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Why quantum mechanics is complex*

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Abstract. The zero-signature Killing metric of a new, real-valued, 8-dimensional gauging of the conformal group accounts for the complex character of quantum mechanics. The new gauge theory gives manifolds which generalize curved, relativistic phase space. The difference in signature between the usual momentum space metric and the Killing metric of the new geometry gives rise to an imaginary proportionality constant connecting the momentum like variables of the two spaces. Path integral quantization becomes an average over dilation factors, with the integral of the Weyl vector taking the role of the action. Minimal $U(1)$ electromagnetic coupling is predicted.

One of the more puzzling aspects of quantum mechanics is its seemingly necessary reliance on complex quantities. The theory's interference effects, probability amplitudes and coupling to unitary gauge interactions all make important use of complex numbers. In this essay we show how a real-valued 8-dimensional geometry can account for these behaviors in a natural way.

This geometry, called biconformal space, arises as a new gauging of the conformal group [1-3]. The gauging is accomplished in three steps. First, counting fixed points identifies *eight* of the conformal transformations as translations, with the remaining homogeneous Weyl transformations (Lorentz transformations and dilations) forming the isotropy subgroup, C_0. Second, we construct an elementary geometry as the quotient, C/C_0, of the conformal group, C, by C_0. This produces a principal fiber bundle over an 8-dimensional manifold (various topologies are allowed). Finally, we generalize to a curved Cartan connection by adding horizontal curvature 2-forms to the group structure equations. The curvature breaks both the translational and inverse translational symmetries while retaining all 15 gauge fields. This contrasts sharply with previous 4-dim gaugings of the conformal group [4 - 9], for which the inverse translational gauge fields are always auxiliary [5, 9].

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To implement these steps we choose the O (4,2) representation of the conformal group, with connection 1-forms ω^A_B (A, B, ... = 0, 1, . . . , 5). Letting boldface or Greek symbols denote forms and (a, b, ...) = (1,...,4), the O (4,2) metric is given by $\eta_{ab} = diag$ (1,1,1,−1) and $\eta_{05} = \eta_{50} = 1$ with all other components vanishing. The covariant constancy of η_{AB} reduces the independent components of ω^A_B into four Weyl-invariant parts: the *spin connection*, ω^a_b, the *solder form*, ω^a_0, the *co-solder form*, ω^0_a, and the *Weyl vector*, ω^0_0 where the spin connection satisfies

$$\omega^a_b = -\eta_{bc}\eta^{ad}\omega^c_d \ .$$

The remaining components of ω^A_B are given in terms of these. We restrict (A,B,...) = (0,1,...,4) in all subsequent equations. The Maurer-Cartan structure equations of the conformal group are simply

$$\mathbf{d}\omega^A_B = \omega^C_B \wedge \omega^A_C \ .$$

Since no finite translation can reach the point at infinity and no inverse translation can reach the origin, the space C/C_0 gives a copy of (noncompact) Minkowski space for each of the two sets of translations. Since the generators of the two types of translation commute modulo the homogeneous Weyl group, the base manifold is conformally the Cartesian product of these two Minkowski spaces.

The generalization to a curved base space is immediate. Adding curvature 2-forms to the structure equations and breaking them into parts based on homogeneous Weyl transformation properties, we have:

$$\begin{aligned}
\mathbf{d}\omega^a_b &= \omega^c_b \wedge \omega^a_c + \omega^0_b \wedge \omega^a_0 - \eta_{bc}\eta^{ad}\omega^0_d \wedge \omega^c_0 + \Omega^a_b \\
\mathbf{d}\omega^a_0 &= \omega^0_0 \wedge \omega^a_0 + \omega^b_0 \wedge \omega^a_b + \Omega^a_0 \\
\mathbf{d}\omega^0_a &= \omega^0_a \wedge \omega^0_0 + \omega^b_a \wedge \omega^0_b + \Omega^0_a \\
\mathbf{d}\omega^0_0 &= \omega^a_0 \wedge \omega^0_a + \Omega^0_0
\end{aligned} \qquad (2.12)$$

The four curvatures Ω^a_b, Ω^a_0, Ω^0_a and Ω^0_0 are called the Riemann curvature, torsion, co-torsion and dilational curvature, respectively. If we set $\omega^0_a = \omega^0_0 = \Omega^0_a = \Omega^0_0 = 0$ we recover a 4-dim spacetime with Riemannian curvature Ω^a_b and torsion Ω^a_0. If we set only $\omega^0_a = \Omega^0_a = 0$, we have a 4-dim Weyl geometry.

