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Generation of kinetic helicity from irrotational motions

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Abstract. Vortical and helical fluctuations are an integral part of any turbulent fluid. A turbulent magnetohydrodynamic system with net kinetic helicity is known to support the growth of a large scale magnetic field through the dynamo mechanism. An analogous effect, called the kinetic α - effect has been shown to give rise to large scale structures in a turbulent hydrodynamic system. Thus, it appears that the mechanism for formation of large scale structures, magnetic or hydrodynamic, hinges on the presence of kinetic helicity, in an essential manner. It has been demonstrated that systems possessing nonzero net kinetic helicity and those with zero net kinetic helicity, but nonzero mean square helicity, can support the formation of large scale structures via the mechanism of inverse cascade of energy. The question, then arises, "what are the mechanisms for the production of small scale vortical and helical fluctuations, especially in astrophysical situations" ?

We examine two mechanisms for the generation of vorticity and helicity from longitudinal velocity fields, like those associated with, for example, acoustic waves, which by themselves are irrotational. We assume the existence of finite amplitude acoustic waves; these could be generated in a highly compressible medium such as obtains in astrophysical situations. We discuss (i) growth of vorticity field through a mechanism, known as the Langmuir circulation, in which the ambient shear in the fluid couples with the longitudinal waves to produce vorticity and helicity and (ii) the conversion of part of the energy contained in a longitudinal velocity field into vortical and helical velocity fields when it propagates through a turbulent medium; this latter mechanism may be seen as the inverse of the Lighthill mechanism where part of the turbulent energy is converted into sound waves.

1. Introduction

Fluid motions are present in the universe at a variety of sites - oceans, planetary and stellar atmospheres, stellar interiors, galactic and intergalactic medium. However, not all the consequences of the inherently turbulent nature of fluids are known inspite of their prolonged study spanning centuries. The novelties in a nonlinear medium, which a fluid essentially is, arise due to the multiplicity of ways in which the interactions among the large number of spatial and temporal scales of motion are specified and systemised. In addition, the dimensionality of a fluid system plays a crucial role in determining its behaviour (Hasegawa 1985). Thus, a 3D system has characteristics which are absent in a 2D system. Further an isotropic 3D system is completely different from an anisotropic 3D system. Perhaps the kinetic helicity, defined as the scalar product of the vorticity and the velocity, is the most discerning property of a turbulent fluid. The presence of helical flows, essentially, endows anisotropy to a 3D system as a result of which the nature of the energy cascade in a turbulent fluid undergoes a complete metamorphosis; infact, to such an extent that a direct cascade (energy flowing from large spatial scales to small spatial scales) changes into an inverse cascade (energy flowing from small spatial scales to large spatial scales) (Levich and Tzvetkov 1985). The inverse cascade, it is conjectured, can lead to the formation of coherent, organized structures in an apparently turbulent medium (Krishan 1991, 1993; Krishan and Sivaram 1991).

The dynamo mechanisms of various types for generation of large scale magnetic field offer an example where small scale velocity and magnetic field fluctuations correlate to produce a large scale magnetic field. The corresponding processes are believed to take place in a purely hydrodynamic medium too, (Sulem et al. 1989), in which, the essential role of kinetic helicity has already been emphasized. For example, the well known α - effect for the generation of a large scale magnetic field operates only if the system possesses a net kinetic helicity. In a hydrodynamic system, however, the inverse cascade or the formation of flows on large spatial scales has been shown to occur, even in the situation where helicity fluctuations whose first moment vanishes, but the second moment does not (Levich and Tzvetkov 1985; Moffat and Tsinober 1992). These topics are the subject of intensive study as they have applications in several different fields. Along with the attempts to understand the formation of large scale magnetic or kinetic structures, it is clearly desirable to investigate the various ways of generating kinetic helicity. In this work, we, first, show the connection between the development of mass density and kinetic helicity, since this connection is evident only when the full nonlinearity of the compressible fluid equations is preserved. This is significant as the amplifications of mass density, through for example, gravitational collapse processes is of a common occurrence in astrophysical situations. We, then, discuss two mechanisms by which a part of the energy contained in irrotational flows, like those associated with acoustic waves can be used to generate and amplify vortical and helical flows.

