Bull. Astr. Soc. India (2001) **29**, 143–151

Critical mass value and secular solutions of the generalised photogravitational Robe's restricted three-body problem

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Received 31 March 2000; accepted 31 May 2001

Abstract. The secular solutions of Robe's restricted three-body problem is dealt in this paper when the first primary is an oblate spheroid-rigid shell, filled with homogeneous, incompressible fluid, with its equatorial plane coincident with the plane of motion and the second primary is a source of radiation. The collinear equilibria have conditional retrograde elliptical orbits around them in the linear sense, while the triangular points have long or short periodic retrograde elliptical orbits for the mass parameter, $0 \leq \mu < \mu_{crit}$, the critical mass parameter, which decreases with the increase in oblateness and radiation force. However, retrograde elliptical periodic orbits exist for the case $\mu = \mu_{crit}$ through special choice of initial conditions, the eccentricity of which increases with oblatenes and decreases with radiation force for oblateness, which is not zero.

Key Words : restricted problem, critical mass, secular solution

1. Introduction

Restricted three-body problem as conceived by Robe (1977) describes the motion of an infinitesimal mass, moving inside the rigid spherical shell, filled with homogeneous, incompressible fluid, which is the first primary and the mass outside the shell at a distance is the second primary. The two primaries move in circular orbits around their centre of mass on account of their mutual attraction and the infinitesimal mass not influencing the motion of the primaries. The Robe model is meant to provide some insight into the problem of small oscillations of the earth's core in the gravitational field of the earth-moon system. But in our model we have assumed further that the rigid shell is an oblate spheroid and the body, located outside, is radiating one and we attempt to study the effects of oblateness and radiation of the primaries on the infinitesimal third point in Robe's problem.

Here we consider the case when the gravitation prevails. The resultant force acting on the infinitesimal mass due to gravitational force of first primary and radiation pressure of the second primary is given by

$F = F_g - F_p = F_g (1 - F_p/F_g) = qF_g$, where $q = (1 - F_p/F_g)$ is the mass reduction factor, which is constant for a given particle.

We obtain the locations of the triangular points, which form isosceles triangle with the distance of the first primary and $L_{4,5}$ being less than one and other distances are unity. We obtain the characteristic equation of the linearised equation of motion, which has four roots. We observe further that the roots are functions of μ, q and A_1 ; μ being the mass parameter.

When the discriminant of the characteristic equation is positive, the roots $\lambda_{1,2} = \pm is_4$, $\lambda_{3,4} = \pm is_5$, are found to be pure imaginary.

The solutions consist of short and long period terms when angular frequencies are s_4 and s_5. The value of mass parameter i. e critical mass, μ_{crit} have been calculated when discriminant becomes zero. The series solutions of the critical mass, μ_{crit} in terms of ϵ, ϵA_1 have been found to be a accurate as we have included terms upto ϵ^2, $\epsilon^2 A_1$, in the series expansion.

Then on substitution of the value of $\mu = \mu_{crit}$ from equation (13) to equation (20) and retaining only linear terms in A_1 and terms ϵ^2 for finding eccentricity we discover that it improves the eccentricity or secular solution. Again we could compare eccentricities or secular solution by taking small values of A_1 and μ in equations (20) and (21).

2. Equations of motion

Following the terminology and notations of Szebehely (1967) and choosing the unit of mass equal to the sum of the masses of the primaries, the unit of length equal to their separation and the unit of time such that the Gaussian constant of gravitation is unity, the perturbed mean motion of the primaries is given by

$$n^2 = 1 + \frac{3}{2} A_1$$

where A_1 is the oblateness coefficient of the bigger primary and

$A_1 = (\overline{AE}_1^2 - \overline{AP}_1^2) / 5R^2$ where AE_1 is the equatorial radius and AP_1 is the polar radius of the bigger primary and (R = 1) is the distance between the primaries.