Horizontality yields curvatures of the form

$$\Omega^A_B = \tfrac{1}{2}\,\Omega^A_{Bcd}\;\omega^c_0\;\Lambda\omega^d_0 + \Omega^{Ac}_{Bd}\omega^d_0\Lambda\omega^0_c + \tfrac{1}{2}\,\Omega^{Acd}_B\,\omega^0_c\;\Lambda\;\omega^0_d\;.$$

Based on the correspondence principle relating biconformal space and phase space ([1], [2]), Ω^A_{Bcd} is called the spacetime term, Ω^{Ac}_{Bd} the cross term and Ω^{Acd}_B the momentum term of Ω^A_B .The fiber symmetry group does not mix the spacetime, cross or momentum terms.

For a *flat* biconformal space, with $\Omega^A_B = 0$, the connection may be put into the form ([1],[2])

$$\omega^0_0 = \alpha_a(x)\,\mathbf{d}x^a - y_a\mathbf{d}x^a \equiv W_a\mathbf{d}x^a \qquad (8a)$$

$$\omega^a_0 = \mathbf{d}x^a \qquad (8b)$$

$$\omega^0_a = \mathbf{d}y_a - (\alpha_{a,b} + W_aW_b - \tfrac{1}{2}W^2\eta_{ab})\mathbf{d}x^b \qquad (8c)$$

$$\omega^a_b = (\delta^a_d\delta^c_b - \eta^{ac}\eta_{bd})\,W_c\,\mathbf{d}x^d \qquad (8d)$$

where (x^a, y_a) are eight independent coordinates on the space and $\alpha_{a,b} \equiv \dfrac{\delta\alpha_a}{\delta x^b}$ denotes the partial of $\alpha_a$with respect to x^b (but not y_b).

Our current discussion centers on the metric structure of biconformal spaces. Although biconformal space is based upon the conformal group of Minkowski space, it does not inherit the Minkowski metric, η_{ab}. Nonetheless, every biconformal space has a natural metric based on the Killing metric. The Killing metric is built from the conformal group structure constants as

$$K_{AB} = C^C_{AD}\,C^D_{BC}$$

where the labels

$$(A,B,\ldots) \in \{\tbinom{b}{a},\tbinom{a}{0},\tbinom{0}{a},\tbinom{0}{0}\}.$$

refer to the Lorentz, translation, inverse translation and dilatation generators, respectively.

A short calculation gives

$$K^{(conf.)}_{AB} = \begin{pmatrix} K^L_{\binom{b}{a}\binom{d}{c}} & 0 & 0 & 0 \\ 0 & 0 & K_{\binom{0}{a}\binom{b}{0}} & 0 \\ 0 & K_{\binom{a}{0}\binom{0}{b}} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where

$$K^{L}_{\binom{b}{a}\binom{d}{c}} = 2(\delta^{b}_{c}\,\delta^{c}_{a} - \eta^{bd}\,\eta_{ac})$$

is the Killing metric for the Lorentz group, and

$$K_{\binom{0}{a}\binom{b}{0}} = \delta^{b}_{a}$$

$$K_{\binom{a}{0}\binom{0}{b}} = \delta^{a}_{b}$$

We now construct a metric on the bundle by expanding ω^{A} in a coordinate basis

$$\omega^{A} = (\omega^{a}_{b}, \omega^{a}_{0}, \omega^{0}_{a}, \omega^{0}_{0}) = \omega^{A}{}_{M}\mathbf{d}x^{M}$$

then defining metric components

$$g_{MN} = K_{AB}\omega^{A}{}_{M}\omega^{B}{}_{N} .$$

Unlike the degenerate Killing metric for the Poincaré group, $K^{(Conf.)}_{AB}$ is nondegenerate both on the bundle, and when projected to the base space. Restricting the indices to the range $M', N' \in \{(\mu), (\nu)\}$; $x^{M'} = (x^{\mu}, y_{\nu})$ the biconformal metric on the base space is

$$g_{M'N'} = K_{A'B'}\omega^{A'}{}_{M'}\omega^{B'}{}_{N'}$$

$$= \omega^{a}_{0\,M'}\,\omega^{0}_{a\,N'} + \omega^{0}_{a\,M'}\omega^{a}_{0\,N'}$$

Understanding the importance of $K_{A'B'}$ first requires understanding the phase space correspondence principle.

