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Periodicity of the generating solutions of the restricted three-body problem in three-dimensional coordinate system

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Abstract. This paper deals with the periodicity of the generating solutions of the restricted three-body problem in three-dimensional coordinate system in KS-variables in the light of Krashinski (1963), Kurcheeva (1973) and Ahmad (1995).

Key Words : restricted three-body problem - KS-transformations - regularization, generating solutions and period

1. Introduction

Krashinski (1963) established generating solutions in the restricted problem of two bodies. Kurcheeva (1973) studied the existence of periodic solutions of the restricted three-body problem representing analytic continuation of Keplerian rectilinear periodic motions. She established periodic and isoperiodic solutions in different cases reducing to the generating one for $\mu=0$. The period of these solutions is analytic function of μ, the mass ratio of the smaller primary to the total mass of the primaries, where $0 \leq \mu \leq ½$.

Ahmad (1995) reproduced the work of Kurcheeva in a well furnished way in the light of Mandelstam and Papalexi (1932). In his work he has shown that the isoperiodic solutions are stable and all the other solutions are unstable. All the above authors made their efforts in planar case only but no one took up the three-dimensional coordinate system.

In the recent era KS-transformations are playing an important role in the field of celestial mechanics. It is an established fact that in celestial mechanics the representation of many formulae in terms of KS-variables is much simpler and more compact than in Levi-Civita's (1906) variables. In three-dimensional space, to test the periodicity and the isoperiodicity of the generating solutions is generally impossible, if the equations of motion be regularized with the help of Levi-Civita's parabolic transformations. It is only possible when the equations of motion in three-dimensional space be regularised by the KS-transformations which transform the three-dimensional space into four-dimensional phase space of real numbers. In this paper, for the first time, we have produced a new system of periodic generating solutions of the restricted three-body problem in terms of KS-variables in three-dimensional synodic coordinate system and also established the period of the generating solutions. In the first and second sections we have established the regularized equations of motion by KS-transformations. In the third section we have produced the periodic generating solutions in KS-variables. In the fourth section, the periodicity of each of the generating solutions have been examined and it is found that all the generating solutions produced in our work, are periodic with the period $\frac{k\pi}{2}$, k is any positive integer.

2. Equations of motion

The equations of motion in canonical form of the infinitesimal mass (x_1, x_2, x_3) moving in the gravitational field of two massive bodies of unequal masses moving in circles in the synodic system, are

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial X_i}, \frac{dX_i}{dt} = -\frac{\partial H}{\partial x_i} \quad (i = 1,2,3) \tag{1}$$

where Hamiltonian is given by

$$H = \frac{1}{2} \sum_{i=1}^{3} X_i^2 + (X_1 x_2 - X_2 x_1) - \mu (x_1 - \frac{1}{2}) - \frac{1-\mu}{r_1} - \frac{\mu}{r_2} \tag{2}$$

$$r_1^2 = x_1^2 + x_2^2 + x_3^2$$

$$r_2^2 = (x_1 + 1)^2 x_2^2 + x_3^2$$

and $X_1 = \frac{dx_1}{dt} - x_2$, $X_2 = \frac{dx_2}{dt} - x_1$, $X_3 = \frac{dx_3}{dt}$, (X_1, X_2, X_3) are generalised momenta corresponding to (x_1, x_2, x_3). All the conditions of the restricted three-body problem remain the same.

3. Regularization

There are many methods of regularization to make the energy function free from the singularity or singularities. Here at collision, the energy function H has two singularities at $r_1 = 0$ (i=1.2). In order to remove the singularity at r_1=0, let us regularize the equations of motion by KS-transformations given by

$$\mathbf{x} = L(\mathbf{q})\,\mathbf{q} \tag{3}$$

where the orthogonal KS-matrix

$$L(q) = \begin{bmatrix} q_1 & -q_2 & -q_3 & q_4 \\ q_2 & q_1 & -q_4 & -q_3 \\ q_3 & q_4 & q_1 & q_2 \\ q_4 & -q_3 & q_2 & -q_1 \end{bmatrix}$$

$L^T(\mathbf{q})L(\mathbf{q}) = (\mathbf{q},\mathbf{q})I = L(\mathbf{q})L^T(\mathbf{q})$, I is the unit matrix, $\mathbf{x} = (x_1,x_2,x_3)$ and $\mathbf{q} = (q_1,q_2,q_3,q_4)$.

