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Entropy of the gravitational field*

Tony Rothman[1] and Peter Anninos[2]

[1] *Physics Department, University of Texas, Austin, TX 78712*
[2] *National Center for Supercomputing Applications, University of Illinois, Urbana, IL 61801*

Abstract. We develop a formulation of the entropy of the gravitational field by adopting the statistical mechanics expression for entropy $S = ln\Omega$, where Ω is the phase space of the field bounded by a Hamiltonian. Phase space is calculated for gravitational waves and radiation and density perturbations in expanding FLRW spacetimes, attributing entropy to a lack of knowledge in the exact field configuration. In all cases, S behaves monotonically as required for a definition of gravitational entropy and is a good measure of inhomogeneity. It also reduces to black-hole entropy under appropriate circumstances.

Thermodynamics and general relativity are two physical theories that, in most respects, seem to have little in common. Thermodynamics is based on the particle concept, relativity on fields; relativity is a time-reversible dynamical theory, while thermodynamics manifests a time-symmetry embodied in the second law. There have been several successful attempts to unite aspects of relativity and thermodynamics, most notably the Bekenstein-Hawking formulation of black-hole entropy and Hawking's discovery of black-hole radiation (Bekenstein 1973; Hawking 1976). On the other hand, many aspects of relativity have not been wedded to a thermodynamic framework. One outstanding problem concerns what is often referred to as the gravitational arrow of time, or the entropy of the gravitational field.

Few attempts have been made to characterize gravitational entropy. The best-known is that of Penrose (1989), who hypothesized that C^2, the square of the Weyl tensor, should increase monotonically as the universe becomes more inhomogeneous. Unfortunately, several studies (Wainwright & Anderson 1984; Bonnor 1987) have cast considerable doubt on Penrose's hypothesis. In any case, C^2 is a local quantity and to characterize the overall inhomogeneity of spacetime, one requires a global quantity.

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We have attempted to tackle this problem in a direct fashion. We take over, if not the thermodynamic definition, then the statistical mechanics definition of entropy and apply it to the gravitational field, i.e.,

$$S = ln\Omega, \tag{1}$$

where S is gravitational entropy and Ω is the volume of phase space for the system bounded by a Hamiltonian, H. (Throughout this essay we use units in which h = c = k = G = 1).

This approach has several immediate advantages. In statistical mechanics one generally calculates the entropy via the partition function, but this requires a knowledge of the temperature, which is ill-defined for dynamical systems. To evaluate the phase space, however, does not require a temperature. Second, it allows us potentially to make contact with the Hamiltonian formalism of relativity, the ADM formalism (Arnowitt, Deser & Misner 1962).

To illustrate the basic procedure, consider a system of N harmonic oscillators with Hamiltonian given by

$$H = \frac{1}{2}\sum_{i=1}^{N} \pi_i^2 + \frac{k}{2}\sum_{i=1}^{N} \phi_i^2 , \tag{2}$$

where the displacement ϕ_i, are the canonical coordinates and $\pi_i \equiv \dot{\phi}_i$ the canonical momenta. The phase space for this system can be evaluated using N-dimensional spherical coordinates with the result

$$\Omega = \frac{(2\pi)^N H^N}{\omega^N \Gamma(N+1)} \tag{3}$$

where $\omega = \sqrt{k}$ is the classical oscillator frequency. The entropy is then found from (1).

Note that this expression for Ω ignores the oscillator phases. This is in fact necessary for the interpretation of $ln\Omega$ as entropy. That is, entropy implies information loss and is not defined if one knows the trajectories of the oscillators in phase space. Therefore, to apply the definition (1) to dynamical systems we must ignore phases or, equivalently, assume they are random. We thus assume each region of phase space will be occupied with equal probability, and the system of oscillators can be regarded as a microcanonical ensemble.

There are several reasons to regard (3) as meaningful. First, one can easily show that In Ω represents the classical limit of Einstein's formula for the entropy of N harmonic oscillators. Second, suppose we wish to construct a black hole out of photons, i.e., quantum oscillators. To do this, the total energy of the oscillator system should equal M, the black hole's mass. This condition implies that $H = M = N\epsilon$, where ϵ is the average energy of an oscillator. We also have $\omega = 2\pi\epsilon = 2\pi/\lambda$, where λ is the photon's wavelength. The minimum energy per oscillator needed to construct the hole corresponds to the longest allowed wavelength, which is of order the Schwarzschild diameter $\lambda \sim 4M$. Let us, however, parameterize the wavelength as $\lambda = fM$. Then $\epsilon = 1/(fM)$ and $N = fM^2$. Now, the number of possible quantum states is actually $\Omega/\hbar^N$. With the previous expression for N and Stirling's formula, Eq (3) yields in our units $S = ln[(2\pi)^N \Omega] = N ln[2\pi e] = fM^2 ln[2\pi e]$. Choosing $f = 4$ results in $S = 4M^2 ln[2\pi e]$, a factor of about 2π lower than the Bekenstein-Hawking value of $8\pi^2 M^2$.

The close agreement between the two results is striking; indeed exact agreement can be obtained by letting $\lambda = 8\pi^2 M = (2T_H)^{-1}$, where T_H is the Hawking temperature. At the same time, however, the phase-space approach makes clear that $S \sim N \sim M^2$.

