

Luminosity Effects in Projected Fractals

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Received 15 May 2001; accepted 6 January 2002

Abstract. The use of two-dimensional catalogues in unraveling the large-scale distribution of extra-galactic objects can reveal more information than has been supposed if the objects have approximate scaling properties such as observations suggest. After a brief general discussion of this issue, we turn to specific examples of projected fractals for the case where the objects studied have a Schechter luminosity function. We analyze the effects of projection on the characteristics of such a fractal distribution. Our results indicate that two-dimensional catalogues of sources could be of value in detecting the effects of luminosity functions as well as of large-scale structure.

Key words: Large-scale structure, fractals

1. Introduction

What we now call large scale structure — the patterns revealed in the spatial distribution of galaxies — was first detected in attempts to quantify the distribution of galaxies on the sky as revealed in position catalogues. Those first studies were mainly statistical and were carried out in terms of correlation functions of the galaxy positions (Neyman, *et al.*, 1953; Neyman and Scott, 1959; Peebles 1980), though one should also note the less objective but revealing studies by de Vaucouleurs (1970). In the early statistical work, the galaxies were considered to be points, as they were in the related studies of catalogues of radio sources under the rubric of $\log N - \log S$ studies (Ryle and Clarke, 1961; Peebles, 1983); that picture is also adopted here. From those positional studies, Mandelbrot (1975) recognized that the galaxy distribution is well modeled by what he has called a fractal distribution * (Mandelbrot, 1982).

In more recent times, three-dimensional galaxy catalogues have been made available in which the three coordinates assigned to each galaxy are two coordinates on the sky

* We use the term *fractal* in its most general sense to include the special cases called multifractals and monofractals, which terms have come into common use in recent years.

and a redshift. Though these data are not as abundant as those in the two-dimensional catalogues, they nevertheless support the findings of approximate self-similarity in the large-scale distribution of galaxies within prescribed scale intervals (Provenzale, 1991). However such conclusions depend on the transformation from redshift space to geometrical space, which is not completely accurate because of the peculiar velocities of the objects in question. Above all, since the data in two-dimensional catalogues are more plentiful than those in three dimensions and are likely to remain so for a time (Cress, et al., 1996), it seems worthwhile to discuss further the ways in which the two-dimensional data may be used to unravel the nature of the three-dimensional distribution of extragalactic objects.

The basic issue in analyzing a two-dimensional catalogue goes back to work on stellar statistics (Trumpler and Weaver, 1953), with some added complications. In stellar statistics, the effect of fractality was not considered much, though in the galaxy case it is important. What was significant in the stellar studies was the effect of the luminosity distribution of the observed objects on the projected distributions, an issue is our main concern here. However, rather than deal with the inverse problem implied by the integral equation known as the fundamental equation of stellar statistics (Trumpler and Weaver, 1953), we shall study the effects of projection by directly simulating the projection of some simple point sets. In this way, we can explicitly study the role of luminosity functions in the projection of fractals. However, for the present, we shall leave out of account some other features of the galaxy case that remain to be considered in detail, namely source evolution and cosmological effects (Spedalere and Schucking, 1980; Ribeiro, 1995). Our main concern then is not so far from that of the study of stellar statistics: disentangling the luminosity function of the observed objects from their complex distribution on the sky. Naturally, the stellar case has difficulties of its own, such as interstellar absorption. Although an analogous issue has been examined for the galaxy distribution by Durrer, et al. (1997), that is also something for another exploration.

One of our motivations for raising these issues comes from our interest in trying to distinguish the different scaling regimes that may exist in the galaxy distribution, though on different length scales. It has been suggested (Murante *et al.*, 1998) that three scaling regimes may be discerned in the distribution of galaxies: on the smallest scales the results are consistent with a distribution of density singularities; in the intermediate range, there is scaling behaviour suggestive of flat structures such as Zeldovitch (1970) favored; on the largest scales, the data indicate a homogeneous distribution of galaxies with nonfractal behavior. This categorization is based on the results of an analysis of three-dimensional catalogues. However, it has to be admitted that those data may as yet not be adequate for clearly deciding such issues and that is why we wish to consider whether we may reliably use two-dimensional catalogues for immediate needs.

