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Hori's method to non-linear coupled oscillators and applications to restricted problem

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Abstract. Hori's method of solving coupled non-linear differential equations has been used to solve restricted problem of three bodies and non-resonance case has been discussed. For a certain set of values of the parameters, the chaotic motion has been observed which is confirmed by a phase plane diagram and time series analysis.

Key words : Lie technique, non-linear, chaotic motion, restricted three body.

1. Introduction

Szebehely (1967) and Contopoulos (1963) used canonical transformation to solve restricted problem of three bodies. Hori (1966, 1967) has developed a method of canonical transformation by using Lie theorem (Lie 1888), which can be applied to the theory of general perturbation to solve coupled non-linear equations. Thereby, he also discussed resonance and non-resonance cases for third order secular perturbations. In the present work, we have used Hori's technique to study coupled nonlinear equations of the restricted problem of three bodies.

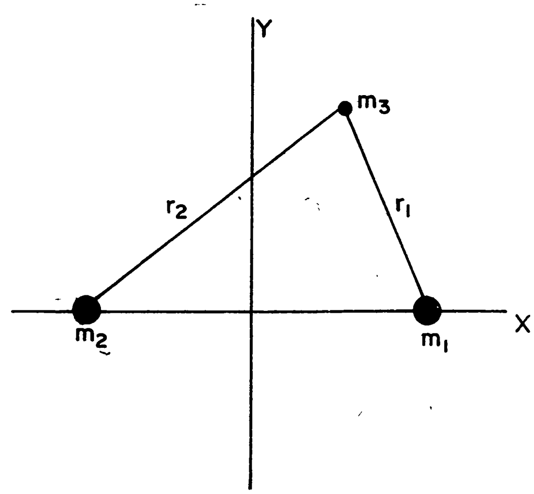

Figure 1. Restricted Three Body problem

These equations of motion, (Szebehely, 1967), are

$$\ddot{X} - \dot{Y} = \frac{\partial \Omega}{\partial X}$$
$$\ddot{Y} + 2\ddot{X} = \frac{\partial \Omega}{\partial Y} \qquad (1)$$

where $\Omega = \frac{1}{2}\left[(1-\mu)r_1^2 + \mu r_2^2\right] + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}$,

$$r_1^2 = (X-\mu)^2+Y^2, \quad r_2^2 = (X+1-\mu)^2+Y^2, \quad \mu = \frac{m_2}{m_1+m_2},$$

and dots $(\cdot)$ represent differentiation w.r.t. time.

Equation (1) implies, the Jacobi integral $\dot{X}^2 + \dot{Y}^2 = 2\Omega - C$.

Expanding (1), as in Ragos and Zagouras (1998), we obtain

$$\ddot{x} - \dot{y} = p_1x+p_2y+p_3x^2+p_4y^2+p_5xy$$
$$\ddot{y} + \dot{x} = q_1x+q_2y+q_3x^2+q_4y^2+q_5xy. \qquad (2)$$

Here p_1, p_2, p_3, p_4, p_5, q_1, q_2, q_3, q_4 and q_5 are constants. Using substitution $x \to x_1$, $y \to x_2$, $\dot{x} \to y_1$, $\dot{y} \to y_2$ and applying the canonical transformation developed by Hori (1966, 1967) in (2), we get equivalent system

$$\frac{dx_j}{dt} = \frac{\partial H}{\partial y_j}, \quad \frac{dy_j}{dt} = -\frac{\partial H}{\partial x_j} \quad (j=1,2) \qquad (3)$$

with Hamiltonian $H = H_o + H_1$ where $H_1 = y_1x_2 - y_2x_1$ and

$$H_o = \frac{1}{2}\left(y_1^2+y_2^2\right) - \frac{1}{2}\left(p_1x_1^2+q_2x_2^2\right) - \frac{1}{3}\left(p_3x_1^3+q_4x_2^3\right) - (p_2+q_1)\,x_1\,x_2$$
$$- \left(p_4+\frac{q_5}{2}\right)x_2^2\,x_1 - \left(\frac{p_5}{2}+q_3\right)x_1^2\,x_2.$$

