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Stability of triangular points in the generalised photogravitational restricted three body problem

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Abstract. We study the effect of oblateness and radiation pressure forces of the primaries on the location and the stability of the triangular points in the restricted three body problem. We observe that the equations of motion and locations of the triangular points are affected by the radiation pressure forces and oblateness of the primaries. It is further seen that these points are stable for $0 \leq \mu < \mu_C$, and unstable for $\mu_C \leq \mu \leq \frac{1}{2}$. It is also seen that for these points the range of stability increases or decreases according as p > or < 0 where p depends upon the radiating and oblateness coefficients.

Key words. Stability, triangular points, generalised, photo gravitational RTBP

1. Introduction

Radzievskii (1950) formulated the photogravitational restricted three body problem. This arises from the classical problem when one of the interacting masses is an intense emitter of radiation. He discussed it for three specific bodies : the Sun, a planet and a dust particle. Chernikov (1970) extended his work by including aberrational deceleration (the Poynting - Robertson effect). He found that despite the absence of a Jacobi integral, the equations of motion admit of particular solutions corresponding to six libration points. He demonstrated the instability of the solutions by Lyapunov's first method.

Schuerman (1980) generalized the restricted three body problem by including the force of radiation pressure and the Poynting - Robertson effect. The Poynting - Robertson effect renders the L_4 and L_5 points unstable on a time scale (T) long compared to the period of rotation of the two massive bodies. For the solar system, T is given by $T \simeq [(1-\beta)^{2/3}/\beta] \times 544$ a^2 year, where β is the ratio of radiation to gravitational forces, and a is the separation between the Sun and the planet in AU.

He also discussed implications for space colonization and a mechanism for producing azimuthal asymmetries in the interplanetary dust complex.

Sharma (1982) studied the linear stability of triangular libration point of the restricted three body problem when the more massive primary is a source of radiation and an oblate spheroid as well. He found that the eccentricity of the conditional retrograde elliptic periodic orbits around the triangular points at the critical mass μ_C increases with the increase in the oblateness coefficient and the radiation force and becomes unity when μ_C is zero.

Simmons et al., (1985) gave a complete solution of the restricted three-body problem. They discussed the existence and linear stability of the equilibrium points for all values of radiation pressures of both luminous bodies and all values of mass ratios.

Ragos and Zagouras (1988) found two families of periodic solutions about the 'out of plane' equilibrium points in the photogravitational restricted three-body problem.

Shaboury (1990) gave a possibility of nine libration points for small values of oblateness in the photogravitational restricted three-body problem when the infinitesimal mass is of an axisymmetric body and one of the finite masses be a spherical luminous body while the other be an axisymmetric non-luminous body.

Todoran (1993) claimed that the "out of plane" equilibrium points (out of the orbital plane of the primaries) in the restricted three-body problem as concerned radiation pressure, do not actually exist. This question was answered by Ragos and Zagouras (1993). Liou and Zook (1995) investigated asteroidal dust ring of micron - sized particles trapped in the 1 : 1 mean motion resonance with Jupiter. They with Jackson(1995) examined the effects of radiation pressure, Poynting-Robertson (PR) drag, and solar wind drag on dust grains trapped in mean motion resonances with the sun and jupiter in the restricted three-body problem. Khasan (1996) studied the existence of libration points and their stability in the photogravitational elliptic restricted three-body problem.

The classical problem of three bodies was generalized by considering the various aspects such as the shape of the bodies, influence of the perturbing forces other than the forces of mutual gravitational etc., to make the problem more realistic. In the solar system, some of the planets, like Saturn and Jupiter are sufficiently oblate. It has been seen that oblateness of the body plays an important role in the restricted three-body problem.

Hence, the idea of the radiation pressure forces together with oblateness of the body raises a curiosity in our mind to study the "stability of triangular points in the generalised photogravitational restricted three-body problem". The problem is photogravitational in the sense that both the primaries are sources of radiation. The problem is generalised in the sense that both primaries are taken as oblate spheroid.

We use A_i (i=1, 2) for oblateness coefficients of the bigger and the smaller primaries respectively such that $0 < A_i \ll 1$ and

$$A_1 = (AE_1^2 - AP_1^2) / 5R^2$$

$$A_2 = (AE_2^2 - AP_2^2) / 5R^2$$

where AE_1 and AE_2 are the equatorial radii, AP_1 and AP_2 being the polar radii of the bigger and the smaller primaries respectively.

We further denote the radiation repulsive force by $q_i (i=1, 2)$, which are given by the equation

$$F_{P_i} = F_g (1 - q_i),$$

F_g being the gravitational attraction forces, $q_i \simeq 1$ and so

$$0 < 1 - q_i \ll 1.$$

Here Poynting - Robertson drag effect is ignored. We are neglecting the perturbation in the potential between m_1 and m_2 due to the radiation pressure, because m_1 is supposed to be sufficiently large.