Before describing how projection effects are to be treated, we devote the next section to a brief outline of how structure in the galaxy distribution may be quantified. Then we turn to the distinction between theoretical predictions and numerical calculations of the effects of projection on the basis of simple examples. To understand this, we must try to minimize the influence of details such as the number of points in the sets and the parameters of the luminosity functions.

Then we go on to investigate the influence of luminosity effects by imposing a Schechter luminosity function on the objects of study and attempt to see how this influences the distribution of the projected objects, or rather of their apparent magnitudes. We conclude that luminosity effects can indeed make themselves felt in the macroscopic parameters of the projected fractals and suggest how these issues may be included in quantitative analyses of two-dimensional catalogues.

2. Quantifying Cosmic Scaling

In a rarefied gas, the usual definition of density breaks down as we go to the limit of small volumes. The situation is even worse for fractals where the density as usually understood is singular, as we shall see presently. Since the correlation functions used in the early analyses of the galaxy distribution (Neyman Scott, 1959; Peebles 1980) are defined in terms of the density, this has led to questioning of the use of correlations for the study of fractal sets (Bhavsar, 1980; Coleman and Pietronero 1992). Fortunately, only a slight change in approach is needed to obtain a more robust characterization of the structures of point sets by way of the *correlation integral* (Grassberger and Procaccia, 1983; Paladin and Vulpiani, 1984; Borgani, 1995). Indeed, the change seems so slight that one may be surprised at the difference it makes. Nevertheless, explicit calculations performed on analytically understood point sets have shown that the correlation integral is the more reliable tool of analysis of the properties of point distributions (Thieberger *et al.* 1990). For these reasons, the correlation integral has been used increasingly in the study of large-scale structure (Provenzale, 1991) and, in this section, we briefly summarize its properties and generalizations. A deeper discussion of this question is provided by Bessis and Fournier (1989).

Consider then a set of N galaxies — or points — for which the correlation integral may be defined as

$$C_2(r) = \frac{1}{N(N-1)} \sum_i \sum_{j \neq i} \Theta(r - |\mathbf{X}_i - \mathbf{X}_j|) \quad (2.1)$$

where Θ is the Heaviside function and the summations are over $N - 1$ galaxies with coordinates \mathbf{X}_j , $j \neq i$ (Grassberger and Procaccia 1983). Modifications of this formula to allow for the effects of the finiteness of the sample may be introduced when dealing with real data (Murante *et al.*, 1997; Kerscher, 1999), but we pass over such technical details here.

We may interpret $C_2(r)$ as $\mathcal{N}(r)/N$ where $\mathcal{N}(r)$ is the average number of galaxies within a distance r of a typical galaxy in the set. As r goes to zero, C_2 should go to zero and, for general distributions, we express this conclusion as $C_2 \propto r^{D_2}$. The exponent D_2 is called the correlation dimension and it is necessarily ≤ 3 for an embedding space of dimension three. When D_2 is not an integer, the distribution is called fractal (Mandelbrot, 1982) and, in that case, the density, $n(r) \propto \mathcal{N}/R^3 \propto r^{D_2-3}$, has a singularity, as mentioned above.

As with correlation functions, it is possible to study higher order statistics. To generalize the correlation integral formalism, one introduces (Paladin and Vulpiani 1987)

$$C_q(r) = \left(\frac{1}{N} \sum_i \left[\frac{1}{N-1} \sum_{j \neq i} \Theta(r - |\mathbf{X}_i - \mathbf{X}_j|) \right]^{q-1} \right)^{\frac{1}{q-1}}, \quad (2.2)$$

where q is a parameter that defines the order of the moment. For $q = 2$, a two-point probability (second-order moment) is evaluated and the standard correlation integral is recovered. For any integer value of q , $C_q(r)$ is the fraction of q -tuples in the set whose members lie within a distance r of one another. For sufficiently small r , C_q will go to zero for $q > 1$ and, for a typically well-behaved set, it will vanish like r^{D_q} , where the index D_q is called a generalized or Renyi dimension (Renyi 1970, Halsey *et al.* 1986).