In [1], [2] it is shown that dilationally flat ($\Omega^{0}_{0} = 0$) biconformal spaces are generalizations of phase space, with the line integral of the Weyl vector, ω^{0}_{0}, equal to the action. The class of superhamiltonian hypersurfaces, $\mathcal{H}(H,\mathbf{p},x^{a}) = 0$ in the $\alpha_{a} = 0$ flat biconformal space are in 1 - 1 correspondence with Hamiltonian systems. The biconformal structure equations imply the existence of curves satisfying Hamilton's equations. Generalizing these dynamical equations by letting $\alpha_{a} \neq 0$ predicts the Lorentz force law for a charged particle in a background electromagnetic field, with $\alpha_{a} = -\frac{q}{\hbar c}\,A_{a}$. This last result rescues Weyl's geometric theory of electromagnetism [10-12]. Weyl equated the Weyl vector to the electromagnetic vector potential, but this results in predictions of unphysical size change. By contrast, equating α_{a} to the vector potential predicts the Lorentz force law and no size change.

But the structure equations of biconformal space go beyond the usual structure of phase space, giving a differential system for the eight solder forms, the spin connection and the Weyl vector. The Killing metric described above also represents new structure, importantly different from the usual metric on phase space.

Consider this metric difference in detail. The usual *phase space* metric follows from the metric on spacetime and the fact that the 4-momentum is proportional to the tangent vector to a curve. This implies that the *same* Minkowski metric acts on both the configuration and momentum parts of phase space :

$$g_{AB}^{Ph.Sp.} = (\eta_{\mu\nu}, \eta^{\mu\nu}) .$$

The two parts of $g_{AB}^{Ph.Sp.}$ are not an 8-dim metric – they are the same metric applied to either the spacetime or the momentum sector separately.

By constrast, the Killing metric $K_{A'B'}$ on biconformal space has eigenvalues ± 1 and zero signature. Suppose we diagonalize $K_{A'B'}$, and demand that the first four eigenvalues, which act on the spacetime coordinates x^{μ} , match the Minkowski metric of spacetime part of $g_{AB}^{Ph.Sp.}$. Then (since each 4-space is Lorentzian) the *only* way to match the remaining four eigenvalues is by contracting the y_{μ} coordinates using *minus* the Minkowski metric. Therefore,

$$g_{\mathrm{AB}}^{Biconf.} = (\eta_{\mu\nu,} - \eta^{\mu\nu})$$

This is the only way to achieve zero signature while keeping the usual Lorentz invariant inner product on both spacetime and momentum space.

Notice that $g_{\mathrm{AB}}^{Biconf.}$ also refers to two 4-dimensional subspaces, but for a different reason. Because we have introduced a *dimensionful* metric, the scaling weight of the spacetime and momentum parts are opposite. The two parts cannot be added together and must be regarded as applying only on appropriate submanifolds.

The essential point we wish to make is that the sign difference between $g_{AB}^{Ph.Sp.}$ and $g_{\mathrm{AB}}^{Biconf.}$ implies the existence of a complex structure relating the geometric variables y_{μ} to the physical variables p_{μ} . In the simplest case, when identifying biconformal coordinates (x^{μ}, y_{ν}) with phase space coordinates (x^{μ}, p_{ν}), we just set $y_{\nu} = \beta p_{\nu}$. Since the constant β must account for the sign difference in equating the norms

$$\eta^{\mu\nu} p_{\mu} p_{\nu} = (-\eta^{\mu\nu}) y_{\mu} y_{\nu} = -\beta^2 \eta^{\mu\nu} p_{\mu} p_{\nu}$$

it is pure imaginary. Also, β must account for the different units of y_{ν} (length $^{-1}$) and p_{ν}

(momentum), with $\hbar$ the obvious choice. Therefore (up to a real dimensionless constant) we can set

$$y_\nu = \frac{i}{\hbar} p_\nu$$

This relationship between the geometric variables of conformal gauge theory and the physical momentum variables is the source of complex quantities in quantum mechanics.

For a more general derivation of the complex structure, suppose we have identified the spacetime coordinates, but the y_μ and p_μ are related by an arbitrary nondegenerate transformation,

$$y_\mu = M_\mu^\nu p_\nu$$

Then for the norms to agree, we require

$$\begin{aligned} \eta^{\alpha\beta} p_\alpha p_\beta &= (-\eta^{\mu\nu}) y_\mu y_\nu \\ &= (-\eta^{\mu\nu})\, M_\mu^\alpha p_\alpha M_\nu^\beta p\beta \end{aligned}$$

or, since p_μ is arbitrary and $\eta^{\mu\nu}$ symmetric,

$$\eta^{\alpha\beta} = (-\eta^{\mu\nu}) M_\mu^\alpha M_\nu^\beta$$

Lowering an index, and turning to matrix notation, we have simply

$$M^t M = -1$$

which means that the real matrix M is normal, hence diagonable by an orthogonal transformation. Let O be the diagonalizing transformation so that $M = O^t DO$, with D diagonal. Then we have

$$M^t M = O^t D^t O O^t D O = O^t D^2 O = -1$$

or

$$D^2 = -1$$

demonstrating the existence of a complex structure. The factor of i used above is the simplest representation for D.

