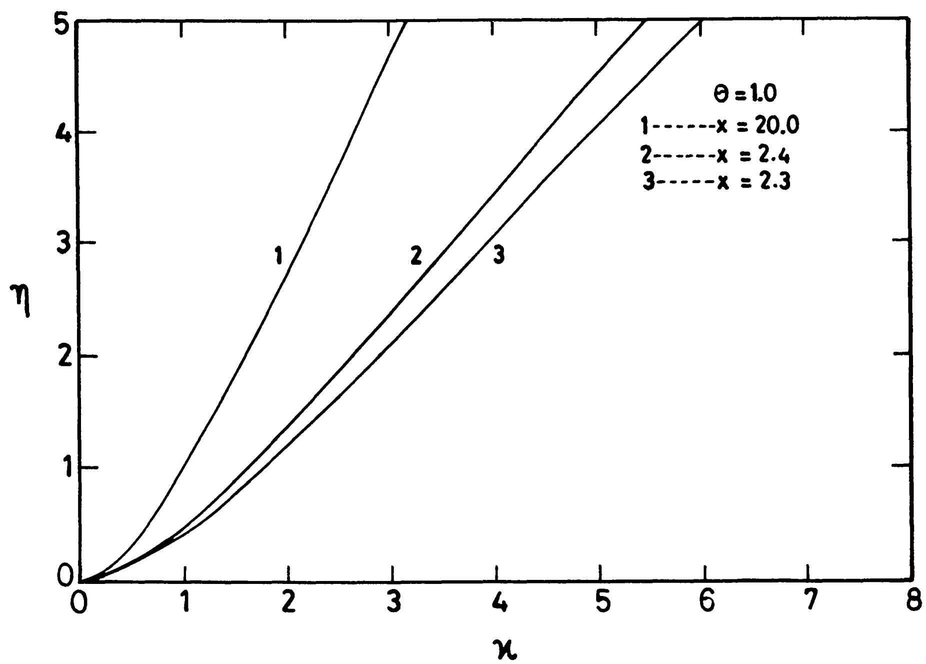

Figure 1. η *vs* χ plot for $\nu = 1, \theta = 1$.

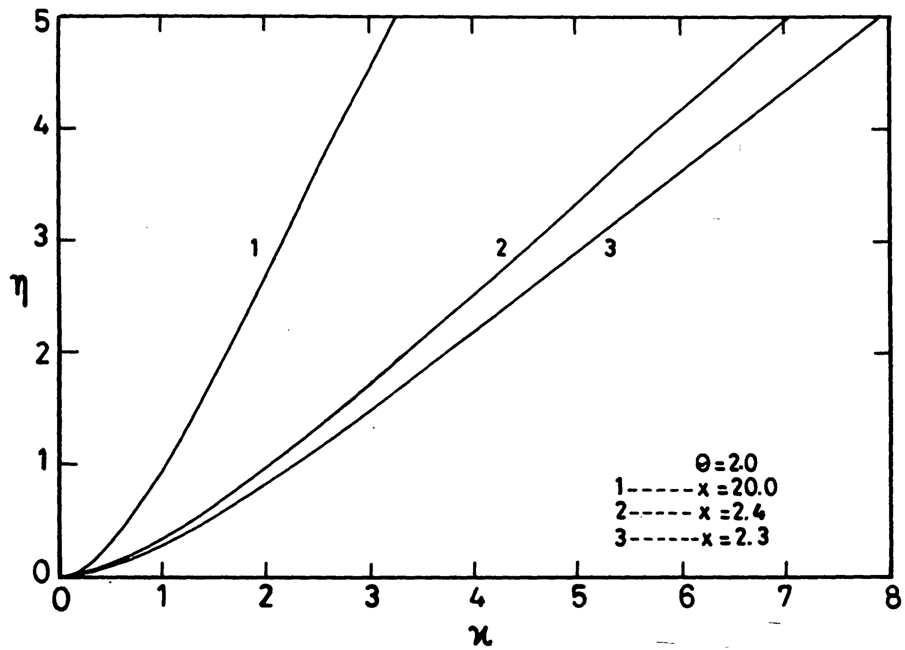

Figure 2. η *vs* χ plot for $\nu = 1, \theta = 2$.

