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SOLAR AND STELLAR CONVECTION

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Abstract

The flow field in stellar convection zones, characterised by high Rayleigh number, low Prandtl number and large Reynolds number, is time dependent and turbulent, consisting of eddies having a range of scale lengths upto the size of the convection zone The mixing lengths approximation has been the most widely used formalism to model such a flow, in spite of all its shortcomings It is possible to test the consistency of this approach as well as to get a good estimate of the mixing length in the framework of linear stability analysis The salient features of stellar convection distinguishing it from the laboratory flows are discussed The underlying physical ideas behind the local and non-local mixing-length theories and tests of the resultant models are outlined Some of the results from other attempts to treat stellar convection using truncated modal analysis and finite difference schemes are briefly summarised

I Onset of Convection in Stars

Energy is transported inside stars mainly by two processes (i) radiative, where photon is scattered or absorbed and reemitted by matter at random until it escapes into space (ii) convective, by the explicit motion of parcels of fluid which carry energy to the cooler outer parts of the star

The favourable mode of energy transport is determined by the thermal structure of the star While analysing the stellar structure it is useful to think in terms of three time scales

1 The nuclear time scale, τ_n, over which the chemical composition of the star changes ap preciably due to nuclear reactions ($\tau_n \sim 10^{10}$ yr for the sun)

2 The Kelvin time scale τ_k, which governs the response time of the star to changing thermal reservoir ($\tau_k \sim 10^7$ yr, for the sun)

3 The dynamical time scale, τ_d, which is the relaxation time when the mechanical equilibrium of the star is disturbed ($\tau_d \sim 1$ hr, for the sun)

The star hould satisfy two conditions in order to remain in a steady state over time period comparable to its nuclear time scale

a Mechanical equilibrium which demands that the outward force of pressure is balanced by the inward force due to gravity everywhere inside the star i e

$$d P/d r = \rho GM (r) / r^2 \tag{1}$$

where P is the total pressure, ρ is the density, r, radius, G, gravitational constant and M(r) is the mass enclosed within the radius r