The explicit representation of (3) are

$$x_1 = q_1^2 - q_2^2 - q_3^2 + q_4^2 \qquad \text{(Steifel \& Schiefle, 1971 p-24)}$$
$$x_2 = 2(q_1q_2 - q_3q_4) \tag{4}$$
$$x_3 = 2(q_1q_3 + q_2q_4)$$

The associated generalized momenta

$$Q_j = \sum_{i-1}^{3} X_i \frac{\partial x_i}{\partial q_j} \qquad (j = 1,2,3,4) \tag{5}$$

(Kurcheeva-1977) and (Steifel & Schiefle, 1971 p-28)

where X_i's are given by

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \frac{2}{|\mathbf{q}|^2} L(\mathbf{q}) \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix} \qquad \mathbf{Q} = (Q_1,Q_2,Q_3,Q_4) \tag{6}$$

(Steifel & Schiefle, 1971 p-239)

Explicitly (6) reads

$$X_1 = \frac{2}{2|\mathbf{q}|^2}(q_1Q_1 - q_2Q_2 - q_3Q_3 + q_4Q_4)$$

$$X_2 = \frac{1}{2|\mathbf{q}|^2}(q_2Q_1 + q_1Q_2 - q_4Q_3 - q_3Q_4) \tag{7}$$

and
$$X_3 = \frac{1}{2|\mathbf{q}|^2}(q_3Q_1 + q_4Q_2 + q_1Q_3 + q_2Q_4)$$

In all the above relations a key role was played by the bilinear relation

$$T(\mathbf{q}.\mathbf{Q}) = (q_4Q_1 - q_3Q_2 + q_2Q_3 - q_1Q_4) = 0 \tag{8}$$

Also
$$r_1 = (\mathbf{q}.\mathbf{q}) = |\mathbf{q}|^2 = q_1^2 + q_2^2 + q_3^2 + q_4^2 = p^2 \tag{9}$$

and
$$r_2^2 = 1 + 2x_1 + |\mathbf{q}|^4(\mathbf{Q}.\mathbf{Q}) = |\mathbf{Q}|^2 \quad = Q_1^2 + Q_2^2 + Q_3^2 + Q_4^2 \tag{10}$$

Let us now connect the physical time t and the pseudo time s by the formula

$$dt = 4\rho^2 ds \qquad (t = 0 \text{ at } s = 0) \tag{11}$$

Thus the regularized canonical equations of motion of the infinitesimal mass given by Kurcheeva (1977) are

$$\frac{dq_i}{ds} = \frac{\partial K}{\partial Q_i}, \quad \frac{dQ_i}{ds} = -\frac{\partial K}{\partial q_i} \quad (i = 1,2,3,4) \tag{12}$$

where the new Hamiltonian K (regularized at $r_1 = 0$ only) is given by

$$\begin{aligned} K = 4\rho^2(H+\frac{1}{2}c) = \frac{1}{2}(Q_1^2+Q_2^2+Q_3^2+Q_4^2) + 2\rho^2(Q_1q_2-Q_2q_1+Q_3q_4-Q_4q_3+c_0) - 4 \\ +4\mu[1-\rho^2 (q_1^2-q_2^2-q_3^2+q_4^2 +\frac{1}{r_2}-\frac{1}{2}c_1 -\frac{1}{2})] \end{aligned} \tag{13}$$

4. Generating solutions

For generating solutions, considering $\mu = 0$, the reduced Hamiltonian is given by

$$K_0 = \frac{1}{2}(Q_1^2+Q_2^2+Q_3^2+Q_4^2) + 2\rho^2(Q_1q_2-Q_2q_1+Q_3q_4-Q_4q_3+c_0)-4 \tag{14}$$

Also the Hamilton equations of motion for the reduced Hamiltonian K_0, are

$$\frac{dq_i}{ds} = \frac{\partial K_0}{\partial Q_i}, \quad \frac{dQ_i}{ds} = -\frac{\partial K_0}{\partial q_i} \quad (i = 1,2,3,4) \tag{15}$$