2. Connection between kinetic helicity and mass density

The equations expressing mass and momentum conservation respectively for an inviscid fluid are :

$$\frac{D\rho}{Dt} \equiv \frac{\partial \rho}{\partial t} + (\vec{U}.\vec{\nabla})\rho = -\rho(\vec{\nabla}.\vec{U}) \tag{1}$$

and

$$\frac{D\vec{U}}{Dt} \equiv \frac{\partial \vec{U}}{\partial t} + (\vec{U}.\vec{\nabla})\vec{U} = -\frac{1}{\rho}\vec{\nabla}P + \frac{1}{\rho}\vec{F}, \tag{2}$$

where $\vec{F}$ is a conservative force density and can be written as, $\vec{F} = -\rho\vec{\nabla}\varphi$; the pressure force term $(-\frac{1}{\rho}\vec{\nabla} P)$ can be expressed as $(-\vec{\nabla} e)$ with

$$e = \int \frac{dP}{\rho} \qquad (3)$$

as the specific enthalpy for adiabatic variations. Equations (2) can then be recast in the form :

$$\frac{\partial \vec{U}}{\partial t} + \vec{\omega} \times \vec{U} + \vec{\nabla} B = 0 \qquad (4)$$

with vorticity $\vec{\omega} = \vec{\nabla} \times \vec{U}$ and the Bernoulli function, $B = \frac{u^2}{2} + e + \varphi$.

From the curl of Equation (4)

$$\frac{D\vec{\omega}}{Dt} \equiv \frac{\partial\vec{\omega}}{\partial t} + (\vec{U}.\vec{\nabla})\vec{\omega} = (\vec{\omega}.\vec{\nabla})\vec{U} - \vec{\omega}(\vec{\nabla}.\vec{U}), \qquad (5)$$

we see that the rate of change of vorticity depends upon the irrotational part of the motion for which $\vec{\nabla}.\vec{U} \neq 0$ assuming that initially there is a nonzero vorticity present. The kinetic helicity h is defined as

$$h = \vec{U}.curl\vec{U}. \qquad (6)$$

With a few algebraic manipulations, we find :

$$\frac{1}{h}\frac{Dh}{Dt} = \frac{1}{\rho}\frac{D\rho}{Dt} + \frac{1}{h}(\vec{\omega}.\vec{\nabla}B'), \qquad (7)$$

where, $B' = U^2 - B$, which shows that the rate of change of kinetic helicity is intimately linked with the rate of change of mass density. This link is lost in the linearized fluid equations. Thus, a phenomenon like the Jean's collapse will be accompanied by changes in kinetic helicity. Since h can take positive or negative values, we observe that $\frac{D\rho}{Dt} > 0$ if $\frac{Dh}{Dt} > 0, h > 0$ and $\frac{Dh}{Dt} < 0$, $h < 0$; i.e. a positive helicity becomes more positive and a negative helicity more negative with the growth of condensation. There is a growth of a rarefaction if the helicity decreases (provided, of course, Bernoulli's term on the right hand side is negligible compared to the compression term).

In conclusion, a compression is accompanied by an increase of helicity whereas a rare faction is accompanied by a decrease of helicity, which is good since helicity aids the organization of magnetic field, matter and motion, perhaps simultaneously too ! Here, we have ignored the contribution from the Bernoulli term. In the absence of compressional processes, the kinetic helicity could still be generated through a variation of the Bernoulli term, a process based on what is known as Crocco's theorem.

We can compare the time scales of variations of h and ρ. Dimensionally speaking :

$$\frac{\vec{\omega}.\vec{\nabla}B'}{h} \simeq \frac{\vec{\omega}.\nabla B'}{U\omega} \simeq \frac{\nabla B'}{U} \simeq \frac{U_{eff}}{L} \simeq \frac{U_{eff}}{c_s}\frac{c_s}{L} \simeq \frac{M}{t_s} \tag{8}$$

where, $U_{eff} = B'/U$ is an effective speed; L is a characteristic scale, c_s, the sound speed, M, the Mach number and t_s the sound propagation time across L. Equation (7) then tells us that

$$\frac{1}{t_h} = \frac{1}{t_\rho} + \frac{M}{t_s} \tag{9}$$

We know from Jean's mechanism that for scales $L \gg \frac{2\pi c_s}{\omega_J}$, the growth time of mass density for Jean's collapse is $t_\rho \simeq \frac{2\pi}{\omega_J}$ where, $\omega_J = \sqrt{4\pi G\rho}$ is the Jean's frequency. Thus, $t_s \gg t_\rho$ and therefore from equation(9), we find $t_h \simeq t_\rho$ i.e. the kinetic helicity and the mass density can grow at comparable rates for not too large a value of the Mach number. For large Mach numbers the contribution from Bernoulli's term should dominate, then