The equations of motion in the dimensionless barycentric-synodic co-ordinate system (x,y)

are
$$\ddot{x} - 2n\dot{y} = \frac{\partial \Omega}{\partial x}$$
$$\ddot{y} + 2n\dot{x} = \frac{\partial \Omega}{\partial y} \qquad (1)$$

Where
$$\Omega = \frac{n^2}{2}(x^2 + y^2) - kr_1^2 + \frac{\mu q}{r_2} + \frac{1-\mu}{r_1} + \frac{(1-\mu)A_1}{2r_1^3} \qquad (2)$$

with $k = \frac{4}{3}\pi\rho_1\left(1 - \frac{\rho_1}{\rho_3}\right)$; $\rho_1 \neq \rho_3$ i.e $k \neq 0$

The Jacobian integral of equations (1) is $\dot{x}^2 + \dot{y}^2 = 2\Omega - c$; c being the Jacobian constant. The curve of zero velocity are given by $2\,\Omega\,(x,y) = c$ and since $\Omega\,(x,y) = \Omega\,(x \pm y)$, the curves are symmetric with respect to x-axis. The singularities of the manifold of the states of motion are located at those points of the curves of zero velocity where $\frac{\partial\Omega}{\partial x} = 0 = \frac{\partial\Omega}{\partial y}$

$$\text{i.e } x\left[(n^2 - 2k) - \frac{\mu q}{r_2^3} - \frac{(1-\mu)}{r_1^3} - \frac{3}{2}\,\frac{(1-\mu)A_1}{r_1^5}\right]$$

$$-2k\,\mu - \frac{(\mu-1)\mu q}{r_2^3} - \frac{(1-\mu)\mu}{r_1^3} - \frac{3}{2}\,\frac{(1-\mu)A_1}{r_1^5}\,\mu = 0 \tag{3}$$

and

$$y\left[(n^2 - 2k) - \frac{\mu q}{r_2^3} - \frac{1-\mu}{r_1} - \frac{3}{2}\,\frac{(1-\mu)A_1}{r_1^5}\right] = 0 \tag{4}$$

3. Equilibrium points location

For $y \neq 0$, Equations (3) and (4) disclose that

$$r_2^3 = q/n^2\,; \quad r_1 = 1 \tag{5}$$

Equations (5) locate the other two points L_4 and L_5. These points forming isosceles triangle with the primaries are known as triangular points; it may he noted that $r_2 \leq 1$.

For $y = 0$, equation (3) determines the location of the collinear points, L_1 $(x_1, 0)$, L_2 $(x_2, 0)$ and L_3 $(x_3, 0)$.

4. Stability of the triangular points

Putting $x = x_0 + \xi$, $y = y_0 + \eta$ in the equations of motion (1) for studying the motion near any of the triangular points, L (x_0, y_0), we get the variational equation as (Sharma, 1987).

$$\ddot{\xi} - 2n\dot{\eta} = \Omega_{xx}\,(x_0, y_0)\,\xi + \Omega_{xy}\,(x_0, y_0)\eta$$

$$\ddot{\eta} + 2n\dot{\xi} = \Omega_{xy}\,(x_0, y_0)\,\xi + \Omega_{yy}\,(x_0, y_0)\eta \tag{6}$$

The characteristic equation of equations (6) is

$$\lambda^4 + (4n^2 - \Omega^0_{xx} - \Omega^0_{yy})\,\lambda^2 + \Omega^0_{xx}\,\Omega^0_{yy} - (\Omega^0_{xy})^2 = 0 \tag{7}$$

At the triangular points L_4 and L_5, we have :

$$\Omega^0_{xx} = f(x - 1 + \mu)^2 + g(x + \mu)^2 - 2k,$$

$$\Omega^0_{xy} = \pm\, y[f(x - 1 + \mu) + g(x + \mu)]$$

$$\Omega^0_{yy} = y^2[f + g] - 2k > 0,$$

where $f = \dfrac{3\mu n^2}{r_2^2} > 0$ and $g = 3(1 - \mu)\,(1 + \dfrac{5}{2}A_1) > 0$

Hence, with $\lambda^2 = \Lambda$ the characteristic equation (7) becomes :

$$\Lambda^2 + (4n^2 - fr_2^2 - g + 4k)\,\Lambda + y^2\, fg - 2k\,[f\, r_2^2 + g - 2k] = 0 \tag{8}$$

$$\text{or, } \Lambda^2 + [n^2 - 3(1 - \mu)A_1 + 4k]\,\Lambda + 9\mu\,(1 - \mu)n^2\,(1 + \frac{5}{2}A_1)\,(1 - \frac{r_2^2}{4}) - 15A_1k + 3\mu A_1k + 4k^2 = 0 \tag{9}$$

We obtain further from the above equation

$$\Lambda_{1,2} = \frac{1}{2}\,[\{3(1 - \mu)A_1 - n^2 - 4k\} \pm \{(3(1 - \mu)A_1 - n^2 - 4k)^2 - 36\,\mu(1 - \mu)n^2\,(1 + \frac{5}{2}A_1)\,(1 - \frac{r_2^2}{4}) - 60A_1k + 12\mu A_1k + 16k^2\}^{1/2}]$$