Thus the equations of motion of the infinitesimal mass in three-dimensional synodic coordinate system can be derived from (9), (14) and (15) as

$$\ddot{q}_1 - 4(\rho^2 + q_1^2 + q_2^2)\dot{q}_2 + 4(q_1q_4 - q_2q_3)\dot{q}_3 - 4(q_1q_3 + q_2q_4)\dot{q}_4 = 4q_1(3\rho^4 - c_0), \tag{16}$$

$$\ddot{q}_2 + 4(\rho^2 + q_1^2 + q_2^2)\dot{q}_1 + 4(q_1q_3 + q_2q_4)\dot{q}_3 + 4(q_1q_4 - q_2q_3)\dot{q}_4 = 4q_2(3\rho^4 - c_0), \tag{17}$$

$$\ddot{q}_3 - 4(q_1q_4 - q_2q_3)\dot{q}_1 - 4(q_1q_3 + q_2q_4)\dot{q}_2 - 4(\rho^2 + q_3^2 + q_4^2)\dot{q}_4 = 4q_3(3\rho^4 - c_0), \tag{18}$$

$$\ddot{q}_4 + 4(q_1q_3 + q_2q_4)\dot{q}_1 - 4(q_1q_4 - q_2q_3)\dot{q}_2 + 4(\rho^2 + q_3^2 + q_4^2)\dot{q}_3 = 4q_4(3\rho^4 - c_0), \tag{19}$$

where (.) denotes the differentiation with respect to the pseudo time s.

Multiplying (16) – (19) respectively by $\dot{q}_1,\dot{q}_2,\dot{q}_3,\dot{q}_4$ and integrating their sum, we get

$$\sum_{i-2}^{4} \dot{q}_1^2 = 4\rho^6 - 4c_0\rho^2 + 8. \tag{20}$$

Here 8 is the constant of integration, taken same as in the two body problem by Krashinski (1963) and in the restricted three-body problem by Ahmad (1955).

Again multiplying (16) — (19) respectively by q_2,q_1,q_4,q_3 and subtracting the sum of the second and fourth from the sum of the first and third and then integrating, we get

$$\dot{q}_1 q_2 - q_1 \dot{q}_2 + \dot{q}_3 \dot{q}_4 - q_3 \dot{q}_4 = 2\rho^4 + b \tag{21}$$

Here b is the constant of integration to be determined by the initial conditions. Now we wish to represent the generating solutions in rotating coordinate system in KS-variables expressed in terms of orbital elements.

If Ω is the longitude of the ascending node, σ is the angular distance of the infinitesimal mass from the node and J is the inclination of the orbital plane of the infinitesimal mass with the plane of motion of the primaries then the compact parametric representation of KS-variables in the non-rotating (sidereal) coordinate system in any plane curve having its orbital plane in general position are given by (Steifel & Schiefle, 1971 p-67).

$$\begin{aligned} q_1 &= \sin \frac{J}{2} \cos \left(\frac{\Omega-\sigma}{2}\right), \quad q_2 = \sin \frac{J}{2} \sin \left(\frac{\Omega-\sigma}{2}\right), \\ q_3 &= \cos \frac{J}{2} \sin \left(\frac{\Omega+\sigma}{2}\right), \quad q_4 = -\cos \frac{J}{2} \cos \left(\frac{\Omega+\sigma}{2}\right), \end{aligned} \tag{22}$$

It may be noted that J and Ω are constants.

The equations of motion from (16)–(19) are in the rotating (synodic) coordinate system and q_i's (22) are in the sidereal system, so to solve the equations of motion one has to change the KS-variables of (22) in rotating cordinate system and accordingly Q_i's.

In terms of orbital elements J, Ω, σ the non-rotating three-dimensional coordinates x_i's are

$$\begin{aligned} x_1 &= \sin^2\left(\frac{J}{2}\right) \cos (\Omega - \sigma) + \cos^2 \left(\frac{J}{2}\right) \cos (\Omega + \sigma) \\ x_2 &= \sin^2\left(\frac{J}{2}\right) \sin (\Omega - \sigma) + \cos^2 \left(\frac{J}{2}\right) \sin (\Omega + \sigma) \\ x_3 &= \sin J \sin \sigma \end{aligned} \tag{23}$$

(Steifel & Schiefle, 1971 p-68)

If x_1x_2-plane rotates about x_3-axis with the mean angular motion n, then at time t, in the rotating three-dimensional coordinate system the position of the infinitesimal mass $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ are given by transformations

$$\begin{aligned} \bar{x}_1 &= x_1 \cos nt + x_2 \sin nt \\ \bar{x}_2 &= - x_1 \sin nt + x_2 \cos nt \\ \bar{x}_3 &= x_3 \end{aligned} \tag{24}$$

