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How Impossible is topology change ?*

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Abstract. It is often stated that topology change is impossible in classical general relativity. In particular, it appears to be widely believed that the pleasure of topology change comes at a fixed price : topology-changing space times must be singular. This perception is wrong. I discuss here both the kinematics and the dynamics of topology change, in order to clarify what precisely the obstacles are, and (with luck) to dispel a few of the more widespread misconceptions about this process. Some of the work presented here extends the work of Geroch and Tipler to a wider class of spacetimes, and some of it offers novelties - such as an explicit example of non-singular 2-dimensional topology change–that have been claimed in the literature to be impossible.

1. Introduction

Can the topology of space change ? A number of people have posed this question [1-26], usually in the context of quantum gravity. In a quantum theory of gravitation, Wheeler [1,2], Hawking [3], and others have argued that we will necessarily have to consider fluctuations not only in geometry but also in topology. Topology change is also interesting for another reason. Wheeler [2], Misner and Wheeler [4], Sorkin [12] and others have suggested that we might be able to regard the particles of ordinary matter as kinks or knots in space. Support for this view comes from results that show that nontrivial topological configurations of space can possess particle-like properties such as mass and charge [4] and half-integral spin [10]. Such theories must accommodate the creation and annihilation of particles by allowing the spatial topology to change. Finally, purely as a question about the classical Einstein theory, we may ask : since geometry evolves in general relativity, might it not be possible that topology does as well?

Topology change is interesting for these reasons. But there are several misconceptions about this process. I discuss here both the kinematics and the dynamics of topology change,

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in an attempt to dispel some of the more widespread of these misconceptions. I also offer some novelties - such as an explicit example of non-singular 2-dimensional topology change, previously thought impossible [20]. A more detailed version of my result is available in a longer paper [26], referred to in this essay as "AB".

2. Topology change : The kinematics

Given two spacelike hypersurfaces S_1 and S_2, possibly of different topologies, under what conditions does there exist an interpolating space-time $\mathcal{M}$ between them? To answer this, we need the *connected sum* technique for constructing new manifolds out of old (fig.1). Using this construction it is possible to show [AB 26] that if no restrictions are placed on the interpolating space-time, then *any* topology-changing process is allowed (fig.2). The construction shown in fig.2 is very arbitrary, and we need to restrict $\mathcal{M}$ in some reasonable manner. The sort of restriction that we want is one that does not allow points of $\mathcal{M}$ to have causal access to holes or to "regions at infinity". This is achieved if we ask that the interpolating space-time be *causally compact* [AB 26} :i.e., for any $p \in \mathcal{M}$, the closures of the future and past of p are compact (fig. 3).

Many open Universes are *causally compact* (AB [26]). In a closed Universe, S_1 and S_2 are both closed hypersurfaces (i.e., compact without boundary) and it is reasonable to require that $\mathcal{M}$ be compact as well. In this case, $\mathcal{M}$ will be causally compact with respect to any Lorentz metric that it admits. To study such situations we can draw on results from cobordism theory [27]. If a compact manifold $\mathcal{M}$ interpolates between closed hypersurfaces S_1 and S_2, then $\mathcal{M}$ is called a *cobordism,* and S_1 and S_2 are called *cobordant.* It is known that when n = 2,3,4,7 or 8, any two closed (and oriented) hypersurfaces are cobordant. In fact, when n = 4 (once, naively, considered the case of greatest physical interest), any two closed hypersurfaces, oriented or not, are cobordant.

Given the existence of an n-dimensional cobordism, $\mathcal{M}$, the next question is : can an appropriate Lorentz metric be put on it ? The conditions for this have been obtained by Reinhart [28] and by Sorkin [13], and they involve restrictions on the Euler characteristic, χ :

$$n \text{ even} \; : \; \chi(\mathcal{M}) = 0.$$
$$n \text{ odd} \; : \; \chi(S_1) = (S_2).$$

The condition when n is odd forbids a number of topological transactions [13], but it does not rule out topology change in general : e.g., S^6 and ($S^4 \times S^2$) $\# T^6$ are cobordant and both have $\chi = 2$ (where S^n is an n dimensional sphere, and T^n an n -dimensional torus). The first of these is simply connected and the second is not, and so the transition between the two represents topology change.

