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The equivalence of precession phenomena in metric theories of gravity*

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Abstract. The requirement of general covariance imparts to metric theories of gravity, such as general relativity, important structural features. A precise mathematical form results, ensuring that computation of observable physical effects in the theory gives the same answers independently of the chosen system of coordinates. This coordinate independence property, in turn, can lead to an equivalence of apparently different physical effects. An important example is provided by the phenomenon of geodetic precession of a gyroscope as it falls freely in the gravitational field of a massive body. A simple argument is presented that demonstrates clearly, without the need for detailed calculation, how geodetic precession of a gyroscope and the effect of frame-dragging are fundamentally equivalent. The argument applies to a general class of metric theories of gravity. There exist potentially important implications of this equivalence for interpreting experiments proposed to test frame-dragging.

Key words : gravitation - relativity

1. Introduction and Summary

General relativity predicts two main effects on the spin of gyroscope : (1) the precession of spin axis due to the motion of the gyroscope in the gravitational field of a massive body, and (2) the precession arising from the "gravitomagnetic" field related to motions of the source itself. W. de Sitter (1916) derived the first effect, referred to as "geodetic" precession, in an analysis of the motion of the Moon around the Earth as the system revolves around the Sun. Because of an argument presented by Schiff (1960a, 1960b), the second effect can be referred to as the "frame-dragged" precession. Both effects are small in the vicinity of the Earth, presenting a considerable challenge to experimentalists. Since the original proposal by Schiff in 1960, an experiment to test both effects precisely has been under development with the support of the National Aeronautics and Space Administration (NASA) (for a history and overview of the experiment, see Everitt (1988)). In this experiment, now well known as Gravity Probe-B (GP-B) (NASA Gravity Probe- A was an experiment to test the gravitational

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redshift effect, verifying the predicted effect to an accuracy of 2 parts in 10^4 (Vessot 1980)), gyroscopes consisting of electrically supported, spherical rotors are to be flown in an Earth orbiting satellite. For a gyroscope in a 650 km radius polar orbit, the geodetic precession is 6600 m arc-sec / yr, while the frame-dragged precession is only 42 m arc-sec/yr. The goal of GP-B is to measure these effects to an accuracy of 1 m arc-sec/yr. A different version of the experiment has been proposed that would use a drag-free satellite design, instead of an electrical suspension system for the rotors (Lange 1995). Studies suggest that this design might be able to deliver improved accuracies by a factor of 10^2 to 10^3, provided it is used in conjuction with a dual-satellite scheme and microarcsecond-level steller astrometry. Highest possible accuracy is desirable not only for verifying the precessions themselves, but for testing other important theoretical predictions (e.g., a potentially small deviation from unity of the post-Newtonian parameter γ due to cosmological relaxation in scalar-tensor field theories (Damour and Nordtvedt 1993a, 1993b). Satellite experiments employing alternative methods for detecting the precessions at the accuracies of GP-B, but without using actual gyroscopes have been proposed (for a review, see Will (1989)) ; for discussion of a proposed experiment with LAGEOS satellites, see Ciufolini and Wheeler (1995).

In the meantime, it has become possible to determine the 19.2 m arc-sec geodetic precession of the lunar perigee predicted by de Sitter to an accuracy of 2% (Shapiro *et al.* 1988; Bertotti, Ciufolini, and Bender 1987). Geodetic precession of pulsar spin-axes might be confirmed eventually in favourable pulsar binary systems (Wolszczan 1991). As we will see in more detail, the interpretation of geodetic precession observations can be expanded in an interesting way. Because of general covariance, the observable precession can be calculated in any convenient reference frame. In a frame in which the massive body is at rest, the predicted precession appears to be purely geodetic in origin, apparently dependent upon the motion of the gyroscope. However, in a frame in which the gyroscope is at rest, the precession can be shown to be purely a consequence of frame-dragging due to the apparent motion of the source. This result can be proven rigorously by deriving the coordinate transformation that is required to give the metric in a frame comoving with the gyroscope (Ashby and Shahid-Saless 1990; Shahid-Saless 1990).