From (23) and (24)

$$\begin{aligned} \bar{x}_1 &= \sin^2 \left(\frac{J}{2}\right) \cos (\Omega - \sigma - nt) + \cos^2 \left(\frac{J}{2}\right) \cos (\Omega + \sigma - nt) \\ \bar{x}_2 &= \sin^2 \left(\frac{J}{2}\right) \sin (\Omega - \sigma - nt) + \cos^2 \left(\frac{J}{2}\right) \sin (\Omega + \sigma - nt) \\ \bar{x}_3 &= \sin J \sin \sigma \end{aligned} \tag{25}$$

From (4) and (9), the inverse of KS-transformations in rotating coordinate system, the variables q_i's can be obtained either from

$$q_1^2 + q_4^2 = \frac{r_1 + \bar{x}_1}{2}$$

$$q_2 = \frac{\bar{x}_2 q_1 + \bar{x}_3 q_4}{r_1 + \bar{x}_1} \tag{26}$$

$$q_3 = \frac{\bar{x}_3 q_1 - \bar{x}_2 q_4}{r_1 + \bar{x}_1} \qquad \text{if } \bar{x}_1 \geq 0$$

or from

$$q_2^2 + q_3^2 = \frac{r_1 - \bar{x}_1}{2}$$

$$q_1 = \frac{\bar{x}_2 q_2 + \bar{x}_3 q_3}{r_1 - \bar{x}_1} \tag{27}$$

$$q_4 = \frac{\bar{x}_3 q_2 - \bar{x}_2 q_3}{r_1 - \bar{x}_1} \qquad \text{if otherwise}$$

Thus by suitable choosing we can easily show either from the combination {(25), (26)} or {(25), (27)} that

$$q_1 = \rho \sin\frac{J}{2} \cos\left(\frac{\Omega - \sigma - nt}{2}\right)$$

$$q_2 = \rho \sin\frac{J}{2} \sin\left(\frac{\Omega - \sigma - nt}{2}\right) \tag{28}$$

$$q_3 = \rho \cos\frac{J}{2} \sin\left(\frac{\Omega + \sigma - nt}{2}\right)$$

$$q_4 = -\rho \cos\frac{J}{2} \cos\left(\frac{\Omega + \sigma - nt}{2}\right)$$

where also $r_1 = \sqrt{x_1^2 + x_2^2 + x_3^2} = q_1^2 + q_2^2 + q_3^2 + q_4^2 = \rho^2$.

From (11) & (28) one can easily find that

$$\sum_{i=1}^{4} \dot{q}_i^2 = \dot{\rho}^2 + \frac{\rho^2}{4}(\dot{\sigma}^2 + 16n^2\rho^4 - 8n\rho^2\,\dot{\sigma}\cos J) \tag{29}$$

and

$$\dot{q}_1 q_2 - q_1 \dot{q}_2 + \dot{q}_3 q_4 - q_3 \dot{q}_4 = 2n\rho^4 - \frac{1}{2}\rho^2\,\dot{\sigma}\cos J. \tag{30}$$

the combination (20, 29) and (21, 30) yield

$$\dot{\rho}^2 + \frac{\rho^2}{4}(\dot{\sigma}^2 + 16n^2\rho^4 - 8n\rho^2\,\dot{\sigma}\cos J) = 8\text{-}4c_0\,\rho^2 + 4\rho^6 \tag{31}$$

and

$$2n\rho^4 - \frac{1}{2}\rho^2\dot{\sigma}\cos J = 2\rho^4 + b. \tag{32}$$

Following Krashinski (1963) and Ahmad (1995) at s=0, ρ=0, (32) gives b=0 and consequently

$$\dot{\sigma} = 4(n - 1)\rho^2 \sec J. \tag{33}$$

From (31) and (33)

$$\dot{\rho}^2 = 8 - 4c_0\rho^2 - 4(n - 1)^2\rho^6 \tan^2 J. \tag{34}$$

As our purpose is to produce periodic generating solutions, so without loss of generality we can choose the ideal frame i.e. n = 1, for the generating solutions in the circular restricted problem of three bodies in three-dimensional synodic coordinate system and hence from (33), $\dot{\sigma} = 0$ i.e. σ = constant = σ_0 (say).

Thus the equation (34) yields

$$\dot{\rho}^2 = 8 - 4c_0\rho^2.$$

On integration $$\rho = \sqrt{\frac{2}{c_0}} \cos(2\sqrt{c_0}\, s - s_0) \tag{35}$$

where s_0 is the constant of integration.