In the even-dimensional case it is the interpolating space-time that is restricted, not the initial and final surfaces. This suggests a further question : if a given cobordism does not have $\chi = 0$, can it be modified in some way so χ now vanishes ? When $n > 2$, this can always be done, whatever the topologies of S_1 and S_2 (AB [26]). In two dimensions the only possibilities with boundaries [13] are the cylinder, with two S^1 boundaries, and the Möbius strip, with one S^1 boundary (fig 4). The space-time of fig. 4b (overlooked in previous work

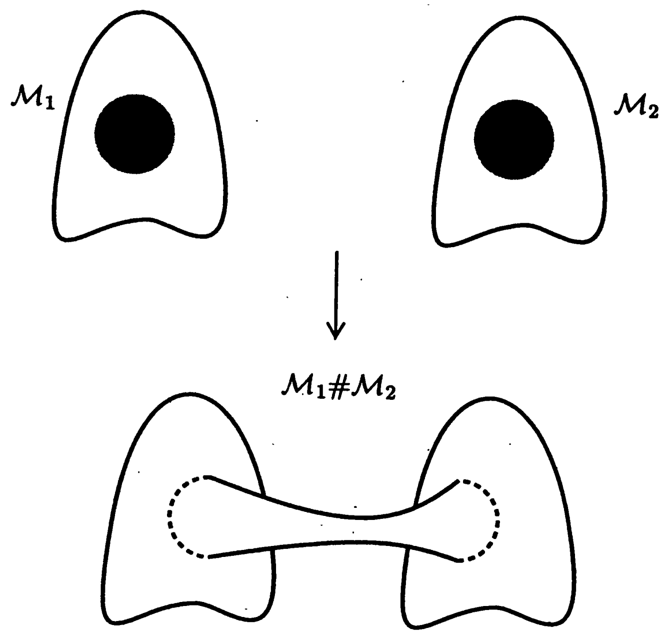

Fig. 1. The connected sum, denoted by #, of two n-dimensional manifolds $\mathcal{M}_1$ and $\mathcal{M}_2$ .An open n-disk - represented by a shaded region above - is removed from each of the manifolds and the edges of the two disks are identified.