The purpose of this essay is to show how the equivalence of geodetic precession and frame-dragging can be easily demonstrated in metric theories of gravity without detailed coordinate transformation calculations, and to shed new light on the theoretical interpretation of related experiments. More specifically, the former goal will be accomplished by inspection of a "fundamental" equation for gyroscopic precession that is common to metric theories. As a result of the preparation of this essay, it became apparent how the equivalence could be broadened to include frame-dragging from rotating sources. This potentially important issue will be considered in the conclusions. For now, our analysis proceeds as follows.

Relative to a local Lorentz frame oriented with respect to the distant stars (henceforth designated by the acronym OLLF) that is comoving with a gyroscope having a velocity $\vec{v}$ with respect to a massive body, the spin three-vector $\vec{S}$ is determined to post-Newtonian order in metric theories of gravity by the equation (where the notation and conventions of Misner, Thorne, and Wheeler (1973) are generally followed throughout this paper) :

$$\frac{d\vec{S}}{d\tau} = \vec{\Omega} \times \vec{S}, \tag{1.1}$$

where the angular velocity of precession $\vec{\Omega}$ is given by

$$\vec{\Omega} = -\frac{1}{2}\vec{v} \times \vec{a} - \frac{1}{2}\vec{\nabla} \times \vec{g} + \left(\gamma + \frac{1}{2}\right)\vec{v} \times \nabla U. \tag{1.2}$$

We use the terminology OLLF, for oriented local Lorentz frame, versus the terminology "quasi-inertial" frame (adopted by Ashby and Shahid-Saless (1990)), to emphasize that this frame is meant to be kept aligned on the distant stars. The first contribution to equation (1.2) is the well-known Thomas precession for a gyroscope that has an acceleration $\vec{a}$ due to non-gravitational forces. Frame-dragging arises from the second term in equation (1.2), which is seen to depend upon the off-diagonal terms in the metric, where $\vec{g} = g_{0j}\vec{e}_j$. Geodetic precession results from the third term, where U is the Newtonian potential of the body, defined positively. The parameter γ (equal to one in general relativity) measures the contribution from the purely spatial components of the metric. A point of clarification is in order here regarding terminology used in the literature. As an example, it has been remarked that geodetic precession "is essentially just the Thomas precession caused by gravitation" (Weinberg 1972) (similarly, see Wilkins (1970), p. 282). This analogy must be invoked to arrive at a correct result in derivations of geodetic precession that treat gravity as a spin-2 fold on a flat background (Schwinger 1974a, 1974b), or that specialize to the case of a uniform field (Parker 1969). Technically, however, as a result of geodesic motion and parallel transport, there is no Thomas precession for a gyroscope that is freely-falling (i.e., experiencing no non-gravitational accelerations).

In the rest-frame of a non-rotating massive body, the off-diagonal components of the metric vanish, in which case a freely falling gyroscope is seen to undergo geodetic precession only. However, equation (1.2) can be used equally as well by an observer who accelerates in such a way that his velocity matches that of the gyroscope at a particular instant. With respect to this accelerated reference frame, the freely-falling gyroscope is temporarily at rest, while the massive body now has a velocity $-\vec{v}$. According to equation (1.2), the gyroscope can undergo only a frame-dragging precession relative to this observer. After correcting this result for Thomas precession, it can be shown that the precession relative to the OLLF comoving with the gyroscope is equivalent in metric theories of gravity to geodetic precession. Conversely, geodetic precession can be interpreted as an effect that is partly due to "spin-orbit" coupling (Schwinger 1974a, 1974b), in which there appears an induced gravitomagnetic field in the rest-frame of the gyroscope (for further discussion, see Thorne (1988)).

In the next section, the full details of the above argument are presented. In particular, it is shown that within the parametrized post-Newtonian (PPN) formalism (Will 1993), the potentials in the off-diagonal components of the metric yield a frame-dragged precession that is equivalent to geodetic precession when the analysis is performed in the accelerated reference frame and then transformed to the comoving OLLF. Concluding remarks appear in Section III.