$$t_s^2 \ll M^2 t_\rho^2$$

Thus the Jean's criterion combined with the domination of Bernoulli's term becomes :

$$M^2 t_\rho^2 \gg t_s^2 > t_\rho^2$$

Under such circumstances, $t_\rho > t_h$ and the time scale of helicity fluctuations becomes shorter than that of the density fluctuations. So we see in both cases in the absence of Bernoulli's term as well as under its dominance, the helicity fluctuations grow along with the density fluctuations. Equation (5) for vorticity can also be written in a form analogous to equation (7) as :

$$\frac{1}{\vec{\omega}}\frac{D\vec{\omega}}{Dt} = \frac{1}{\rho}\frac{D\rho}{Dt} + \frac{(\vec{\omega}.\vec{\nabla})\vec{U}}{\vec{\omega}} \tag{10}$$

from which we conclude that mass density, vorticity and helicity can grow at comparable rates.

3. The Langmuir circulation

It has been recognized, now, for sometime that the development of vortices at the interface of two fluids can enhance the transfer of energy from one fluid to the other say through the Kelvin - Helmholtz instability. The term Langmuir circulation, specifically, applies to the growth of cellular circulation pattern in the form of alternate right and left handed helical flows in the upper layers of wind blown lakes and oceans. (Langmuir 1938, Leibovitch and Ulrich 1972). The axes of these wind driven helical motions are horizontal and parallel to the direction of the wind. The wind excites acoustic waves at the water surface. The acoustic waves produce what is known as the Stokes Drift of the fluid elements. The initial vortex elements in the fluid now experience stretching and rotation due to the Stokes Drift. In an ideal fluid the vortex lines remain frozen to the matter as do the magnetic field lines in an ideal magnetohydrodynamic system. Thus, if a material line coinciding with the vortex line undergoes extension over a portion of its length, mass conservation implies that the cross-section of the associated vortex tube must decrease. It has been shown that the direction and the magnitude of $\vec{\omega}$ in a fluid element, individually change with time in the same way as the direction and magnitude of the vector $\vec{\delta l}$ representing a material line element which at some initial time t_o was chosen to be parallel to the local vorticity (Batchelor 1967). Therefore, the vorticity in a fluid element must satisfy the relation :

$$\frac{\vec{\omega}(t)}{|\vec{\omega}(t_o)|} = \frac{\vec{\delta l}(t)}{|\vec{\delta l}(t_o)|} \tag{11}$$

using which we can determine $\vec{\omega}(t)$, if we know $\vec{\delta l}(t)$ corresponding to the Stokes Drift speed induced by the wind driven acoustic waves. In the astrophysical context turbulent, gas clouds of different densities and velocities could be the counterparts of wind and water. Following Leibovitch and Ulrich (1972), we represent the acoustic field by a velocity potential ψ expressed as

$$\psi = \sigma \frac{a}{k} e^{kz} \{ \sin[k(x\cos\theta + y\sin\theta) - \sigma t] + \sin[k(x\cos\theta - y\sin\theta) - \sigma t] \} \tag{12}$$

We would like to find the Lagrangian velocity $\vec{U}_L$ of a fluid element induced by the potential field given in equation (12) which represents plane waves propagating at angles ($\pm\theta$) to the x axis. Let $\vec{U}_L(\vec{r}_o,t)$ be the velocity of a fluid element whose position at time t = 0 is $\vec{r}_o$. At later times, its position is given by

$$\vec{r} = \vec{r}_o + \int_0^t \vec{U}_L(\vec{r}_o,t')\, dt'. \tag{13}$$

The velocity at the point $\vec{r}$ is $\vec{U}_L(\vec{r}_o,t)$ in the Lagrangian frame and $\vec{U}_E(\vec{r},t)$ in the Eulerian frame. Therefore, we get,

$$\vec{U}_L(\vec{r}_o,t) = U_E[\vec{r}_o + \int_0^t \vec{U}_L(\vec{r}_o,t')\, dt'\ t]$$

$$= \vec{U}_E(\vec{r}_o,t) + \{(\int_0^t \vec{U}_L(\vec{r}_o,t')\, dt').\ \vec{\nabla}\}\ \vec{U}_E(\vec{r},t)\ |\ \vec{r} = \vec{r}_o, \tag{14}$$

$$+ \ldots\ldots\ldots\ldots$$

using the Taylor Series expansion. To the first order $U_L(\vec{r}_o,t) = U_E(\vec{r}_o,t)$. Thus replacing $\vec{U}_L(\vec{r}_o,t')$ by $U_E(\vec{r}_o,t')$ in the integrand in equation (14), we obtain Stokes Drift to second order as :

$$\vec{U}_L(\vec{r}_o,t) = \vec{U}_E(\vec{r}_o,t) + \{(\int_0^t \vec{U}_E(\vec{r}_o,t')\, dt').\ \vec{\nabla}\}\ \vec{U}_E(\vec{r},t)\ |\ \vec{r} = \vec{r}_o. \tag{15}$$