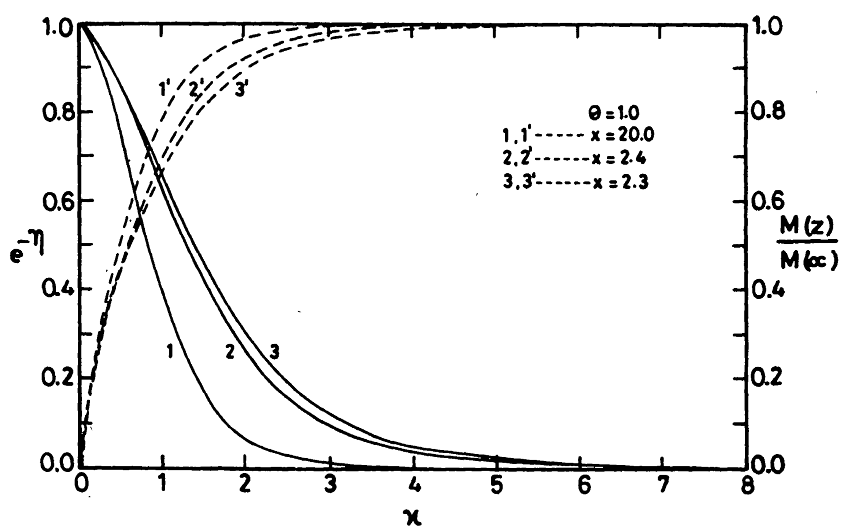

Figure 3. Distribution of mass m_0 with height. Variation of $e^{-\eta}$ and $M(Z)$ with χ for $\nu = 1, \theta = 1$. Solid curve corresponds to $e^{-\eta}$ and dashed curve to $M(z)$.

are in thermal equilibrium, though this formula is derived not from Wielen's derivation, but due to the thermal collision of the stars. The difference between the two mechanisms is quite distinct. Thermalization leads to an equipartition of energy, giving $\langle v^2(m)\rangle \propto m^{-1}$, but for such a condition to be valid, sufficiently long time, *i.e.* longer than the collisional relaxation time must elapse, after the system has been "prepared". For stellar systems, these collisional relaxation times are very large, being 10^{12}–10^{14} years, which is longer than the age of the galaxies. An exhaustive study of these phenomena can be found in the review article by Chandrasekhar (1943). Wielen's mechanism on the other hand, considers random walk in velocity space. This leads

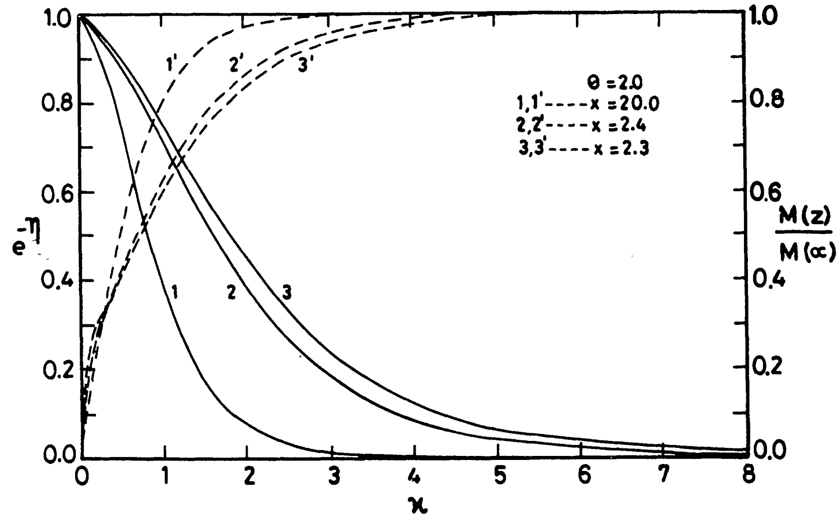

Figure 4. Distribution of mass m_0 with height. Variation of $e^{-\eta}$ and $M(Z)$ with χ for $\nu = 1, \theta = 2$. Solid curve corresponds to $e^{-\eta}$ and dashed curve to $M(z)$.