2. Geodetic precession as a frame-dragging effect

With respect to an observer who at a particular instant is accelerating with $\vec{a} = -\vec{\nabla} U$, but whose velocity temporarily matches that of a freely-falling gyroscope, the gyroscope precesses due to frame-dragging only, with precessional angular velocity

$$\vec{\Omega}' = -(1/2)\ \vec{\nabla} \times \vec{g}, \tag{2.1}$$

where $\vec{g}$ is to be evaluated in the accelerated frame. This frame is assumed to be nonrotating. To post-Newtonian order,

$$g_{0j} = \Delta V_j + \Delta' W_j, \tag{2.2}$$

where the potentials V_j and W_j are defined as

$$V_j(\vec{x}) = \int \frac{\rho(\vec{x}')\vec{v}'_j}{|\vec{x}-\vec{x}'|^3} d^3x', \tag{2.3a}$$

$$W_j(\vec{x}) = \int \frac{\rho(\vec{x}')\vec{v}'.(\vec{x}-\vec{x}')(x-x')_j}{|\vec{x}-\vec{x}'|^3} d^3x'. \tag{2.3b}$$

Without affecting the results, possible contributions to equation (2.2) from motion with respect to a preferred reference frame have been neglected for the sake of clarity. The parameters Δ and Δ' are given in the PPN formalism by the expressions.

$$\Delta = -(1/2)\ (4\gamma + 3 + \alpha_1 - \alpha_2 + \zeta_1 + 2\xi), \tag{2.4a}$$

$$\Delta' = -\ (1/2)\ (1 + \alpha_2 - \zeta_1 + 2\xi), \tag{2.4b}$$

where the preferred-frame parameters α_1 and α_2 are to be set to zero. For the case in which the body as a whole has only an apparent translational velocity $-\vec{v}$ (i.e., no rotation or internal motions), expansion of V_j and W_j in powers of 1/r yields to lowest order

$$V_j = -\ U v_j, \tag{2.5a}$$

$$W_j = -U \hat{n}_j (\vec{v}\ .\ \hat{n}), \tag{2.5b}$$

where $U = M/r$ and the unit vector $\hat{n}$ points from the gyroscope towards the body. Equations (2.1), (2.2) and (2.5) give

$$\vec{\Omega}' = -\ (1/2)\ (\Delta + \Delta')\ \vec{v} \times \vec{\nabla} U. \tag{2.6}$$

The precessional velocity of the gyroscope relative to the comoving OLLF can be found by bringing up a nonrotating local freely-falling frame (LFFF) which is instantaneously at rest relative to the body. Relative to this LFFF, the basis vectors of the accelerated frame are Thomas precessing with angular velocity

$$\vec{\omega} = -(1/2)\ \vec{v} \times \vec{a} = (1/2)\, \vec{v} \times \vec{\nabla} U. \quad (2.7)$$

Therefore, the precessional velocity of the gyroscope relative to the LFFF is given by

$$\vec{\Omega} = \vec{\Omega}' - \vec{\omega}. \quad (2.8)$$

A simple boost from the LFFF to the comoving OLLF does not alter this result to post Newtonian order. Equations (2.7) and (2.8) are thus seen to yield for a freely-falling gyroscope the result.

$$\vec{\Omega} = -(1/2)\,(\Delta + \Delta' + 1)\ \vec{v} \times \vec{\nabla} \vec{U}. \quad (2.9)$$

For Δ and Δ' given by equation (2.4), this result is seen to be equivalent to the purely geodetic precession predicted by equation (1.2).

3. Conclusions

The above argument demonstrates clearly the intimate relationship that exists in metric theories of gravity between geodetic precession and frame-dragging. It has been shown here how purely geodetic precession can readily be interpreted as a consequence of frame-dragging. Thus, the verification of the de Sitter precession of the Moon at the 2% level is also seen to test the terms responsible for frame-dragging, implying that $|\Delta + \Delta'| = 4 \pm 0.02$ according to equation (2.9). This theoretical interpretation is appropriate for generally covariant, metric theories of gravity. Although not considered here, it is expected that a violation of Lorentz invariance would disturb the equivalence of these precession effects. This is suggested by the manner in which different reference frames having a relative velocity enter into the above analysis. Further work on this issue could reveal interesting consequences, and show how potential improvements could be obtained with precession experiments for testing Lorentz invariance. It has been emphasized elsewhere how terms connected with frame-dragging that arise in metric theories are vitally necessary for cancellation of counterterms in certain calculations in order for correct observational predictions to result (Nordtvedt 1988a, 1988b).