Substituting for $\vec{U}_E(\vec{r}_o,t) = \vec{\nabla}\ \psi(\vec{r}_o,t)$ and averaging over a wave period, we obtain

$$\vec{U}_L(\vec{r}_o,t) = \vec{U}_E(\vec{r}_o,t) + 4\sigma a^2 k\cos\theta e^{2kz} \cos^2(ky_o\sin\theta)\hat{x}), \tag{16}$$

where $\hat{x}$ is a unit vector. In the absence of the wave field, the velocity $\vec{U}_E(\vec{r}_o,t)$ of the fluid may be a result of convective and gravitational forces operating in astrophysical fluids such as molecular clouds. Let us, as an example, take it as a stationary profile $U_x(z)$. The material line element $\delta\vec{l}(t)$ and the vorticity $\vec{\Omega}(t)$ are therefore given by

$$\delta\vec{l}(t) = \vec{U}_L(\vec{r}_o,t)t + \delta\vec{l}(t_o) \tag{17}$$

and

$$\vec{\omega}(t) = \frac{\delta\vec{l}(t)}{\delta y_o}\frac{\partial U_x(z)}{\partial z}, \tag{18}$$

where we have taken $|\delta\vec{l}(t_o)| = \delta y_o$ and $|\vec{\omega}(t_o)| = \frac{\partial U_x}{\partial z}$

The vorticity $\vec{\omega}$ has x and y components and only the x component grows linearly with time as :

$$\omega_x(t) = \frac{\partial U_x}{\partial z}\left[U_x(z) + 4\sigma a^2 k\cos\theta e^{2kz}\cos^2(ky_o\sin\theta)\right]\frac{t}{\delta y_o}. \tag{19}$$

We see that in the absence of the irrotational velocity field at t=o, the fluid element had only y component of vorticity given by $\frac{\partial U_x}{\partial z}$. At a later time, due to the motion induced by the acoustic field, the fluid element has now acquired an x component of vorticity too which grows linearly with time. The fluid element thus acquires a net helicity h given by

$$h = \omega_x(t)\left[U_x(z) + 4\sigma a^2 k\cos\theta e^{2kz}\cos^2(ky_o\sin\theta)\right] \tag{20}$$

Note that the fluid had zero helicity initially. So, the helicity can be generated with the help of an irrotational motion.

4. The inverse Lighthill mechanism

The Lighthill mechanism refers to the production of sound waves in a turbulent fluid where a part of the energy contained in the essentially rotational motion of the turbulence is converted into irrotational motions associated with acoustic fields. Can, the inverse process occur too? In this section, we show the conversion of a part of the acoustic energy into rotational motion during the propagation of sound through a turbulent medium (Belyan et al. 1991). We use the standard fluid equations for an ideal, compressible fluid for studying the conversion of irrotational motions associated with sound waves into rotational motions.

The equations are :

$$\frac{\partial \vec{U}}{\partial t} + (\vec{U}.\vec{\nabla})\vec{U} = -\frac{c_s^2}{\gamma}\nabla \ln P, \tag{21}$$

$$\frac{1}{\gamma}\frac{d}{dt} \ln P = -\vec{\nabla}.\vec{U}, \tag{22}$$

where γ is the ratio of the specific heats, P is the pressure and c_s is the sound speed. Let the turbulence be described through a velocity field $\vec{U}_o$ and pressure P_o satisfying equations(22). The turbulent field is disturbed such that

$$\vec{U} = \vec{U}_o + \vec{U}_1, \qquad P = P_o + P_1 \tag{23}$$

where

$$\vec{U}_1(\vec{r},t) = \vec{U}_1(\vec{r})e^{-i\sigma t}, \tag{24}$$

$$P_1(\vec{r},t) = P_1(\vec{r})e^{-i\sigma t}.$$

The linearized forms of Equations (21) and (22) are :