2. Equations of motion

Let m_1 and m_2 be the masses of the bigger and smaller primaries, m is the mass of the third infinitesimal body. We assume that both the primaries are oblate spheroids and radiating as well. Let R be the distance between the primaries, r_1 and r_2 be the distance of m_1 and m_2 from m respectively.

The potential V between m_1 and m_2 is given as

$$V = -G\, m_1\, m_2 \left[\frac{1}{R} + \frac{(A_1 + A_2)}{2R^3} \right] \tag{1}$$

Let (x, y) be the coordinate of m_2 with respect to m_1, its equations of motion are.

$$\ddot{x} = -\frac{m_1 + m_2}{m_1 m_2}\frac{\partial V}{\partial x}$$

$$\ddot{y} = -\frac{m_1 + m_2}{m_1 m_2}\frac{\partial V}{\partial y} \tag{2}$$

The particular solutions of (2) can be written as R = constant, X= R cos θ, Y = R sin θ, θ=nt, where the mean motion n is given as

$$n^2 = G\frac{(m_1 + m_2)}{R}\left[\frac{1}{R^2} + \frac{3}{2}\frac{(A_1 + A_2)}{R^4}\right] \tag{3}$$

Let (x, y) be the coordinate of the third body in a rotating coordinate system with the origin at 0 and the line joining the primaries being the x-axis and the line perpendicular to it being the y-axis.

The kinetic energy T of m is given as

$$T = T_0 + T_1 + T_2,$$

with

$$T_0 = \frac{1}{2} mn^2 (x^2 + y^2),$$

$$T_1 = mn (x\dot{y} - \dot{x}y),$$

$$T_2 = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2).$$

The potential V between m and m_1 and, m and m_2 is given as

$$\overline{V} = - Gm \, [m_1 (\frac{q_1}{r_1} + \frac{A_1 q_1}{2r_1^3}) + m_2 (\frac{q_2}{r_2} + \frac{A_2 q_2}{2r_2^3})] \tag{4}$$

where

$$r_1^2 = (x - x_1)^2 + y^2,$$

$$r_2^2 = (x - x_2)^2 + y^2,$$

$(x_1,0)$ and $(x_2,0)$ are coordinates of m_1 and m_2 respectively.

Let the modified potential energy be

$$\bar{U} = \bar{V} - T_0,$$

$$\overline{U} = -Gm \, [m_1 (\frac{q_1}{r_1} + \frac{A_1 q_1}{2r_1^3}) + m_2 (\frac{q_2}{r_2} + \frac{A_2 q_2}{2r_2^3})] - \frac{1}{2} mn^2 (x^2 + y^2)$$

The Lagrangian can be put in the form

$$L = T_2 + T_1 - \bar{U},$$

The equations of motion of the third body are

$$\ddot{x} - 2n\dot{y} = -\frac{1}{m} \frac{\partial \bar{U}}{\partial x}$$

$$\ddot{y} - 2n\dot{x} = -\frac{1}{m} \frac{\partial \bar{U}}{\partial y} \tag{5}$$

Now, we choose the unit of mass equal to the sum of the primary masses. For this we take $m_1 = 1-\mu$ and $m_2 = \mu$, where μ is the ratio of the mass of the smaller primary to the total mass of the primaries and $0 \le \mu \le \frac{1}{2}$. The unit of length is taken as equal to the distance between the primaries and the unit of time is so chosen that the gravitational constant G is unity.

Let the origin be the bary - centre of mass m_1 at $(x_1,0)$ and m_2 at $(x_2, 0)$. Then we have $x_1 = \mu$ and $x_2=-(1-\mu)$.

Therefore, the equations (5) can be written as

$$\ddot{x} - 2n\dot{y} = \frac{\partial U}{\partial x}$$

$$\ddot{y} - 2n\dot{X} = \frac{\partial U}{\partial y} \tag{6}$$

with

$$U = \frac{1}{2} n^2(x^2+y^2) + q_i\left(\frac{(1-\mu)}{r_1} + \frac{\mu q_2}{r_2} + \frac{A_1 q_1(1-\mu)}{2r_1^3} \right.$$

$$\left. + \frac{A_2 q_2 \mu}{2r_2^3} \right) \tag{7}$$

$$r_1^2 = (x - \mu)^2 + y^2,$$

$$r_2^2 = (x+1 - \mu)^2 + y^2, \tag{8}$$

$$n^2 = 1 + \frac{3}{2} (A_1 + A_2) \tag{9}$$

Multiplying the first equation by $2\dot{x}$ and second equation by $2\dot{y}$ of (6) and then adding them, we have

$$\frac{d}{dt}(\dot{x}^2 + \dot{y}^2) = 2\frac{dU}{dt}$$

Its integration gives

$$\dot{x}^2+\dot{y}^2 = 2U - C \tag{10}$$

where C is the Jacobian constant.