Now, we use the following transformation

$$x_j = \xi_j + \frac{\partial S_1}{\partial \eta_j} + \frac{\partial S_2}{\partial \eta_j} + \frac{1}{2}\left\{\frac{\partial S_1}{\partial \eta_j}, S_1\right\} + O(\varepsilon^3)$$
$$y_j = \eta_j - \frac{\partial S_1}{\partial \xi_j} - \frac{\partial S_2}{\partial \xi_j} - \frac{1}{2}\left\{\frac{\partial S_1}{\partial \xi_j}, S_1\right\} + O(\varepsilon^3). \qquad (4)$$

Here, S_1, S_2 are functions of (ξ_j, η_j), $(j=1,2)$ and $\{\ ,\ \}$ is represents the Poisson's bracket.

The equation of motion in ξ and η can be written as

$$\frac{d\xi_j}{dt} = \frac{\partial H^*}{\partial \eta_j}, \qquad \frac{d\eta_j}{dt} = -\frac{\partial H^*}{\partial \xi_j} \quad (j=1,2) \tag{5}$$

The solution of (5) is given by

$$\xi_j = c_j \, \mathrm{Cos}(K_j\tau + c_j'), \ \eta_j = -K_j c_j \, \mathrm{Sin}(K_j\tau + c_j'), \qquad (j=1,2)$$

where K_1, K_2, c_1, c_2, c_1', c_2' are constants.

2. Non-resonance case

When K_1 is not commensurable with K_2, then we get

$$x_1 = \xi_1 + \frac{\varepsilon(2K_1 - K_2)\eta_2}{K_1 (K_1^2 - K_2^2)} + O(\varepsilon^2), \quad x_2 = \xi_2 + \frac{\varepsilon(2K_1 - K_2)\eta_1}{K_1 (K_1 - K_2)} + O(\varepsilon^2)$$

3. Time series analysis

To study the evolution of the system, we have done time series analysis. The result has been indicated in Fig. 2, which shows, after the transient part, motion is chaotic. This is in good agreement with that by Smith and Szebeheley (1993). They have drawn Poincare's surface of section to show chaotic motion. Fig. 3 is a phase plane plot where again we observe the trajectories are random or chaotic.

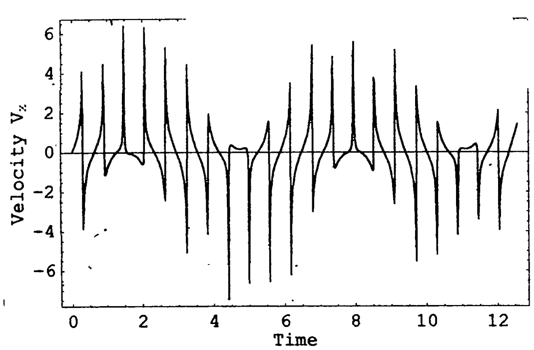

Figure 2. Time series analysis

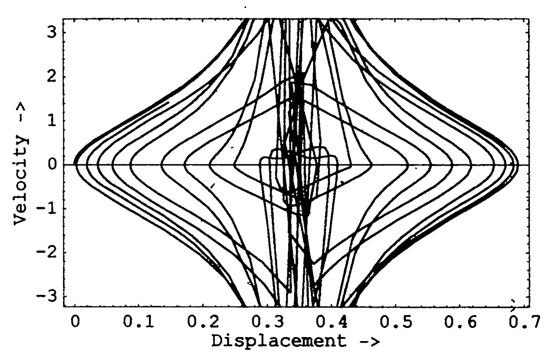

Figure 3. Phase plane trajectories

4. Conclusion

Through this study, solutions for non-resonance case has been obtained. It has been observed that the Lie technique is a powerful tool to investigate problems comprising coupled non-linear equations. We have applied this technique to solve restricted problem. However, one may apply it to solve many other problems also. The evolutionary equations indicate chaotic motion for a particular set of values of the parameters involved. We have observed chaotic motion for $0 < \mu \leq \frac{1}{2}$, which is in accordance with the established principles of restricted problem. This is a clear indication that our solar system is not the paradigm of order and regularity.

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