$$i\sigma\vec{U}_1 = c_s^2\vec{\nabla}\Pi + (\vec{U}_0.\vec{\nabla})\vec{U}_1 + (\vec{U}_1.\vec{\nabla})\vec{U}_0, \tag{25}$$

$$i\sigma\Pi = \vec{\nabla}.\vec{U}_1 + (\vec{U}_0.\vec{\nabla})\Pi, \tag{26}$$

where $\Pi = P_1/\gamma P_0$ and temperature perturbations are ignored. The perturbed velocity $\vec{U}$ is made up of two parts : the first $\vec{U}_i$ due to the incident acoustic wave and the second $\vec{U}_s$ due to the scattered velocity field which has both rotational and irrotational parts. Thus,

$$\vec{U}_1 = \vec{U}_i + \vec{U}_s, \tag{27}$$

and correspondingly,

$$\Pi = \Pi_i + \Pi_s,$$

where $\vec{U}_i$ and Π_i satisfy the conditions :

$$\vec{\nabla} \times \vec{U}_i = 0, \tag{28}$$

$$i\sigma\vec{U}_i = c_s^2\vec{\nabla}\Pi_i\ , \tag{29}$$

$$i\sigma\Pi_i = \vec{\nabla}.\vec{U}_i, \tag{30}$$

and

$$\sigma^2 = k^2 c_s^2 , \tag{31}$$

in the absence of the turbulent field. We now split the scattered field into its vortical part $(\vec{U}_v, \Pi_v)$ and the non-vortical part $(\vec{U}_\phi, \Pi_\phi)$. Since, we are interested in the vortical part, we will not consider the nonvortical part any more.

Taking the curl of equation (25) we get

$$\vec{\nabla} \times \vec{U}_v = -\frac{i}{\sigma}\nabla \times [(\vec{U}_o . \vec{\nabla})\vec{U}_i + (\vec{U}_i . \vec{\nabla})\vec{U}_o]. \tag{32}$$

From equations (25) and (29), we find

$$\vec{U}_v + \frac{i}{\sigma} c_s^2 \vec{\nabla} \Pi_v = -\frac{i}{\sigma}[(\vec{U}_o . \vec{\nabla})\vec{U}_i + (\vec{U}_i . \vec{\nabla})\vec{U}_o]. \tag{33}$$

From equation (26), we find

$$i\sigma \Pi_v = (\vec{U}_o . \vec{\nabla})\Pi_i. \tag{34}$$

Equations (32), (33) and (34) determine the vortical part of the scattered field, given the incident field $(\vec{U}_i, \Pi_i)$ and the turbulent field, $(\vec{U}_o, \Pi_o)$. The vortical part $\vec{U}_v$ is found to be

$$\vec{U}_v = \frac{i}{\sigma^3} c_s^2 \vec{\nabla}\{(\vec{U}_o . \vec{\nabla})(\vec{\nabla} . \vec{U}_i)\} - \frac{i}{\sigma}\vec{\nabla}(\vec{U}_i . \vec{U}_o) + \frac{i}{\sigma}(\vec{U}_i \times \vec{\Omega}_o). \tag{35}$$

The total vorticity $\vec{\omega}$ of the fluid is given by

$$\vec{\omega} = \vec{\nabla} \times (\vec{U}_o + \vec{U}_i + \vec{U}_v)$$

$$= \vec{\omega}_0 + \vec{\omega}_v , \tag{36}$$

where $\vec{\omega}_0$ and $\vec{\omega}_v$ are the vorticities associated respectively with the turbulent field and the scattered field. Using the identity

$$\vec{\nabla} \times [\vec{\nabla}(\vec{U}_i . \vec{U}_o)] = 0 = \vec{\nabla} \times [\vec{U}_i \times (\vec{\nabla} \times \vec{U}_o)] + \vec{\nabla} \times [(\vec{U}_i . \vec{\nabla})\vec{U}_o + (\vec{U}_o . \vec{\nabla})\vec{U}_i], \tag{37}$$

we see from equation (32) that

$$\vec{\omega}_v = \frac{i}{\sigma}\vec{\nabla} \times [\vec{U}_i \times \vec{\omega}_o]. \tag{38}$$

Expressing the velocity field $\vec{U}_i$ of the acoustic wave as

$$\vec{U}_i = i\vec{k}\,\varphi_i e^{i\vec{k}.\vec{r}} \tag{39}$$

We find that the average of $\vec{\omega}_v$ over a wavelength of the sound wave vanishes. Thus, vorticity fluctuations with spatial scales smaller than the wavelength of the sound wave have been generated. The total helicity h of the fluid can be estimated as :

$$h = \frac{1}{2}[(\vec{\omega}_0^* + \vec{\omega}_v^*).(\vec{U}_o + \vec{U}_i + \vec{U}_v) + C.C.] \tag{40}$$