A fractal whose dimensions are all the same (D_q independent of q) is called a homogeneous fractal, or monofractal. The more general cases with D_q depending on q are called multifractals. Some authors reserve the term fractal for the special case of a monofractal but, as mentioned at the outset, we here retain the general sense of the term, with the multifractal as a particular case. For small r , in leading order, a plot of $\log C_q$ versus $\log r$ is approximately linear and the slope is D_q . The dimensions are therefore easily found if the data are adequate, which may not be the case for the three-dimensional distribution of galaxies. Some hints of the next term in the expansion for small r — the lacunarity function (Thieberger *et al.*, 1990) — have been found for the galaxy distribution (Provenzale *et al.*, 1997), though it has yet to be confirmed that the observed fluctuations about the linear plot are intrinsic to the geometrical distribution of galaxies.

3. Projecting Fractals

Having dealt with the polemical and pedagogical aspects of this problem, we may now turn to our basic question: how are the fractal properties of a distribution of points in three-dimensional space transformed by projection of the set onto a subspace like the celestial sphere? The main features of this question can already be seen in projecting from two dimensions into one dimension and, as this version of the problem is more economical of computing resources, we adopt it here for our illustrative examples. The work for the full case of projection from three dimensions is very much the same, as we have verified by performing a few explicit tests. By way of introduction to these problems the reader may wish to look at the analytic study of a specific case, performed by Mandelbrot (1975; see also Dogterom and Pietronero, 1991).

Before describing results obtained with actual examples, we recall some of the relevant mathematical theorems. As in most sciences, we must then face the uncomfortable gap that arises between theory and practice because the quantities most easily measured, here the D_q , for $q > 1$, are not those that conform best to mathematical usage. For the simplest systems, that is for the monofractals for which D_q is independent of q , these distinctions do not arise, while for most practical cases they are slight. So we shall not enter here into the details; these are discussed in Beck and Schloegl (1993), for example.

An empirically useful characterization of a set of points is by way of its fractal dimension D evaluated in a space of dimension \mathcal{D} . To define this, we first find the smallest

number of balls — the mathematician's word for the interiors of spheres of dimension D — of diameter ℓ that are needed to cover all the points. As $\ell \rightarrow 0$, this number will vary like ℓ^{-D} . The quantity D is the fractal dimension of the set of points; it is also called the box-counting dimension and it is almost always the same as D_0 . Computing it from data is less reliable than determining D_2 but, if we know the recipe by which a mathematically conceived set is constructed, it is not difficult to find the exact values of D_0 and to compare them with those determined from the exponents found by the use of correlation functions (Thieberger, *et al.* 1990).

For subtle reasons, for some sets that we will rarely encounter in real life, the box-counting procedure just described may not work and so the definition may be relaxed to allow balls of all sizes in a covering of the set by balls. For each such generalized covering, an effective dimension may be obtained and the least upper bound of these is called the Hausdorff dimension, D_H . It may be shown that $D_H \leq D_0$. The Hausdorff dimension is the one that many mathematicians work with and, of course, they also prefer to be more precise in defining it. However, we almost never need to distinguish D_H from D_0 (Mainieri, 1993), which has an intuitive appeal and brings us closer to the way that de Vaucouleurs (1970) thought about these things than to the more conventional statistical treatments of galaxy distributions. (When asked how he happened to employ such a seemingly mathematical approach, de Vaucouleurs (private communication) explained that this was simply what he had observed in the sky.) Although many of the theorems about fractal sets are stated in terms of the Hausdorff dimension, Sauer and Yorke (1995) showed that the theorems which apply to Hausdorff dimension apply also to the correlation dimension, which may be shown to be a lower bound on the Hausdorff dimension.

Since the actual examples we study here concern solely projections from a two-dimensional point set onto a one-dimensional space such as a line or a circle, we denote the original set by S_2 and the projected set by S_1 . Then we inquire into what can be said about the relation between these two sets.