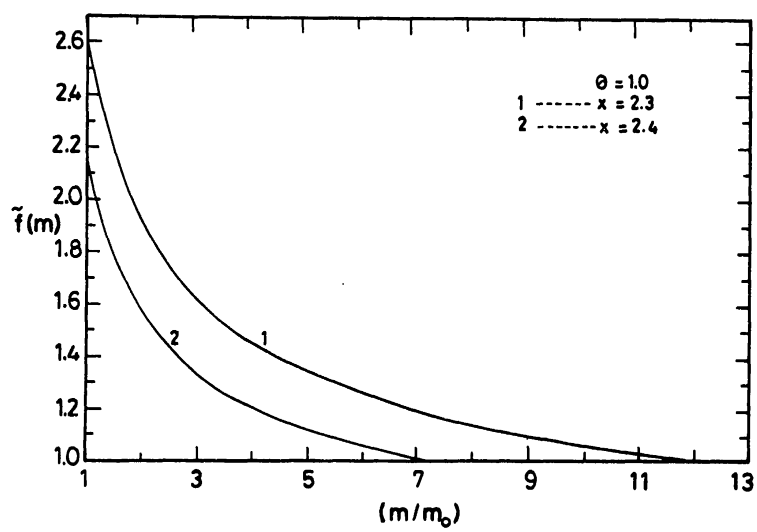

Figure 5. Variation of $\tilde{f}(m)$ with m for $\nu = 1$, $\theta = 1$.

to stochastic acceleration of stars, if the relaxation time of the random forces be shorter than the age of the system. However, both the mechanisms lead to a power—law $\langle v^2(m) \rangle \propto m^{-\Delta}$. For the case of thermal equilibrium, $\Delta = 1$, while Δ is different for stochastic acceleration, being decided by the mass-consumption rate, as is given by Equation (25). For the sake of completeness, we have included $\Delta = 1$ case (*i.e.* thermal equilibrium) in the numerical work.

The numerical computational results show that the mass distribution is crucially dependent upon the mass spectrum as also on the velocity dispersion. This is seen from the variation of η with χ. We also present the variation of $e^{-\eta}$ with χ, $e^{-\eta}$ being proportional to the number density of stars of mass m_0.

The results given here justify the contention that the distribution of stars above the galactic plane require to be fitted to more generalized formulae given here. At

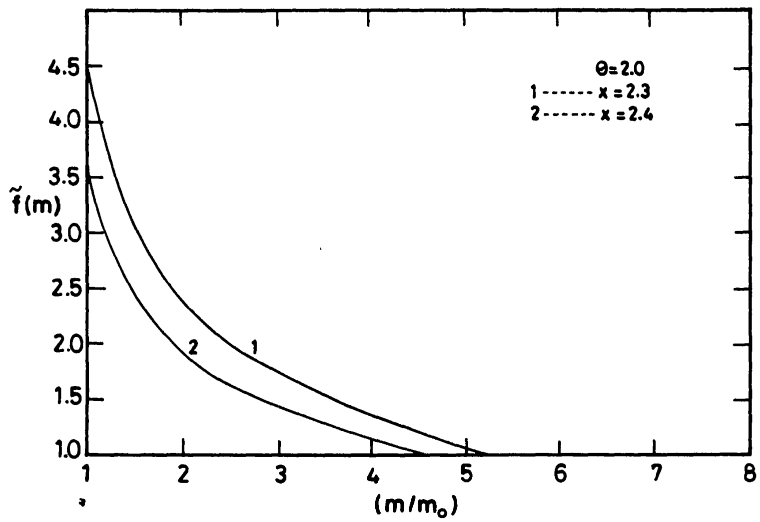

Figure 6. Variation of $\tilde{f}(m)$ with m for $v = 1$, $\theta = 2$.

present, several kinds of formulae are being tried in empirical sense, (Van der Kruit 1988; Hill, Hilditch & Barnes 1979; Kuijken & Gilmore 1989). However, a rigorous formula as given here, is expected to give a better fit. In the observational context, study of the quantity $\tilde{f}_i(m)$ may serve as sensitive tool to find the mass spectrum.

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