It is observed that the roots of the equation (8)

$$\lambda_1 = \Lambda_1^{1/2},\ \lambda_2 = -\,\Lambda_1^{1/2},\ \lambda_3 = \Lambda_2^{1/2},\ \lambda_4 = -\,\Lambda_2^{1/2} \tag{10}$$

are function of μ, q and A_1 and their nature depends upon the nature of the discriminant

$$\Delta = [n^2 + 4k - 3(1 - \mu)A_1]^2 - 36\mu(1 - \mu)n^2\,(1 + \frac{5}{2}A_1)\,(1 - \frac{r_2^2}{4}) - 60A_1\,k + 12\mu A_1\,k + 16k^2.$$

The following three cases arise for discussion :

(i) When Δ is positive, $\Lambda_{1,2}$ are negative and the roots (10) as

$$\lambda_{1,2} = \pm i(-\,\Lambda_1)^{1/2} = \pm\, is_4,\ \lambda_{3,4} \pm i\,(-\,\Lambda_2)^{1/2} = \pm is_5$$

show that the triangular points are linearly stable. The solution of equations (6) in this case can easily be seen to consist of short and long period terms with angular frequencies s_4 and s_5 respectively.

(ii) When Δ is negative, the real parts of two of the four roots (10) are positive and equal; hence, the equilibria are unstable.

(iii) When $\Delta = 0$, both the values of the four roots (10) in pairs are equal. So the solution of the variational equations contain secular terms and consequently the triangular point is unstable. However, with suitable selection of initial conditions, periodic motion can be achieved in the linear sense, which approaches the equilibrium point asymptotically.

5. Critical mass

The discriminant of quadratic equation (9) is zero

when $$[9A_1^2 + 36n^2(1 + \frac{5}{2}A_1)(1 - \frac{r_2^2}{4})]\mu^2 - [18A_1^2 - 6n^2A_1 - 12A_1k + 36n^2(1 + \frac{5}{2}A_1)(1 - \frac{r_2^2}{4})]\mu + n^4 + 9A_1^2 - 6n^2A_1 + 8kn^2 + 36A_1k = 0 \quad (11)$$

When $A_1 = 0$, Equation (11) coincides with that of Chernikov (1970).

Solution of Equation (11) for $0 \le \mu \le 1/2$ is

$$\mu_{crit} \frac{\alpha - \beta^{1/2}}{\Upsilon} \quad (12)$$

where $$\alpha = (1 - \frac{q^{2/3}}{4})(1 + 3A_1) - \frac{A_1k}{3}$$

$$\beta = [(1 - \frac{q^{2/3}}{4})(1 + 3A_1) - \frac{A_1k}{3}]^2 - 4[(1 - \frac{q^{2/3}}{4})(1 + 3A_1) + A_1]$$

$$[\frac{1}{36} - \frac{A_1}{12} + \frac{2}{9}k + \frac{4}{3}A_1k] = 0$$

$$\Upsilon = 2[(1 - \frac{q^{2/3}}{4})(1 + 3A_1) + A_1]$$

Let $q = 1 - \epsilon$, where ϵ is small. Restricting computations with linear terms in A_1 upto quadratic terms in ϵ and retaining terms upto $\epsilon^2 A_1$, we obtain.

$$\alpha = [\frac{3}{4} + \frac{1}{6}\epsilon + \frac{1}{36}\epsilon^2] + [(\frac{9}{4} - \frac{k}{3}) + \frac{1}{2}\epsilon + \frac{1}{12}\epsilon^2]A_1$$

$$\beta = [(\frac{23}{48} - \frac{2}{3}k) + (\frac{25}{108} - \frac{4}{27}k)\epsilon + (\frac{43}{648} - \frac{2}{81})\epsilon^2] + [(\frac{235}{72} - \frac{133}{18}k) + (\frac{3}{2} - \frac{13}{9}k)\epsilon + (\frac{5}{12} - \frac{13}{54}k)\epsilon^2]A_1$$

$$\Upsilon = [\frac{3}{2} + \frac{1}{3}\epsilon + \frac{1}{18}\epsilon^2) + [\frac{13}{2} + \epsilon + \frac{\epsilon^2}{6}]A_1$$

and the expression of μ_{crit} in (12) becomes

$$\mu_{crit} = [\frac{2}{3}\{\frac{3}{4} - (\frac{23}{48} - \frac{2}{3}k)^{1/2}\} + 3\{\frac{1}{6} - (\frac{25}{108} - \frac{4}{27}k)^{1/2}\}\epsilon$$

$$+ 18\{\frac{1}{36} - (\frac{43}{648} - \frac{2}{81}k)^{1/2}\}\epsilon^2] + [\frac{2}{13}\{(\frac{9}{4} - \frac{k}{3}) - (\frac{235}{72} - \frac{133}{18}k)^{1/2}\}$$

$$+ \{\frac{1}{2} - (\frac{3}{2} - \frac{13}{9}k)^{1/2}\}\epsilon + 6\{\frac{1}{12} - (\frac{5}{12} - \frac{13}{54}k)^{1/2}\}\epsilon^2]A_1 \tag{13}$$