For a general 3D model of the turbulent medium, and an oblique sound wave with $\vec{k} = (k_x, o, k_z)$, the components of $\vec{\omega}_v$ are found to be :

$$\omega_{vx} = \frac{\varphi_i}{\sigma}[i(k_z^2\omega_{ox} - k_x k_z \omega_{oz}) + (k_x \frac{\partial \omega_{ox}}{\partial x} + k_z \frac{\partial \omega_{ox}}{\partial z})]e^{i\vec{k}.\vec{r}}$$

$$\omega_{vy} = \frac{\varphi_i}{\sigma}[i(k_x^2 + k_z^2)\omega_{oy} + (k_x \frac{\partial \omega_{oy}}{\partial x} + k_z \frac{\partial \omega_{oy}}{\partial z})]e^{i\vec{k}.\vec{r}} \tag{41}$$

$$\omega_{vz} = \frac{\varphi_i}{\sigma}[i(k_x^2\omega_{oz} - k_x k_z \omega_{ox}) + (k_x \frac{\partial \omega_{oz}}{\partial x} + k_z \frac{\partial \omega_{oz}}{\partial z})]e^{i\vec{k}.\vec{r}}$$

The components of $\vec{U}_v$ are :

$$U_{vx} = \frac{2\varphi_i}{\sigma}[i(k_x^2 U_{ox} + k_x k_z U_{oz} + k_x \frac{\partial U_{ox}}{\partial x} + k_z \frac{\partial U_{oz}}{\partial x} + \frac{k_z \omega_{oy}}{2}]e^{i\vec{k}.\vec{r}}$$

$$U_{vy} = \frac{2\varphi_i}{\sigma}[k_x \frac{\partial U_{ox}}{\partial y} + k_z \frac{\partial U_{oz}}{\partial y} - \frac{k_z}{2}\omega_{ox} + \frac{k_x \omega_{oz}}{2}]e^{i\vec{k}.\vec{r}} \tag{42}$$

$$U_{vz} = \frac{2\varphi_i}{\sigma}[i(k_x k_z U_{ox} + k_z^2 U_{oz}) + (k_x \frac{\partial U_{ox}}{\partial z} + k_z \frac{\partial U_{oz}}{\partial z} - \frac{k_x \omega_{oy}}{2}]e^{i\vec{k}.\vec{r}}$$

and the helicity h averaged over a wavelength is given by

$$h = h_0 + h_1 \tag{43}$$

where

$$h_0 = \frac{1}{2}(\vec{\omega}_0^*.\vec{U}_o + C.C.) \tag{44}$$

and

$$h_1 = \frac{1}{2}(\vec{\omega}_v^*.\vec{U}_v + C.C)$$

$$= \langle \frac{2\varphi_i^2}{\sigma^2}[\frac{\partial U_{ox}}{\partial x}(k_x^2\frac{\partial \omega_{ox}}{\partial x} + k_xk_z\frac{\partial \omega_{ox}}{\partial z}) + \frac{\partial U_{oz}}{\partial x}(k_xk_z\frac{\partial \omega_{ox}}{\partial x}$$

$$+ k_z^2\frac{\partial \omega_{ox}}{\partial z}) + \frac{w_{oy}}{2}(k_xk_z\frac{\partial \omega_{ox}}{\partial x} + k_x^2\frac{\partial \omega_{ox}}{\partial z}$$

$$+ \frac{\partial U_{ox}}{\partial y}(k_xk_z\frac{\partial \omega_{oy}}{\partial z} + k_x^2\frac{\partial \omega_{oy}}{\partial x}) \tag{45}$$

$$+ \frac{\partial U_{oz}}{\partial y}(k_xk_z\frac{\partial \omega_{oy}}{\partial x} + k_z^2\frac{\partial \omega_{oy}}{\partial z}) + \frac{1}{2}(k_x\omega_{oz} - k_z\omega_{ox}) \times$$

$$\times(k_x\frac{\partial \omega_{oy}}{\partial x} + k_z\frac{\partial \omega_{oy}}{\partial z}) + \frac{\partial U_{ox}}{\partial z}(k_xk_z\frac{\partial \omega_{oz}}{\partial z} + k_x^2\frac{\partial \omega_{oz}}{\partial x})$$

$$+ \frac{\partial U_{oz}}{\partial z}(k_xk_z\frac{\partial \omega_{oz}}{\partial x} + k_z^2\frac{\partial \omega_{oz}}{\partial z}) - \frac{\omega_{oy}}{2}(k_x^2\frac{\partial \omega_{oz}}{\partial x} + k_xk_z\frac{\partial \omega_{oz}}{\partial z})\rangle$$