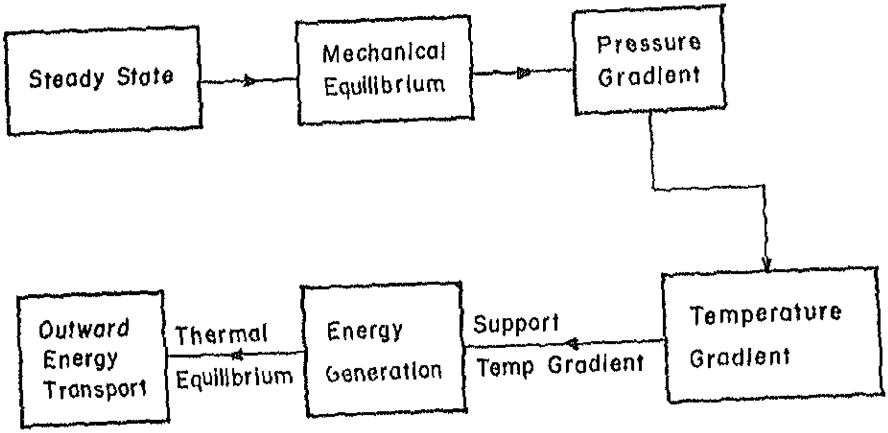

Fig.1 A portrayal of the physical ideas determining the tellar structure

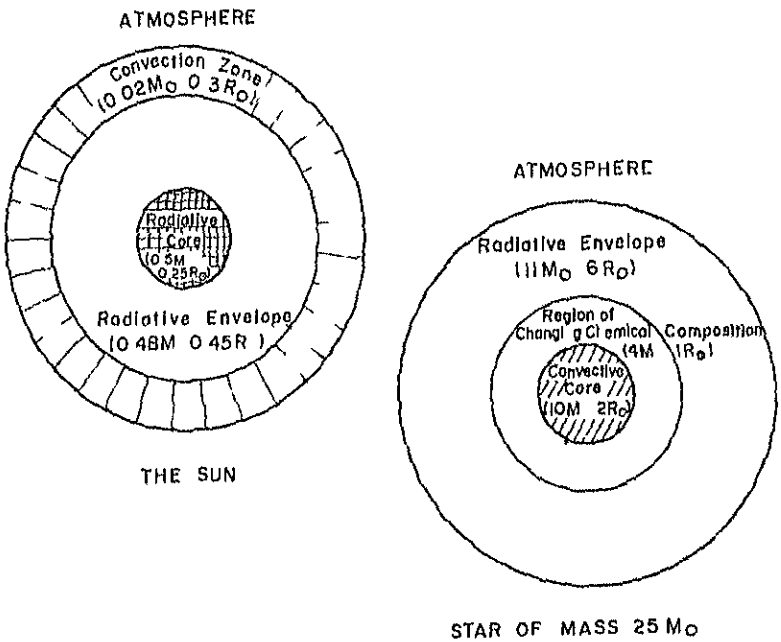

Fig.2 Structure of a solar type star contrasted with that of a massive star in similar evolutionary status

b Thermal equilibrium requiring that the energy generated in the stellar intrior is transported outwards i.e.

$$d L(r)/d r = 4\pi \epsilon \rho r^2 \qquad (2)$$

where L(r) is the net energy generated within the radius r and ϵ is the rate of energy generation per unit mass per unit time. Figure 1 gives a physical idea of the factors governing the stellar structure.

Generally the energy generated in the stellar interior is transported outwards by radiation. However under the following two circumstances this process alone cannot transport the entire energy.

1. When there is a large nuclear energy generation rate.
2. When hydrogen (or some time even helium) is partially ionized, the opacity of the matter in the stellar envelope increases sharply inward.

In the above cases the temperature gradient becomes unstable against convection. In stars once convection sets in the medium becomes turbulent and eddies having a range of size upto the thickness of the convection zone are excited. They transport energy in the convective layer. In massive stars the core is convective while the envelope is radiative or convective depending upon the evolutionary status. However, the envelope of the less massive stars has a convection zone. A schematic representation of the structure of typical solar type star and massive star is given in Figure 2. The solar type stars have a convection zone starting almost from the photosphere. The granulation (small scale intensity pattern consisting of bright granules separated by dark lanes) and supergranulation (cells having large horizontal outflow) are considered to be the manifestation of convection in the subphotospheric layers of the sun (cf. notes). Main properties of the solar convection zone are given in Table 1.

Table 1

Properties of Solar Convection Zone

Outer radius	1R$_\odot$ (7 × 10^5 km)
Inner radius	~0.7 R$_\odot$ (5 × 10^5 km)
Depth of the Convection Zone	~0.3 R$_\odot$ (2 × 10^5 km)
Temperature at the base	~2 × 10^6 K
Density at the base	~0.2 gm/cc
Observed Convective Velocity at the Photosphere	~1.3 km/sec

II Condition for Radiative Equilibrium

A region of the star is said to be in radiative equilibrium when the energy transport is entirely by radiation. This happens in a region where convection is absent. When is the temperature gradient unstable against the onset of convection?

Consier the density profile of a star of uniform chemical composition given in Figure 3a (from Cox and Giuli, 1968). Let a fluid element be disturbed slightly so that its density decreases (represented by the downward arrow). The fluid element moves outwards quasistatically due to the buoyancy force,

$$F = \Delta \rho g$$

(where $\Delta\rho$ is the decrease in density and g is the acceleration due to gravity). If it does not exchange heat with the surroundings, its trajectory can be determined by noting that