We now generalize the approach to cosmological systems by considering tensor perturbations in a spatially flat but expanding spacetime

$$ds^2 = a^2(\eta)[-d\eta^2 + (\delta_{ij} + h_{ij}(\eta, z))dx^i dx^j], \tag{4}$$

where η is the conformal time, $a(\eta)$ is the expansion scale factor, and $h_{ij} << \delta_{ij}$. Assuming singly polarized waves in the transverse traceless gauge, one can reduce the discrete Hamiltonian for the perturbations to the form

$$H = \frac{1}{2} \sum_{i=1}^{N} \left(\pi_i^2 + \phi_i'^2 - \frac{\ddot{a}}{a} \phi_i^2 \right), \tag{5}$$

where $(\dot{\cdot}) \equiv = d/d\eta$, $(\cdot') \equiv d/dz$, $\phi = ah/\sqrt{32\pi}$, and h ($\equiv h_{xx} = -h_{yy}$) represents the single degree of freedom for the + polarization state.

The gradient term in Eqn. (5) can be approximated as a nearest-neighbour potential in the Hamiltonian: $\phi_i'^2 \approx (\phi_i - \phi_{i+1})^2$, which results in a phase-space identical to (3) but for insignificant numerical factors. The third term in Eqn. (5) is similar to the harmonic oscillator potential but has a time-dependent spring "constant" $\equiv \ddot{a}/a$ and is preceded by a(–) sign. This negative sign results in an inverted potential; i.e. a reflection barrier and a phase space that is unbounded. Nevertheless, if cutoffs are imposed on the spatial displacements and momenta, one can evaluate phase space by means of hypergeometric functions with the result $\Omega \propto H^N (\ddot{a}/a)^{-N/2}$ (Rothman & Anninos 1997).

Phase space is calculated at each spacelike hypersurface given the values of H and $\ddot{a}/a$ on that slice; it is thus parameterized by the conformal time of the different spatial slices. Assuming a spatial dependence for h of the form e^{ikz} (up to an arbitrary random phase), and a matter dominated universe, we have $a \propto \eta^2$ and $h \propto \eta^{-3/2} J_{\pm 3/2}(k\eta) e^{ikz}$, where $J_{\pm 3/2}$ are Bessel functions.

The Hamiltonian (5) in the subhorizon limit $k\eta >> 1$, is then simply the harmonic oscillator Hamiltonian $H \propto \pi^2 + k^2\phi^2$. As expected H and therefore Ω oscillate in time at constant amplitude. To remove the oscillations we may define a 4–Hamiltonian, which is an average over several oscillations. ^4H will be strictly constant. For superhorizon scales, $k\eta <<$ 1, spatial gradients are negligible and we have $H \propto \pi^2 - \ddot{a}\phi^2/a$. Therefore

$$H \propto \begin{cases} \eta^2, \\ \eta^{-4}, \end{cases} \quad , \quad \Omega \propto \begin{cases} \eta^{3N}, & \text{for "growing modes"}, \\ \eta^{-3N}, & \text{for decaying modes.} \end{cases} \tag{6}$$

Because the superhorizon perturbations are strongly coupled to the background curvature, the increasing phase space for the growing modes reflects the increasing rate of expansion. The decaying modes, on the other hand, change at a rate faster than the universe expands, and Ω thus decreases.

Gauge–invariant calculations of phase space for both radiation and dust perturbations in flat, open and closed universes can be carried out in a similar fashion as for the case of gravitational waves. The results for open and flat, dust-filled models with scalar perturbations are (Rothman and Anninos 1997).

$$\Omega_{\text{open}} \propto \begin{cases} \text{Constant}, \\ e^{-2N\eta}, \end{cases} \quad , \quad \Omega_{\text{flat}} \propto \begin{cases} \eta^{5N}, & \text{for growing modes}, \\ \eta^{-5N}, & \text{for decaying modes.} \end{cases} \tag{7}$$

The open model solutions are given in the late time $\eta >> 1$ limit. (At early times, the flat, open and closed models are identical). One can show that the above growing mode solutions reflect the behavior in the density fluctuations $\delta\rho/\rho$ and that they therefore represent the state of gravitational collapse; i.e. oscillators increasing their displacement due to collapse, rather than expansion of the universe. As expected, the late time behavior of phase space in the closed model shows a faster growth rate in the open case.

In all cases we have examined, Ω is found to be a monotonic function of time and therefore behaves as required for gravitational entropy. The increase in entropy, apart from a "fiducial" increase due to the expansion of the universe, is associated with gravitational collapse and, therefore, an increase in inhomogeniety. In addition to providing a good measure of inhomogeniety, our entropy function also appears naturally to reduce to the entropy familiar from other circumstances. Although we have performed the inhomogenous calculations in the perturbative limit, we have also examined the fully nonlinear Bianchi IX cosmology using the ADM formalism. It appears that even in this homogeneous case, if one regards anisotropy as the long-wavelength limit of inhomogeneity, the entropy can be interpreted sensibly.

To conclude, the phase space of the gravitational field appears to open a wide range of investigations and to forge another link between relativity and statistical mechanics.

References

[1] Arnowitt R., Deser S., Misner C.W., 1962, Gravitation: An introduction to Current Research, ed. L. Witten, (John Wiley, New York).

[2] Bekenstein J.D., 1973, Phys. Rev. D 7, 2333.

[3] Bonnor W.B., 1987, Physics Letts. A 122, 305.

[4] Hawking S.W., 1976, Phys. Rev. D 13, 191.

[5] Penrose R., 1989, in Fourteenth Texas Symposium on Relativistic Astrophysics, ed. E.J. Fenyves (New York Academy of Sciences, New York).

[6] Rothman T., Anninos P., 1997, Phys. Rev. D, 55, 1948.

[7] Wainwright J., Anderson P.J., 1984, Gen. Rel. Grav. 16, 609.