There are two further points to be discussed in identifying the proper correspondence between the physical variables of phase space and geometric variables of biconformal space.

First, η_{ab} does not respect the scale invariance of biconformal space as $K_{A'B'}$ does, but scales with weight +2. This means that there is no way to introduce it directly into a *general* biconformal space. However, the use of diagonalizing coordinates

$$\mathbf{u}^a = \frac{1}{\sqrt{2}}(\omega^a_0 + \eta^{ab}\omega^0_b)$$
$$\mathbf{v}_a = \frac{1}{\sqrt{2}}(\omega^0_a - \eta_{ab}\omega^b_0)$$

is possible nonetheless because this transformation is *symplectic*, preserving

$$\mathbf{d}\omega^0_0 = \omega^a_0 \wedge \omega^0_a$$

The closed, nondegenerate $\mathbf{d}\omega^0_0$ corresponds to the symplectic form in phase space. So, while the definite conformal weight properties of the coordinates ($x\mu$,y_v) are lost, the dynamical equivalence to phase space is maintained. Therefore, while the diagonalization masks the effect of scale changes, it is still dynamically correct in the symplectic, flat case or *locally* in the general case.

The second issue is the *existence* of spacetime as an integrable 4-dim submanifold of flat biconformal space. While the flat structure equations are involute, we must ask if the involution still exists in the $\mathbf{u}^a$,$\mathbf{v}^a$ basis. Substituting into the structure equation for ω^a_0 we find

$$\mathbf{du}^a = \frac{1}{\sqrt{2}}\mathbf{d}(\omega^a_0 + \eta^{ab}\omega^0_b)$$
$$= \mathbf{u}^b\omega^a_b + \eta^{ab}\mathbf{v}_b\omega^0_0$$

where we have used the antisymmetry of the spin connection, $\eta^{ab}\omega^c_b = -\eta^{cb}\omega^a_b$. Therefore, $\mathbf{u}^a$ is in involution if $\omega^0_0 = W_a\mathbf{u}^a$ so that the spacetime manifold exists provided only that the $\mathbf{u}^a = 0$ foliation is regular. The condition $\omega^0_0 = W_a\mathbf{u}^a$ means that 4-dim character of the action integral emerges together with spacetime. Interestingly, this involution reduces the rewritten $\mathbf{u}^a = 0$ structure equations to those of a constant curvature momentum 4-space with radius on the order of the Compton wavelength. With the required factor of i in converting to momentum variables, the radius in fact becomes a wavelength.

Finally, we give a brief description of one way to see how the path integral formulation of quantum mechanics corresponds to a statement about measurement in biconformal space using *real* variables (x^a, y_a). In [13] and [14] we show how other properties of quantum systems are automatically present in biconformal and other scale-invariant spaces. For the present, note that the form of the Weyl vector for a large class of solutions [1 - 3] including

the flat case above is

$$\omega^0_0 \alpha_a(x)\mathbf{d}x^a - y_a\mathbf{d}x^a = -(\frac{q}{\hbar c} A_a(x)\mathbf{d}x^a + y_a)\mathbf{d}x^a .$$

Identifying the integral of ω^0_0 as the action and expressing y_a in terms of the momentum we find

$$\int\omega^0_0 = -\frac{i}{\hbar}\int (p_a - i\frac{q}{c} A_a)\mathbf{d}x^a$$

We therefore correctly find both the proper minimal $U(1)$ coupling to the electromagnetic field, and the proper complex coefficient for the exponent. The average over $exp\ (-\frac{i}{\hbar}S)$ becomes an average over $exp(\int \omega^0_0)$. But $exp\ (\int_C \omega^0_0)$ is precisely the dilation factor along the path C, so the path integral amounts geometrically to an averaging of size-change factors for different paths.

The reason that these two path averages can give the same results is the difference in signature between phase space and biconformal space. Recall how, in the early days of relativity, the coordinate $\tau = ict$ was used to give speacetime a Euclidean signature. Eventually it was found easier and more general to use an indefinite metric. Here we see the same effect in phase space, where our standard assumption of the same inner product on the configuration and momentum sectors of the space has meant that we are effectively using a purely imaginary coordinate, $p_a = -i\hbar y_a$. By changing the signature, we can eliminate the "i", and work in real, geometric variables.

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