Generally speaking, one finds that $D_h(S_1) = 1$, if $D_h(S_2) > 1$, and that $D_h(S_1) = D_h(S_2)$, if $D_h(S_2) < 1$, as might have been expected intuitively. The theorems on which this rough summary is based were already proven by Besicovitch (1939). Moreover, Radons (1993) has shown that, if $D_\infty(S_2) < 1$, not all $D_q(S_1)$ are the same for different q . Hunt *et al.* (1997) proved a theorem whose relevant part for us is that, under almost all linear transformations, from two dimensions to one dimension, $D_2(S_1) = \min[1, D_2(S_2)]$. In their paper, they give a nonpathological example showing that for $q > 2$ an analogous equality does not hold.

Though the results just outlined as representing the theory of the projection of fractals suggest how well-developed that theory is, we must stress that it typically applies only when very large numbers of points are involved. Since it is sometimes true that the results of asymptotic treatments do not accurately reflect the outcome of a real situation with a finite number of points in the set, we have carried out some explicit examples of the effects of projection on point sets. We now describe these. In choosing such examples, we must be aware that projection onto a circle and onto a plane are fundamentally different. When we project onto a plane here, we do it unidirectionally, while our projections onto the unit circle will be along radii extending from the center. These differences demand

special care in the numerical work. The sectors used in the projections onto the circle must be such that a high enough density of points is encountered in each if we are to obtain reproducible results. And, in comparing different cases, one should take care to use the same angles of projection.

To check on our procedures, we have verified that in projecting homogeneous two-dimensional sets onto lines we obtain the theoretical results to good accuracy, even for a sample of moderate size. For fractals, however, the situation is more complicated and to standardize the procedures we have worked with Cantor fractals constructed as follows. We first make a Cantor set by decimation. That is, from the unit interval $[0, 1]$ we remove the open set $(\frac{1}{a}, 1 - \frac{1}{a})$, where $a > 1$. This leaves the two outer segments $[0, \frac{1}{a}]$ and $[1 - \frac{1}{a}, 1]$. The choice of the parameter a , which is 3 for the standard Cantor set, is left open for the moment. A fractal set is then constructed by repeating the procedure on the two remaining segments and continuing in this way with each of the remaining segments in successive generations. The result is a monofractal whose D_q is $\log 2 / \log a$ for all q .

To construct a two-dimensional fractal we formed the Cartesian product of two one-dimensional fractals, each with its own value of a . We designate the set constructed in this way as $S2 = (a_1, a_2)$. Such composite sets are also monofractals whose dimension is the sum of those of the two Cantorial sets (Falconer, 1990):

$$D(S2) = \frac{\ln 2}{\ln a_1} + \frac{\ln 2}{\ln a_2} . \quad (3.1)$$

This fractal typically produces a different fractal dimension according to the line on which it is projected.

Our calculations on these sets were performed using a procedure called iterated function schemes (Barnsley, 1988). Results for the projection of three such multiplicative fractal sets are presented in the following two tables. In Table 1, we give the projected dimensions for the three sets, designated (a_1, a_2) , chosen to illustrate the projection effects for cases with small, medium and large values of $D(S2)$. In this table we also show that projections onto a circle or onto a (generic) line give, to good approximation, the same results, though care was taken to choose the appropriate angles.

For the smallest number of points used in our analysis, we performed the calculations with many different angles to limit our sample. Because of the nature of projecting onto a circle, we obtained different projected dimensions for different sets of angles. We then selected that set which gave us the largest dimension. By using this procedure we eliminate problematic regions (i.e. sparse regions) in our projection. We then adopted this set of angles for the remainder of our calculations.

In Table 2, we show the convergence of the dimensions with increasing numbers of points in the set, for the case of D_2 . In seeking convergence, we doubled the number of points when going from one line to the next. Our results indicate that for the fractal $S2 = (6.54, 3)$ we do not get the value of two for $D(S1)$ as suggested in the aforementioned work of Hunt *et al.* (1997). Of course, we are dealing with finite sets and, from these, it is not clear that we can achieve convergence to the value predicted by theory. We also give in Table 2 the cases of projecting a line and projecting a homogeneous (randomly

distributed) two-dimensional set on a circle. It is interesting that we cannot distinguish between the results of these two projections.

Table 1: The Projected Dimensions.