In order to appreciate the contents of the expressions for vorticity (Equation 41) and helicity (Equation 45), we consider the case of 2D turbulence with $\vec{U}_o = (U_{ox}(x,y), U_{oy}(x,y), o)$, $\vec{\omega}_o = (o,o,\omega_{oz}(x,y))$ and a linearly polarized sound wave with $\vec{k} = (o,o,k_z)$. We find

$$\omega_{vx} = \omega_{vy} = \omega_{vz} = h_0 = h_1 = 0 \tag{46}$$

The case of 2D turbulence and a sound wave with $\vec{k} = (k_x,o,o)$ gives :

$$\omega_{vx} = \omega_{vy} = 0,$$

$$\omega_{vz} = \frac{\varphi_i}{\sigma}[ik_x^2\,\omega_{oz} + k_x\frac{\partial \omega_{oz}}{\partial x}]e^{ik_x x}.$$

The mean enstrophy

$$\langle\vec{\omega}_v.\vec{\omega}_v^*\rangle=\langle \frac{|\vec{U}_i|^2}{c_s^2}\ \omega_{oz}^2(1 + \frac{1}{k_x^2L_\omega^2})\rangle \tag{47}$$

where $L_\omega = [\frac{1}{\omega_{oz}^2}(\frac{\partial \omega_{oz}}{\partial x})^2]^{-\frac{1}{2}}$ is the scale length of variation of ω_{oz}. We see that the enstrophy is enhanced by the square of the Mach number $M^2 = |U_i|^2 / c_s^2$ on scales L_ω much larger than the wavelength of the sound wave and enhanced by $M^2(k_x^2L_\omega^2)^{-1}$ on scales L_ω much smaller than the wavelength. The helicity is still zero.

The case of 2D turbulence and a plane polarized oblique sound wave with $\vec{k} = (k_x,o,k_z)$ gives :

$$\omega_{vx} = -i\frac{\varphi_i}{\sigma}k_x k_z \omega_{oz} e^{i\vec{k}.\vec{r}},$$

$$\omega_{vy} = 0, \tag{48}$$

and

$$\omega_{vz} = \frac{\varphi_i}{\sigma}\,[ik_x^2\omega_{oz} + k_x\frac{\partial\omega_{oz}}{\partial x}\,]e^{i\vec{k}.\vec{r}},$$

The mean enstrophy

$$\langle\vec{\omega}_v.\vec{\omega}_v^*\rangle = \langle\,\frac{|U_{ix}|^2}{c_s^2}\;\omega_{oz}^2(1 + \frac{1}{k^2L_w^2})\,\rangle \tag{49}$$

with

$$k^2 = k_x^2 + k_z^2$$

depends only on the x component of the amplitude of the sound wave. The helicity h_l is still zero. Thus the generation of helicity needs a full 3D turbulence.

Let us, therefore, consider a full blown 3D turbulence with $\vec{U}_o = (U_{ox}(x,y,z), U_{oy}(x,y,z), U_{oz}(x,y,z))$ and $\vec{\omega}_o = (\omega_{ox}, \omega_{oy}, \omega_{oz})$ and a linearly polarized sound wave with $\vec{k} = (o,o,k_z)$ we find :

$$\omega_{vx} = \frac{\varphi_i}{\sigma}\,[ik_z^2\omega_{ox} + k_z\frac{\partial\omega_{ox}}{\partial z}\,]e^{ik_z z}$$

$$\omega_{vy} = \frac{\varphi_i}{\sigma}\,[ik_z^2\omega_{oy} + k_z\frac{\partial\omega_{oy}}{\partial z}\,]e^{ik_z z} \tag{50}$$

$$\omega_{vz} = \frac{\varphi_i}{\sigma}\,[k_z\frac{\partial\omega_{oz}}{\partial z}]e^{ik_z z}$$