It remains for future experiments, such as GP-B, to test directly frame-dragging produced by the gravitomagnetic field of a rotating source. however, the above considerations suggest how geodetic precession tests could apply to this case, as well. Consider a gyroscope to be orbiting just above the surface of a spherical, non-rotating massive body. This allows the radius of the orbit to be set equal to the radius of the body. We have seen how geodetic precession only would occur. However, a frame rotating at the orbital period of the gyroscope can be used instead. In this frame, the body would now be rotating in the opposite sense, whereas the gyroscope would appear to be perfectly at rest. A gravitomagnetic field would exist in this frame, giving rise to frame-dragging. It must still be possible to calcualte the same precession using the metric in this rotating frame. This approach would permit the equivalence of geodetic precession and frame-dragging from a rotating source to be established. To the best of our knowledge, an equivalence to frame-dragging from a rotating source has not been noted before (e.g., only apparent translation was treated by Ashby and Shahid-Saless (1990). Useful steps in this direction have been taken, however, by virtue of derivations of the metric in a rotating frame (e.g., see Nelson (1987)), and extensions that included massive sources to post-Newtonian order (Nelson 1985, 1990). This additional equivalence became apparent to the author only during the preparation of this essay. A detailed analysis will be presented elsewhere. Once this aspect is firmly established, we will be able to conclude definitively that frame-dragging has been verified in metric theories of gravity up to the present accuracy of 2% of geodetic precession tests.

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References

Ashby N. B. Shahid-Saless, 1990, Phys. Rev. D, 42, 1118.
Bertotti B. I. Ciufolini P.L., Bender, 1987, Phys. Rev. Lett., 58, 1062.
Ciufolini I., Wheeler J.A., 1995, Gravitation and Intertia, Princeton Univ. Press, 1995.
Damour T., Nordtvedt K., 1993a, Phys. Rev. Lett., 70, 2217.
Damour T., Nortdvedt K., 1993b, Phys. Rev. D 48, 3436.
de Sitter W., 1916, MNRAS, 77, 155.
Everitt C.W.F., 1988, in Near Zero, eds J.D. Fairbank et al., W.H. Freeman, Sect. VI.3.
Lange B., 1995, Phys. Rev. Lett., 74, 1904.
Misner C.W., Thorne K.S., Wheeler J.A., 1973, Gravitation, W.H. Freeman, Sect. 40.7.
Nelson R.A., 1985, Gen. Relativ. and Gravit., 17, 637.
Nelson R.A., 1987, J. Math. Phys., 28, 2379; erratum, 1994, ibid., 35, 6224.
Nelson R.A., 1990, Gen. Relativ. and Gravit., 22, 431.
Nordtvedt K., 1988a, Phys. Rev. Lett., 61, 2647.
Nordtvedt K., 1988b, Intl. J. Theor. Phys., 27, 1395.
Parker L., 1969, Am. J. Phys., 37, 309.
Schiff L.I., 1960a, Phys. Rev. Lett., 4, 215.
Schiff L.I., 1960b, Proc. Natl. Acad. Sci., 46, 871.
Schwinger J., Am. J. Phys., 1974a, 42, 507.
Schwinger J., Am. J. Phys., 1974b, 42, 510.
Shahid-Saless B., 1990, Gen. Relativ. Gravit., 22, 1147.
Shapiro I.I. et al., 1988, Phys. Rev. Lett., 61, 2643.
Thorne K., 1988, in Near Zero, eds J.D. Fairbank et. al., W.H. Freeman, Sect, VI.2.
Vessot R.F.C. et. al., 1980, Phys. Rev. Lett., 45, 2081.
Weinberg S., 1972, Gravitation and Cosmology, Wiley p. 237.
Wilkins D.C., 1970, Ann. Phys., 61, 277.
Will C.M. 1989, in Fourteenth Texas Symposium on Relativistic Astrophysics, Ann. New York Acad. Sci., 571, 288.
Will C.M., Theory and Experiment in Gravitational Physics, 1993 (Rev. Ed.), Cambridge Univ. Press.
Wolszczan A., 1991, Nature, 350, 688.