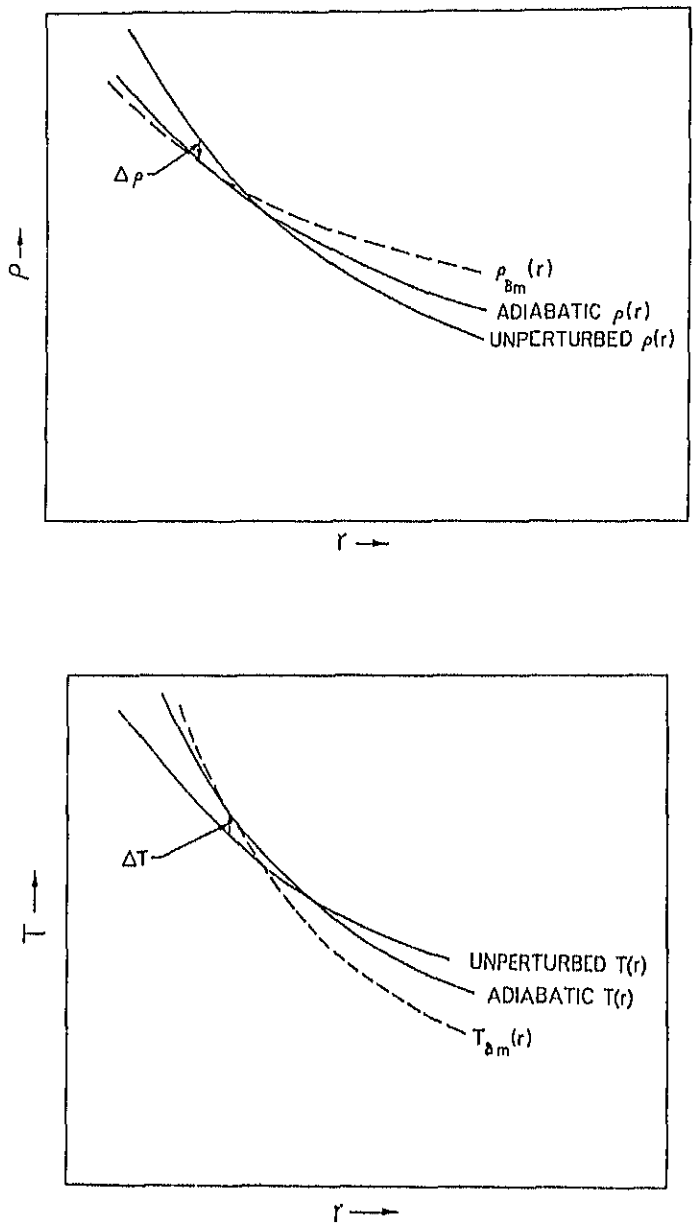

Fig.3 Schematic density and temperature profile in a convectively stable layer (from Cox and Giuli, 1968)

a the entropy of the mobile fluid element is conserved (adiabatic motion)

b it maintains pressure equilibrium with its new surroundings at each instance of time (in view of the smallness of the dynamical time) This trajectory is displayed by the dashed curve passing through the end of the arrow The fluid parcel can attain a new equilibrium if its trajectory crosses the curve denoting the steady state profile since its density as well as pressure will be equal to the equilibrium value at the cross over point In practice the fluid will oscillate about its new equilibrium position as a gravity mode (cf Antia, this proceedings) until its excess energy is dissipated The heat exchange only makes the system more stable Note that the curve representing the actual trajectory (denoted as perturbed) meets the equilibrium profile earlier

Hence from physical arguments we conclude that the density stratification is stable against convection if the outward density gradient is decreasing more steeply compared to its adiabatic value, i e

$$(d\rho/d\,r)_{pert} > (d\rho/d\,r)_{ad} > (d\rho/d\,r)_{unpert} \tag{3}$$

Using the equation of state for the medium the corresponding relation for the temperature profile can be obtained (but it is not so basic as the condition on density) Figure 3b portrays the physical situation Noting that $d\,T/d\,r$ is always negative, the following mathematical relation can be derived

$$\left[\frac{d\ \ln\ T}{d\ \ln\ P}\right]_{ad} > \left[\frac{d\ \ln\ T}{d\ \ln\ P}\right]_{eqbm} \tag{4}$$