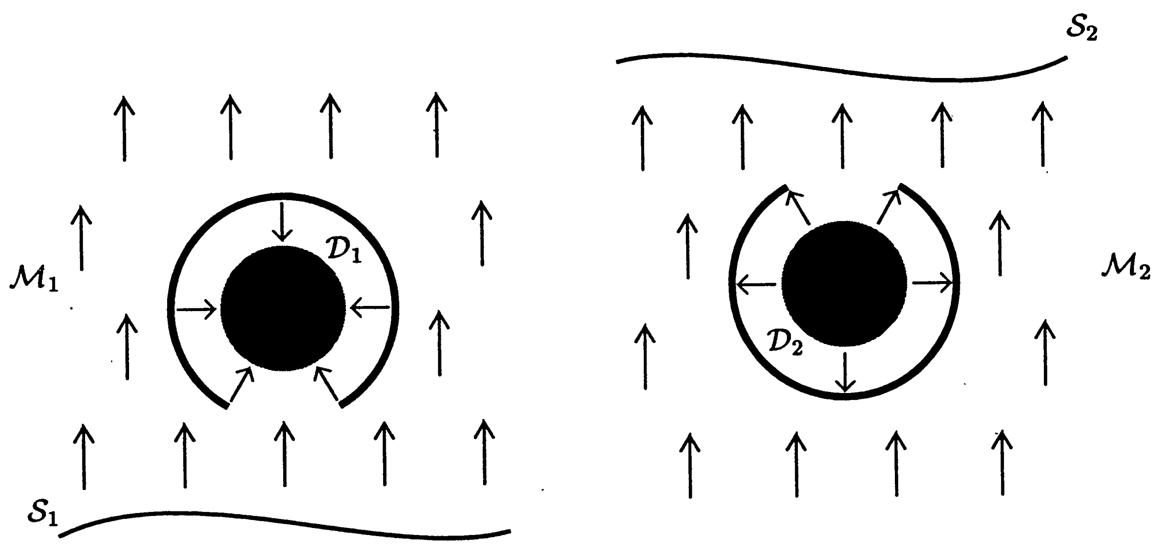

Fig. 2. How an interpolating spacetime may be set up between a pair of hypersurfaces S_1, and S_2, of arbitrary topology. First set up two separate spacetimes, $\mathcal{M}_1 = S_1 \times [0, \infty)$ and $\mathcal{M}_2 = S_2 \times [-\infty, 0]$, as shown, with the arrows indicating the direction of time. Delete the discs $\mathcal{D}_1$ and $\mathcal{D}_2$ (shown shaded), and the almost-complete shells surrounding them (shown as thick arcs). Identify the edges of the two discs, using the standard connected-sum construction, to get a spacetime $\mathcal{M} = \mathcal{M}_1 \# \mathcal{M}_2$ that interpolates between S_1 and S_2.

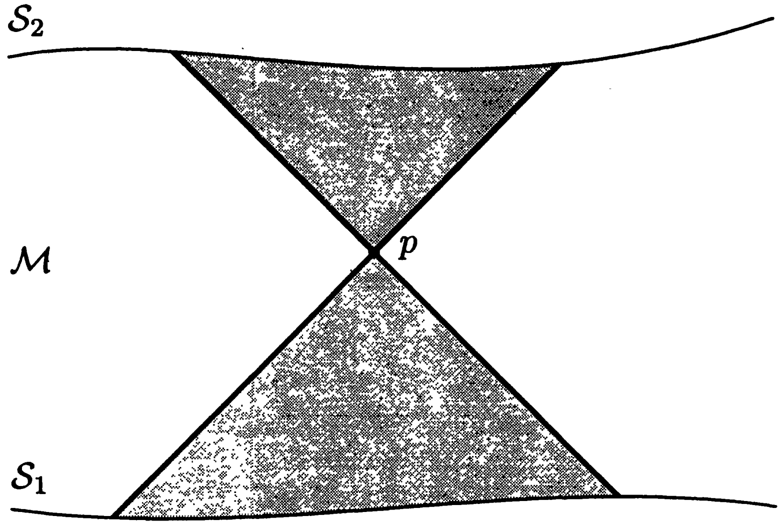

Fig. 3. A causally compact space-time $\mathcal{M}$. Points p between S_1 and S_2 cannot receive signals from, or send signals to, either regions at infinity or "holes" in the spacetime.