Set	D_{th}	D_2	D_4	D_6	N
(6.54,3.)	1.000	0.89	0.86	0.84	62000
(6.54,3.)*	1.000	0.88	0.83	0.79	62000
(15.,14.)	0.519	0.52	0.50	0.50	58000
(3.,2.68)	1.334	0.98	0.98	0.97	58000

*This case is projection on a line.

Table 2: Variation of the calculated D_2 with N.*

N	(6.54,3.)	(15.,14.)	(3.,2.68)	line	2D random
15	.876±.007	.520±.007	.969±.003	.972±.001	.975±.002
30	.883±.003	.519±.002	.978±.002	.985±.001	.987±.001
60	.885±.003	.519±.002	.984±.001	.993±.001	.993±.001
120	.888±.001	.518±.002	.987±.001	.996±.001	.996±.001

*The number of points, N, is given in thousands

4. Luminosity effects.

In considering the distribution on the celestial sphere of galaxies or of radio sources the role of the intrinsic luminosities of the observed objects must be allowed for (Maurogordato and Lachièze-Rey, 1991). Such effects appear in $N(> f)$, the number of galaxies detected whose flux density exceeds a given value, f (Peebles, 1993). This function is a more reliable indicator of the dimension of the set under study than the dimension obtained directly from projection without regard to luminosity effects.

As is often done in first approximation in the stellar case, we assume that the luminosity function of the objects in our study is independent of position and of time. Though such dependences (especially the latter) may be important and ought to be included in more advanced studies, they involve uncertainties and, in any case, we prefer to take things one at a time as we learn how the various influences make themselves felt.

Then, for a homogeneous distribution of galaxies with no evolution, we have (Peebles, 1993)

$$N(> f) = K f^{-1.5} \quad (4.1)$$

The constant, K , in this relation incorporates the luminosity function, which for galaxies, is often chosen to be the function (Schechter, 1976),

$$\sigma \propto w^\alpha \exp(-w) \quad (4.2)$$

where $w = L/L_*$, L is the luminosity, L_* is some characteristic luminosity and observations give $\alpha = -1.07 \pm 0.05$.

Equation (4.1) is the consequence of integrating all the points over the range from zero to infinity. In that case, the value of α influences only the constant and not the

power of f in (4.1). In the case where one can write down the fractal distribution in the same way as the homogeneous one, the equation becomes (Peebles, 1993)

$$N(> f) = kf^{-\beta} . \quad (4.3)$$

where β can be approximated by $D_2/2$. The derivation is approximate and does not allow for lacunarity oscillations (Provenzale et al., 1997) that are washed out in projection according to Durrer *et al.* (1997).

In the calculations described here, we project a two dimensional space onto a one-dimensional one. We nevertheless keep the same luminosity versus distance law as for the three dimensional case, namely

$$f = \frac{L}{4\pi r^2} . \quad (4.4)$$

Therefore, instead of eq. (4.1), we obtain the form

$$N(> f) = Kf^{-1} . \quad (4.5)$$

We also consider the case of projecting a line onto a circle. To explain the results in this case, we have to go into more detail concerning how one obtains the equations. According to Peebles (1993) we can write

$$N(> f) = \int_0^\infty r^2 dr \psi(Ar^2 f) \quad (4.6)$$

where

$$\psi(x) = n \int_x^\infty w^\alpha \exp^{-w} dw \quad (4.7)$$

and α is the exponent from Eq. (4.2).

In the case of the projection of a line, where the variation in r is not very large (for example, for the case of Table 2: $84 > r > 71$), the limits will depend on f . Therefore we cannot assume that we can incorporate the luminosity function into K as in the other cases. Now K will depend on the luminosity function, that is, on the value of α .

To illustrate our point let us assume that we could approximate the different r values by their means. Then we could introduce a δ -function into the integral with argument $(r - r_m)$ so that

$$N(> f) = \int_0^\infty r dr \delta(r - r_m) \psi(Afr^2) = r_m \psi(Afr_m^2) . \quad (4.8)$$

Thus we can expect the power of f to be influenced by α , as confirmed by our numerical calculations. It is therefore convenient to write

$$N(> f) = \text{const. } f^\beta \quad (4.9)$$

and to evaluate the parameters in this expression from the calculations.