and

$$h_l = \langle\,\frac{2\varphi_i^2k_z^2}{\sigma^2}[\frac{\partial U_{oz}}{\partial x}\frac{\partial\omega_{ox}}{\partial z} + \frac{\omega_{oy}}{2}\frac{\partial\omega_{ox}}{\partial z} + \frac{\partial U_{oz}}{\partial y}\frac{\partial\omega_{oy}}{\partial z}\,]$$

$$- \frac{\omega_{ox}}{2}\frac{\partial\omega_{oy}}{\partial z} + \frac{\partial U_{oz}}{\partial z}\frac{\partial\omega_{oz}}{\partial z}]\,\rangle \tag{51}$$

which further simplifies to

$$h_l = \frac{2\,|\,\vec{U}_i\,|^2}{\sigma^2}\,\langle[S_{xz}\frac{\partial\omega_{ox}}{\partial z} + S_{yz}\frac{\partial\omega_{oy}}{\partial z} + S_{zz}\frac{\partial\omega_{oz}}{\partial z}\,]\,\rangle \tag{52}$$

where

$$S_{ij} = \frac{1}{2}\left[\frac{\partial U_{oi}}{\partial x_j} + \frac{\partial U_{oj}}{\partial x_i}\right]$$

are the strain-rate fluctuations-the symmetric part of the deformation rate $(\partial U_{oi} / \partial x_j)$.

We recall that in the equation for vorticity :

$$\frac{\partial \omega_i}{\partial t} + U_j \frac{\partial \omega_i}{\partial x_j} = \omega_j S_{ij} \tag{53}$$

the term $\omega_j S_{ij}$ represents amplification and rotation of the vorticity vector by the strain rate S_{ij}. We note that the helicity h_1 depends on the averaged straining of the vorticity derivative along the direction of propagation of the sound wave. The helicity is generated on length-scales comparable or larger than the wavelength of the acoustic field. We could ask, if in the absence of full blown turbulence, a shear flow could couple with the sound wave to produce helicity. We find that a shear flow $U_{ox}(y,z)$ and a plane polarized sound wave $\vec{k} = (k_x, o, k_z)$ can support the amplification of helicity. For this case, we find from equation (45)

$$h_1 = \frac{|\vec{U}_i|^2}{c_s^2} \frac{k_x k_z}{k^4} \left\langle \frac{\partial U_{ox}}{\partial y} \frac{\partial^2 U_{ox}}{\partial z^2} - \frac{\partial U_{ox}}{\partial z} \frac{\partial^2 U_{ox}}{\partial y \partial z} \right\rangle \tag{54}$$

This case is an example where the initial non-helical flow couples with an irrotational flow $\vec{U}_i$ to produce a helical flow $\vec{U}_v$ with helicity h_1 given by equation (54). The magnitude of the helicity increases as square of the Mach number, and more helicity is produced if the wavelength of the acoustic field is larger than the spatial scale of the ambient turbulence or the shear flow.

Conclusion

We have demonstrated that a nonlinear phenomenon supports interrelationships among the vorticity, the helicity and the mass density. This is significant as, often, while discussing gravitational collapse on various astrophysical scales, the vortical and helical motions are either ignored or treated in a perturbative manner whereby they lose or have their roles diminished. We have demonstrated that a density growth is necessarily accompanied by an helicity enhancement and indeed, it is possible to convert a part of the energy of the acoustic field into that of helical motion. Since helicity is so crucial to the formation of large scale magnetic or kinetic fields, it is gratifying that more of it is produced where it is needed more i.e. in enhanced density regions. Such regions in the form of stars, clusters of stars, galaxies and clusters of galaxies reveal a hierarchical organization throughout the universe. Of course, the seed vorticity is needed, as is the seed mass density for building up large scale structures. In fact, the need for kinetic helicity for formation of a large scale structure is no different from that for generation of a large scale magnetic field. We have further discussed two mechanisms to show how irrotational flows could be converted into rotational and helical flows. We have given an example of a helical flow resulting from an initial vortical but non-helical flow. In various astrophysical situations such as the aftermath of big bang, the collisions of galaxies, the explosion of stars or the turbulent convection zones of stars, one expects the existence of such sheared or turbulent flows on the one hand and acoustical fields, on the other, in a variety of compressible media. It is tempting to speculate that once the kinetic helicity is generated and later amplified in a collapsing compressible medium, this will also be accompained by the creation of magnetic helicity and the turbulent magnetic field which could later be ordered on scales comparable with that of the acoustic field.

We have attempted to demonstrate that the longitudinal modes can generate kinetic helicity on scales of the order of the acoustic wavelengths.

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