Expanding the expression under the small bracket binomially and neglecting the small powers of k and then simply reducing them to decimal we get

i.e

$$\begin{aligned}\mu_{crit} &= [(0.0385211 + 0.3210289k)\\ &- (0.9433756 - 0.4618802\,k)\,\epsilon\\ &- (4.1368092 - 0.8626621\,k)\,\epsilon^2]\\ &+ [(0.0682121 + 0.2633248\,k)\\ &- (0.7247448 - 0.5896918\,k)\,\epsilon\\ &- (3.3729833 - 1.1188614\,k)\,\epsilon^2]\,A_1\end{aligned} \tag{14}$$

The above equation shows that the variability of critical value μ_{crit} of the mass parameter also depends upon the perturbations caused by oblateness effect and radiation of the respective primaries, however small.

6. Secular solutions

When $\Delta = 0$, we note that Λ_1 and Λ_2 are equal and negative and the roots (10) of characteristic equation (8) are pure imaginary and equal in pairs.

$$\lambda_{1,3} = i\kappa^{1/2},\ \lambda_{2,4} = -i\kappa^{1/2} \tag{15}$$

where $\kappa = \frac{1}{2}\{1 + \frac{3}{2}(1 - 2\mu)A_1\} > 0$

Rotating the co-ordinate axis at L_4, 5 at an angle θ through the transormation

$$\xi = \overline{\xi}\,\mathrm{Cos}\,\theta - \overline{\eta}\,\mathrm{Sin}\,\theta\,,\ \eta = \overline{\xi}\,\mathrm{Sin}\,\theta + \overline{\eta}\,\mathrm{Cos}\,\theta$$

and subsitituting in equations (6), we have :

$$\ddot{\bar{\xi}} - 2\,n\,\dot{\bar{\eta}} = \bar{\lambda}^*_2\,\bar{\xi}, \qquad \ddot{\bar{\eta}} + 2n\,\dot{\bar{\xi}} = \bar{\lambda}^*_1\,\bar{\eta}, \tag{16}$$

where $\bar{\lambda}^*_{1,2} = (2n^2 - \kappa) \pm 2n\,(n^2 - \kappa)^{1/2}$

The double roots (15) give secular terms in the solution of equations (16) and the triangular points are unstable.

As in Sharma and Subba Rao (1979), the solution of equations (16) can be obtained with the aid of Laplace transformation as :

$$\bar{\xi} = [\bar{\xi}_0 + \{(\kappa + \lambda^*_1)\,\bar{\xi}_0 / 2\kappa - n\bar{\lambda}^*_1\eta_0 / \kappa\}t]\mathrm{Cos}\,\kappa^{1/2}t$$

$$+ [\{n\bar{\lambda}^*_1\eta_0 / \kappa^{3/2} - (\bar{\lambda}^*_1 - \kappa)\,\bar{\xi}_0 / 2\kappa^{3/2}\}$$

$$+ \{(\kappa + \bar{\lambda}^*_2)\bar{\xi}_0 / 2\kappa^{1/2} + n\,\bar{\eta}_0 / \kappa^{1/2}\}t]\mathrm{Sin}\,\kappa^{1/2}t$$

$$\bar{\eta} = [\bar{\eta}_0 + \{(\kappa + \bar{\lambda}^*_2)\,\dot{\bar{\eta}}_0\,2\kappa + n\bar{\lambda}^*_2\bar{\xi}_0 / \kappa\}t]\mathrm{Cos}\,\kappa^{1/2}\,t$$

$$+ [\{(\kappa - \dot{\bar{\lambda}}^*_2)\dot{\bar{\eta}}_0 / 2\kappa^{3/2} - n\bar{\lambda}^*_2\bar{\xi}_0 / \kappa^{3/2}\}$$

$$+ (\kappa + \bar{\lambda}^*_1)\,\bar{\eta}_0 / 2\kappa^{1/2} - n\dot{\bar{\xi}}_0 / \kappa^{1/2}\}t]\ \mathrm{Sin}\,\kappa^{1/2}\,t. \tag{17}$$

where $\bar{\xi}_0, \bar{\eta}_0, \dot{\bar{\xi}}_0, \dot{\bar{\eta}}_0$ are initial position and velocity components at $t = 0$. The solution (17) is unstable due to the presence of the secular terms in t. If, however, arbitrary initial conditions $\bar{\xi}_0, \bar{\eta}_0$ are combined with special velocity components.