Table 3: Dependence of $N(> f)$ on f .

Set type	Specification	D_{th}	β_{th}	β_{cal}	α
random	2D	2.00000	-1.000	-0.99 ± 0.01	-1.25
random	2D	2.00000	-1.000	-1.00 ± 0.01	-3.25
random	1D	1.00000	—	-2.0 ± 0.1	-1.25
random	1D	1.00000	—	-2.7 ± 0.1	-3.25
cantor	(6.54,3.)	1.00003	-0.500	-0.51 ± 0.03	-1.25
cantor	(6.54,3.)	1.00003	-0.500	-0.51 ± 0.02	-3.25
cantor	(4.7,2.7)	1.14575	-0.573	-0.57 ± 0.02	-1.25
cantor	(7.,4.)	0.85621	-0.428	-0.41 ± 0.05	-1.25

In Table 3, we compare the β obtained for the different sets on which we performed our calculations. The values are obtained by least square fit of $N(> f)$ for different f values. As f increases, the number of rejected points increases. The higher the r_{max} the fewer retained points remain out of a given sample. The larger f , the steeper this decrease is. In the calculations used for Table 3, we kept the number of projected points at the constant value of 40000 throughout by appropriately increasing the number of points processed as the number of rejected points rose. Even a smaller sample gives quite reasonable results.

As our R_{max} is perforce finite, we have to take care not to include too small values of f . If this point is ignored we obtain a more complicated dependence of $N(> f)$ on f . To get the appropriate range where R_{max} does not influence the results, we look for the R for which almost no projected points reach the unit circle, for a given f . Only those f that are not eliminated because of R_{max} are then included in the calculations. We also examined the influence of the results on α (Eq. (4.2)). As might be expected, α does not influence the results except in the case of projecting a line onto a circle. Our results are in good agreement with the analysis described by Peebles (1993).

5. Conclusion

We conclude that extensive two-dimensional catalogues of extraterrestrial objects can reveal more about the statistics of these objects than may have been supposed. We have illustrated the possibilities with a simple exercise in fractal projection that shows how the existing mathematical results on this subject may be brought to bear on the study of large-scale structure. Though we have included only one feature not explicitly mentioned in most mathematical studies, namely the luminosity distribution, we see already the emergence of an interesting aspect of the projection process. To appreciate the usefulness of this result, we should recall that, in the projection of a homogeneous (or randomly distributed) set, there is no real difference in the dimension between the original and the projected dimension once the change in embedding dimension is allowed for. However, if a suitable luminosity function is folded into the projection process, pronounced differences emerge. This makes it possible to distinguish among sets with different dimensions after projection through the influence of the luminosity function.

Thus even when plentiful three-dimensional data become available, projection effects will be worth studying since they avoid some of the problems in the measurement and interpretation of redshifts, while still revealing the influence of the luminosity function. For example, the three regimes mentioned in Murante *et al.* (1998), give appreciably different results on projection if the luminosity effects are included. It is also possible to include the effects of source evolution or of galaxy merger by making the parameters in the Schechter distribution time-dependent.

In conclusion, we thank Antonello Provenzale for his critical reading of the manuscript. We are also grateful to S. Bhavsar, I. Grenier J. Yorke for providing some references. E.A.S. would like to express his gratitude to the Tata Institute for Fundamental Research for hospitality in March, 1999 with special thanks to Kumar Chitre for his many kindnesses during that visit.