Thus from (28)

$$\begin{aligned} q_1 &= \rho \sin\frac{J}{2} \cos\left(\frac{\alpha - t}{2}\right), \\ q_2 &= \rho \sin\frac{J}{2} \sin\left(\frac{\alpha - t}{2}\right), \\ q_3 &= \rho \cos\frac{J}{2} \sin\left(\frac{\beta - t}{2}\right), \\ q_4 &= -\rho \cos\frac{J}{2} \cos\left(\frac{\beta - t}{2}\right), \end{aligned} \tag{36}$$

and from (11), (15) and (36)

$$\begin{aligned} Q_1 &= \dot{\rho} \sin\frac{J}{2} \cos\left(\frac{\alpha - t}{2}\right), \\ Q_2 &= \dot{\rho} \sin\frac{J}{2} \sin\left(\frac{\alpha - t}{2}\right), \\ Q_3 &= \dot{\rho} \cos\frac{J}{2} \sin\left(\frac{\beta - t}{2}\right), \\ Q_4 &= -\dot{\rho} \cos\frac{J}{2} \cos\left(\frac{\beta - t}{2}\right), \end{aligned} \tag{37}$$

where $\alpha = \Omega - \sigma_0$ & $\beta = \Omega + \sigma_0$ are constants as Ω and σ_0 are constants.

From (11) $$t = \frac{1}{c_0\sqrt{c_0}}\left[4\sqrt{c_0}\, s + \sin(4\sqrt{c_0}\, s - 2s_0)\right] + s_1 \qquad (38)$$

where s_1 is the constant of integration and $c_0\sqrt{c_0} = n = 1$ the mean angular motion. Hence the equations (36) & (37) together with (38) represent the generating solutions of the system (12).

5. Periodicity

With the conditions of generating solutions, we have

$$\rho(s) = \sqrt{2}\cos(2s - s_0), \qquad (c_0 = 1) \qquad (39)$$

$$\dot{\rho}(s) = -2\sqrt{2}\sin(2s - s_0), \qquad (40)$$

and $$t(s) = 2[2s + \sin 2s . \cos 2(s - s_0)]. \qquad (41)$$

Let us introduce a number $s^* = \frac{k\pi}{2}$, k is any positive integer, then

$$\rho(s^* + s) = \sqrt{2}\cos[2(s^*+s) - s_0] = (-1)^k \rho(s)$$

$$\dot{\rho}(s^* + s) = -2\sqrt{2}\sin[2(s^*+s) - s_0] = (-1)^k \rho(s)$$

$$t(s^* + s) = 4(s^*+s) + 2\sin 2(s^*+s)\cos 2(s^*+s - s_0)].$$

$$= 2k\pi + t(s).$$

$$\cos\frac{1}{2}[\alpha - t(s^* + s)] = \cos\frac{1}{2}[\alpha - (2k\pi + t(s)]$$

$$= \cos\left[k\pi - \frac{\alpha - t(s)}{2}\right]$$

$$= (-1)^k \cos\left[\frac{\alpha - t(s)}{2}\right]$$

Similarly, $$\cos\frac{1}{2}[\beta - t(s^* + s)] = (-1)^k \cos\left[\frac{\beta - t(s)}{2}\right].$$

$$\sin\frac{1}{2}[\alpha - t(s^* + s)] = (-1)^k \sin\left[\frac{\alpha - t(s)}{2}\right]$$

$$\sin\frac{1}{2}[\beta - t(s^* + s)] = (-1)^k \sin\left[\frac{\beta - t(s)}{2}\right]$$

Therefore, $$q_1(s^*+s) = \rho(s^*+s)\sin\left(\frac{J}{2}\right)\cos\frac{1}{2}[\alpha - t(s^*+s)]$$

$$= (-1)^k \rho(s)\sin\left(\frac{J}{2}\right)(-1)^k \cos\left[\frac{\alpha - t(s)}{2}\right]$$

$$= \rho(s)\sin\left(\frac{J}{2}\right)\cos\left[\frac{\alpha - t(s)}{2}\right]$$

$$= q_1(s)$$

Therefore, $q1(s^*+s) = q_1(s)$

$$(i = 1,2,3,4) \tag{42}$$

Also $Q_i(s^*+s) = Q_i(s)$.

Thus q_1's and Q_1's given in (36) and (37) represent the periodic generating solutions with the period $s^* = \frac{k\pi}{2}$, where k is any positive integer.

6. Conclusion

From the above discussions we conclude that in the circular restricted problem of three bodies in three-dimensional rotating coordinate system the periodic generating solutions are possible only when the orbital elements J,Ω,σ are constants i.e., when synodic frame of reference is ideal and the mean motion n and unperturbed Jacobi's constant c_0 are unity.

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