$$\dot{\bar{\xi}}_0 = \kappa^{1/2}\,\bar{\eta}_0 / m, \quad \dot{\bar{\eta}}_0 = -\,\kappa^{1/2}\,m\bar{\xi}_0, \tag{18}$$

where $m = (\kappa + \bar{\lambda}^*_2)^{1/2} / (\kappa + \bar{\lambda}^*_1)^{1/2}$

the secular terms in (17) are eliminated and we obtained

$$\bar{\xi} = \bar{\xi}_0\,\mathrm{Cos}\,\kappa^{1/2}\,t + (\bar{\eta}_0 / m)\mathrm{Sin}\,\kappa^{1/2}\,t \tag{19}$$

$$\bar{\eta} = \bar{\eta}_0\,\mathrm{Cos}\,\kappa^{1/2}\,t - m\bar{\xi}_0\,\mathrm{Sin}\,\kappa^{1/2}\,t$$

Eliminating t from Equations (19), we get

$$m^2 \bar{\xi}^2 + \bar{\eta}^2 = m^2 \bar{\xi}_0^2 + \bar{\eta}_0^2$$

which represents an ellipse with eccentricity $(1-m^2)^{1/2}$. At $\bar{\xi} = \bar{\xi}_0$, $\bar{\eta}_0 = 0$, the velocity components in (18) are

$$\dot{\bar{\xi}}_0 = 0,\ \dot{\bar{\eta}}_0 = -\kappa^{1/2} m\bar{\xi}_0,$$

and since $k^{1/2}$ m > 0, the sign of $\dot{\bar{\eta}}_0$ is negative of ξ_0; which proves that the periodic orbits, described by equation (19), are retrograde. The eccentricity

$$e = (1 - m^2)^{1/2} = \frac{(\bar{\lambda}_1^* - \bar{\lambda}_2^*)^{1/2}}{(\kappa + \bar{\lambda}_1^*)^{1/2}} = [\frac{2(n^2 - k)^{1/2}}{n+(n^2 - k)^{1/2}}]$$

$$\text{or } \frac{2}{2-e^2} = 1 + \frac{(n^2 - \kappa)^{1/2}}{n} = 1 + \frac{1}{\sqrt{2}}(1 + \frac{3}{2}\mu A_1),$$

$$\text{or } e^2 = 2(\sqrt{2} - 1)\ [1 + \frac{3}{\sqrt{2}}(\sqrt{2} - 1)\ \mu A_1],$$

$$\text{or } e = 2^{1/2}(\sqrt{2} - 1)^{1/2}\ [1 + \frac{3}{2\sqrt{2}}(\sqrt{2} - 1)\ \mu A_1] \tag{20}$$

Substituting the value of $\mu = \mu_{crit}$ from equation (13) into (20) and retaining only linear terms in A_1, we get

$$e = 2^{1/2}(\sqrt{2} - 1)^{1/2}\ [1 + \frac{3(\sqrt{2}-1)}{2\sqrt{2}}\{(\frac{1}{2} - \frac{2}{3}\sqrt{\frac{23}{48} - \frac{2}{3}k})$$

$$+ (\frac{1}{2} - 3\sqrt{\frac{25}{108} - \frac{4}{27}k})\ \epsilon + (\frac{1}{2} - 18\sqrt{\frac{43}{648} - \frac{2}{81}k})\epsilon^2\}A_1] \tag{21}$$

The equation (21) shows that the eccentricity e increases with oblateness and decreases with radiation for non-zero oblateness.

7. Conclusion

The above study of the critical value μ_{crit} of the mass parameter and secular solution in the Robe's three body problem reveals that the range of stability of the triangular points increases and decreases according as variability in oblatness and radiation of the primaries, considered in this model.

Acknowledgements

The authors are thankful to Prof. Bhola Ishwar, Dept. of Mathematics, B.R.A. Bihar University, Muzaffarpur, India for his useful suggestion for the preparation of this paper.

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