References

- Barnsley M.F., 1988, *Fractals Everywhere*, Addison-Wesley, Reading, MA.
 Besicovitch A.S., 1939, *Math. Annalen*, 116, 349.
 Becker C. and Schloegl F., 1993, *Thermodynamics of chaotic systems*, Cambridge Univ. pp. 97-101.
 Bessis D. and Fournier, J.-D., 1989, *Phys. Lett. A*, 140, 331.
 Borgani S., 1995, *Phys. Rep.*, 251, 1.
 Coleman P.H., Pietronero L. and Sanders R.H., 1988, *Astron. & Astrophys.*, 200, L32.
 Coleman P.H. and Pietronero L., 1992, *Phys. Rep.*, 231, 311.
 Cress, C. M., Helfand, D. J., Becker, R. H., Gregg, M. D. and White, R. L., 1996, *Ap. J.* 473, 7.
 Cress, C. M. and Kamionkowski, M., 1998, *MNRAS*, ???.
 Dogterom M. and Pietronero L., 1991, *Physica A*, 1991, 171, 239.
 Durrer, R., Eckmann, J.-P., Labini, S. L., Montouri, M. and Pietronero, L., 1997, *Europhys. Lett.*, 40, 491.
 Falconer K., 1990, *Fractal Geometry* (Chichester, UK: John Wiley and Sons)
 Feder J., 1988, *Fractals*, Plenum Press, N.Y., U.S.A.
 Grassberger P. and Procaccia I., 1983, *Phys. Rev. Lett.*, 50, 346.
 Groth E.J. and Peebles P.J.E., 1977, *Astrophys. J.*, 217, 385.
 Halsey T.C., Jensen M.H., Kadanoff L.P., Procaccia I. and Shraiman B.I., 1986, *Phys. Rev. A*, 33, 1141.
 Hunt B.R. and Kaloshin V.Yu., 1997, *Nonlinearity*, 10, 1031.
 Kerscher M., 1999, *A&A*, 343, 333.
 Mainieri, R., 1993, *Chaos*, 3, 119.
 Mandelbrot B.B., 1975, *Comptes Rendus* 280A, 1551.
 Mandelbrot B.B., 1982, *The Fractal Geometry of Nature* (San Francisco: Freeman).
 Martinez V.J. and Jones B.J.T., 1990, *Mon. Not. R. Astron. Soc.*, 242, 517.
 Maurogordato, S. and Lachièze-Rey, M., 1991, *Ap. J.*, 369, 30.
 Murante G., Provenzale A., Borgani S., Campos A. and Yepes G., 1996, *Astroparticle Phys.*, 5, 53.
 Murante G., Provenzale A. Spiegel E.A. and Thieberger R., 1997, *Mon. Not. R. Astron. Soc.*, 291, 585.
 Murante G., Provenzale A. Spiegel E.A. and Thieberger R., 1998, *Annals of the New York Academy of Sciences*, 867, 258.
 Neyman J., Scott, E.L. and Shane, C.D., 1953, *Astrophys. J.*, 117, 92.
 Neyman J. and Scott E.L. 1959, *Hdbch d. Phys.*, 53, Berlin: Springer, p. 416, ff.
 Paladin G. and Vulpiani A., 1984, *Nuovo Cimento Lett.*, 41, 82.
 Paladin G. and Vulpiani A., 1987, *Phys. Rep.*, 156, 147.
 Peebles P.J.E., 1980, *The Large Scale Structure of the Universe* (Princeton: Princeton Univ. Press).
 Peebles P.J.E., 1993, *Principles of Physical Cosmology* (Princeton: Princeton University Press) p. 215.
 Provenzale A., 1991, in *Applying Fractals in Astronomy*, A. Heck and J.M. Perdang Eds. (Berlin: Springer).

- Provenzale A., Spiegel E.A. and Thieberger R., 1997, *Chaos*, 7, 82.
Radons G., 1993, *J. of Stat. Phys.*, 72, 227.
Renyi A., 1970, *Probability Theory* (Amsterdam: North Holland).
Ribeiro, M. B, 1995, *Ap. J.*, 441 477.
Ryle, M. and Clarke, R. W., 1961, *MNRAS*, 122, 329.
Sauer T.D. and Yorke J.A., 1995(?), in *Ergodic Theory and Dynamical Systems*, ???
Schechter, P., 1976, *Ap. J.*, 203, 297.
Spedalere, R. and Schucking, E. L., 1980, *Astron. J.*, 85, 586.
Thieberger R., Spiegel E.A. and Smith L.A., 1990, in *The Ubiquity of Chaos*, S. Krasner Ed. (AAAS Press).
Trumpler, P. J. and Weaver, H. F., 1953 *Statistical Astronomy* (Berkeley and Los Angeles: University of Calif.)
Zeldovitch Ya.B., 1970, *Astrofizika(A)* 6, 319.