3. Location of triangular points

The locations of triangular points are the solutions of

$$\frac{\partial U}{\partial x} = 0, \ \frac{\partial U}{\partial y} = 0, \ \ y \neq 0 \text{ i.e.,}$$

$$x[n^2 - \frac{1-\mu}{r_1^3} q_1 - \frac{\mu q_2}{r_2^3} - \frac{3}{2} A_1 q_1 \frac{(1-\mu)}{r_1^5} - \frac{3}{2} \frac{A_2 q_2 \mu}{r_2^5}]$$

$$+ \frac{(1-\mu)\mu\, q_1}{r_1^3} - \frac{\mu(1-\mu) q_2}{r_2^3} + \frac{3}{2} \frac{A_1 q_1 \mu (1-\mu)}{r_1^5}$$

$$- \frac{3}{2} \frac{A_2 q_2 \mu (1-\mu)}{r_2^5}] = 0 \quad (11)$$

and

$$n^2 - \frac{(1-\mu)\, q_1}{r_1^3} - \frac{\mu q_2}{r_2^3} - \frac{3}{2} \frac{A_1 q_1 (1-\mu)}{2 r_1^5} - \frac{3}{2} \frac{A_2 q_2 \mu}{r_2^5} = 0$$

which give

$$\frac{q_1}{r_1^3} - \frac{q_2}{r_2^3} + \frac{3}{2} \frac{A_1 q_1}{r_1^5} - \frac{3}{2} \frac{A_2 q_2}{r_2^5} = 0 \quad (12)$$

After re-writing the second equation of (11) and making use of (12), we have

$$n^2 - \frac{q_1}{r_1^3} - \frac{3}{2} \frac{A_1 q_1}{r_1^5} = 0 \quad (13)$$

Combining (12) and (13), we have

$$n^2 - \frac{q_2}{r_2^3} - \frac{3}{2} \frac{A_2 q_2}{r_2^5} = 0 \quad (14)$$

Knowing r_1 and r_2 from the equations (13) and (14), the coordinates of the triangular points are found by solving equations (8) for x and y.

The exact co-ordinates of the triangular points corresponding to L_4 and L_5 are given by

$$x = \mu - \frac{1}{2} + \frac{r_2^2 - r_1^2}{2},$$

$$y = \pm \left[\frac{r_1^2 + r_2^2}{2} - \frac{1}{4} - \left(\frac{r_2^2 - r_1^2}{2}\right)^2\right]^{1/2} \tag{15}$$

When the primaries are neither radiating nor oblate spheroids i.e A_i=0, q_i=1 (i=1, 2), the solutions of the equations (13) and (14) are $r_i = 1$.

Therefore we can assume that solutions of (13) and (14) are

$$r_i = 1 + \epsilon_i$$

where ϵ_i's are very small.

Restricting only linear terms in ϵ_i, A_i, 1 - q_i and coupling terms in A_1q_1 and A_2q_2, we have

$$\epsilon_1 = \frac{1}{3}\left[-\frac{3}{2}(A_1+A_2) - (1-q_1) + \frac{3}{2}A_1q_1\right],$$

$$\epsilon_2 = \frac{1}{3}\left[-\frac{3}{2}(A_1+A_2) - (1-q_2) + \frac{3}{2}A_2q_2\right],$$

$$r_1 = 1 - \frac{A_1+A_2}{2} - \frac{1-q_1}{3} + \frac{1}{2}A_1q_1,$$

$$r_2 = 1 - \frac{1}{2}(A_1 + A_2) - \frac{1}{3}(1 - q_2) + \frac{1}{2}A_2q_2 \tag{16}$$

putting the values of r_i in (15), we get

$$x = \mu - \frac{1}{2} + \frac{1}{3}(1 - q_1) - \frac{1}{3}(1 - q_2) + \frac{1}{2}(A_2q_2 - A_1q_1),$$

$$y = \pm \sqrt{3}\left[\frac{1}{2} - \frac{1}{3}(A_1 + A_2) + \frac{1}{6}(A_1q_1 + A_2q_2) - \frac{1}{9}(1 - q_1) - \frac{1}{9}(1 - q_2)\right] \tag{17}$$

4. Stability of the triangular points

The characteristic equation of the variational equations corresponding to (6) is

$$\lambda^4 - (U^0_{xx} + U^0_{yy} - 4n^2)\,\lambda^2 - (U^0_{xy})^2 + U^0_{xx}\,U^0_{yy} = 0 \tag{18}$$

where the second partial derivatives of U are denoted by subscripts, and the subscript 0 indicates that those derivatives are to be evaluated at the triangular point (x_0, y_0), coordinates of L_4.