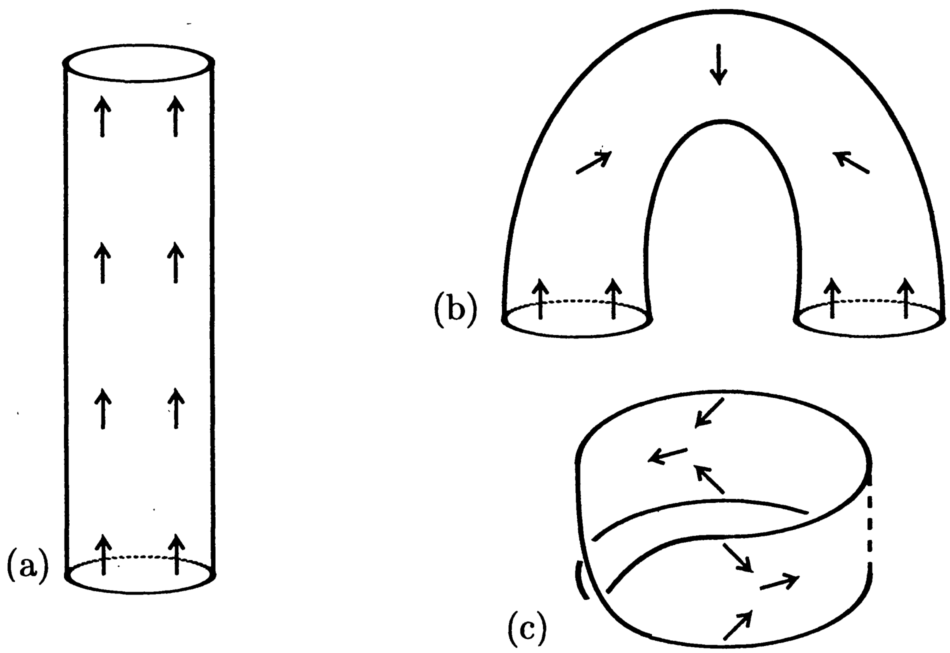

Fig. 4. Examples of 2-dimensional Lorentzian cobordisms. The manifolds in (a) and (b) are cylinders, and the one in (c) a Möbius strip. (The spacetime of fig. (c) is due to Sorkin [13].) The arrows V^a represent the future direction of time. The pictures in (b) and (c) are examples of topology change. They represent, respectively, the transitions $S^1 \cup S^1 \rightarrow 0$ which may be thought of as universe pair-annihilation, and $S^1 \rightarrow 0$ which may be thought of as universe self-annihilation. Reversing V^a gives the reverse transitions: universe pair-creation and universe self-generation.

[13]) and fig. 4c are explicit examples of Lorentzian topology change.

To summarize these results, if n is odd or if n =2, there are examples of topology change. If n is even and is greater than 2, all topology changes are possible. (This result was found previously with slightly different methods [28].)

The topology-changing space - times of fig. 4b and fig. 4c both have closed timelike curves. Geroch has shown that this pathology exists in any topology-changing closed Universe, as well as in some open ones [5,6]. Geroch's result can be extended to the wider class of causally compact spacetimes (AB [26]). No assumptions, beyond causal compactness, are made in the proof. The price of topology change at the kinematical level is, therefore, not singularities, but time travel. This means that *topology change is classically an inherently unpredictable phenomenon : data on the initial surface can never suffice to fully determine what the final surface will be.*

3. Topology change : The dynamics

The future boundary of a region of predictability is called a Cauchy horizon. The mere existence of a Cauchy horizon is problematic, but the situation here is even worse : under mild conditions on the matter energy-momentum tensor, *but no further conditions (such as geodesic completeness),* Einstein's equation implies that the Cauchy horizon cannot exist. This was shown for closed Universes and for some open ones by Tipler [7,8]. The result may also be proved under slightly less restrictive conditions than the ones Tipler used, and for the wider class of causally compact spacetimes (AB [26]). In other words, in a very wide class of spacetimes topology change is simply forbidden as a dynamical process.

4. Topology change : The misconceptions

- *Topology change is intrinsically incompatible with a Lorentzian metric.* The examples of section II show that this perception is not true.

- *Two-dimensional topology changes is necessarily singular.* This perception [20] appears to be based on studies [14, 16] of the $S^1 \cup S^1 \rightarrow S^1$ transition (the so called "trousers topology").

In this case there is a singularity, but the examples of section II show that there are also non-singular topology-changing spacetimes in two dimensions (albeit with closed timelike curves).

- *Closed-universe topology change leads to closed timelike curves only if some suitable energy condition holds.* Neither Geroch's original theorem, nor the extension of it to causally compact spacetimes [AB 26], assume anything about the energy-momentum tensor, or indeed about a field equation - the results are purely kinematical.

- *Topology change may be realized without closed timelike curves in closed universe if the metric is allowed to be singular.* The truth of this depends on the definition of a singularity. If the standard incomplete-geodesic definition is used, then this statement is not true. As long as the causal compactness condition is met, causality violations have to occur when the topology changes, even if incomplete geodesics are admitted.

- *Topology change may be dynamically realized in closed universes if the metric is allowed to be singular.* The comments under the previous misconception apply here as well.

5. Conclusions

I began by asking if the topology of space could change : Kinematically, the answer is "Yes, but..." (i.e., examples exist, but they involve a breakdown of predictability). Dynamically, the answer is flatly "No". Despite this, the kinematical existence of topology-changing spacetimes may still be significant when decisions have to be made about what paths are to be included in the path-integral approach to quantum gravity.

It is important not to view either the kinematical or the dynamical results too dogmatically. Different conclusions may easily be drawn, if different assumptions are made. For example, a weakening of the assumptions so as to allow degenerate metrics is known to lead to topology change [13, 19, 21, 29].

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