$$U^0_{xx} = \frac{3}{4} + a_1 + \mu b_1 \ ,$$

$$U^0_{yy} = \frac{9}{4} + a_2 + \mu b_2 \ ,$$

$$U^0_{xy} = \sqrt{3}\,[-\frac{3}{4} + a_3 + \mu(\frac{3}{2} + b_3)] \ ,$$

with

$$a_1 = \frac{1}{4}\,[\,\frac{15}{2}(A_1 + A_2) - 2(1-q_1) + 4(1-q_2) + 6(A_1q_1 - A_2q_2)]$$

$$b_1 = \frac{1}{4}\,[6(1-q_1) - 6(1-q_2) + 12(A_2q_2 - A_1q_1)] \ ,$$

$$a_2 = \frac{1}{4}\,[\,\frac{21}{2}(A_1 + A_2) + 6(A_1q_1 + A_2q_2) + 2(1-q_1) - 4(1-q_2)] \ ,$$

$$b_2 = \frac{1}{4}\,[6(1-q_2) - 6(1-q_1)] \ ,$$

$$a_3 = \frac{1}{12}\,[-\frac{33}{2}(A_1 + A_2) + 2(1-q_1) - 4(1-q_2)] + 6A_2q_2 - 12A_1q_1] \ ,$$

$$b_3 = \frac{1}{12}\,[33(A_1 + A_2) + 2(1-q_1) + 2(1-q_2)] + 6(A_1q_1 - A_2q_2)] \tag{19}$$

Here each of $|a_i|$, $|b_i|$ is very small as $|A_i| << 1$, $|1-q_i| <<1$, (i=1,2,3). Putting the values of U^0_{xx} , U^0_{yy} , U^0_{xy} found above and the value of n^2 from (9) in (18), the characteristic equation becomes

$$\lambda^4 - \lambda^2\,[\mu(b_1 + b_2) - 1 + a_1 + a_2 - 6\,(A_1 + A_2)\,]$$
$$- \frac{3}{4}\,[3(3 + 4b_3)\,\mu^2 + (-9 - 3b_1 - b_2 + 12b_3$$
$$- 6b_3)\mu - (3a_1 + a_2 + 6a_3)\,] = 0$$

Its roots are

$$\lambda^2 = \frac{1}{2}\,[\,-1 + \mu(b_1+b_2) + a_1 + a_2 - 6(A_1 + A_2) \pm \sqrt{\Delta}], \tag{20}$$

where Δ is the discriminant and

$$\Delta = 9(3 + 4b_3)\,\mu^2 - (27 - 36a_3 + 11b_1 + 5b_2 + 18b_3)\mu$$
$$+ 1 - 11a_1 - 5a_2 - 18a_3 + 12(A_1 + A_2) \qquad (21)$$

Now

$$\frac{d\Delta}{d\mu} < 0 \text{ for } 0 < \mu < \frac{1}{2},\ (\Delta)_{\mu=0} \simeq 1,\ (\Delta)_{\mu=\frac{1}{2}} \simeq -\frac{23}{4}$$

Therefore Δ is a strictly decreasing function of μ in the closed interval (0, ½) and has values of opposite signs at the end points $\mu = 0$ and $\mu = ½$. Consequently, there is one and only one value of μ say μ_C in the open interval (0, ½) for which Δ vanishes.

There are three possible cases :

(i) When $0 \leq \mu < \mu_C$, Δ is positive, the values of λ^2 given by (20) are negative and all the four roots of the characteristic equation are distinct pure imaginary. This shows that the triangular point in question is stable.

(ii) When $\mu = \mu_C$, Δ is zero. Both the values of λ^2 given by (20) are same. So the solutions of the variational equations contain secular terms and consequently the triangular point is unstable.

(iii) When $\mu_c < \mu \leq ½$, Δ is negative. This indicates that the real parts of two of the characteristic roots are positive and so the triangular point is unstable. Hence for $0 \leq \mu < \mu_C$, we have stability and for $\mu_C \leq \mu \leq ½$, we have instability.

5. Critical mass

The critical value μ_C of the mass parameter is a root of the equation $\Delta = 0$, and we find

$$\mu_C = \mu_0 + P$$

with

$$\mu_0 = \frac{1}{2}\left(1 - \sqrt{\frac{23}{27}}\right) = 0.03852 \;.......,$$

$$P = -\frac{2}{9\sqrt{69}}(A_1+A_2) - \frac{2}{27\sqrt{69}}(1-q_1) - \frac{1}{9}\left(1 + \frac{11}{\sqrt{69}}\right)A_1q_1$$
$$- \frac{2}{27\sqrt{69}}(1-q_2) + \frac{1}{9}\left(1 - \frac{11}{\sqrt{69}}\right)A_2q_2$$

The range of stability increases or decreases or remains unchanged according as $P \gtreqless 0$.

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