(We shall denote the logarithmic temperature gradient with respect to pressure [d ln T/ d ln P] by the symbol ∇)

This is the famous Schwarzschild criterion for stability against convection One can convince oneself that the stratification is unstable against convection if the density profile is flatter than the corresponding adiabatic profile, or,

$$\nabla > \nabla_{ad}$$

(For mathematical treatment of the condition for radiative equilibrium refer to notes 2)

In the presence of a chemical composition gradient the corresponding condition can be derived to take the form (Ledoux, 1947)

$$\left[\frac{d\ln T}{d\ln P}\right]_{eqbm} < \left[\frac{d\ln T}{d\ln P}\right]_{ad} - \left[\frac{(\partial\ \ln P/\partial\ \ln\mu)_{T,\rho}}{(\partial\ \ln P/\partial\ \ln T)_{\mu\,\rho}}\right] \left[\frac{d\ln\mu}{d\ln P}\right] \tag{5}$$

(where μ is the mean molecular weight), if we assume that the chemical diffusion time scale for the species is large compared to the dynamical time However, in view of the other possible instabilities $\nabla < \nabla_{ad}$ is probably the more appropriate criterion even in this case as indicated by theoretical considerations (cf Kato, 1966) as well as detailed numerical treatment of stellar convection (cf Xiong, 1981)

III Salient features of Stellar Convection

Let us define a few characteristic numbers of the convection zone

1 The Rayleigh number which can be thought of as the ratio of the forces due to buoyancy and viscosity,

$$R = \frac{g\, d^3}{\nu\, k_{rad}} \frac{\Delta T}{T} \tag{6}$$

where d is the typical length scale (or a scale height) of the convection zone, ΔT is the excess temperature diffeence, ν is the kinematic viscosity and k_{rad} is the thermometric conductivity of the medium In typical stellar convection zones, $R \sim 10^{20}$ 10^{24}

2 Prandtl number which is the ratio of the kinematic viscosity to the thermal diffusity

$$\sigma = \frac{\nu}{k_{rad}} \tag{7}$$

The stars generally have very low Prandtl number, typically of the order of 10^{-7} to 10^{-9} as compared to of the order of unity for the terrestrial fluids (σ should not be confused with the turbulent Prandtl number, which is a measure of the efficiency of momentum exchange between the turbulent eddies as compared to the efficiency of their thermal exchange)

3 Reynolds number which is the ratio of efficiency of momentum exchange between turbulent eddies to the momentum exchange by molecular viscosity

$$R_e = \frac{W\, L}{\nu} \tag{8}$$

where W is the typical speed of the fluid and L, its scale length In the stellar convection zones, R_e is the order of 10^{10}

From theoretical studies and laboratory experiments it is known that a stratified medium becomes convectively unstable when the Reyleigh number approaches $\sim 10^3$ The the exact value depends on the boundary condition but it is not sensitive to the Prandtl number The critical Rayleigh number at which convection sets in and the wave number of the most unstable convective mode (see, notes 1 for the discussion of normal mode) at the critical Rayleigh number are displayed in Table 2 As the Rayleigh number is

Table 2

Onset of Convection in Plane Parallel Fluid Layer for various boundary conditions (Chandrasekhar, 1961)

Boundary condition	Critical Rayleigh number	Wavenumber of maximally growing convective mode for a layer of unit depth
Free Free	657 5	2 22
Rigid Free	1100 7	2 68
Rigid Rigid	1707 8	3 12

increased convection becomes progressively more unsteady When the Prandtl number is in the viscinity of 0 7 (as in the case of dry air) the development of instability is as follows (cf Spiegel 1971)

a	$R \leq 22\,600$	steady cellular pattern (rolls)
b	$R \sim 50\,000$	time dependent but periodic flow
c	$R \sim 10^6$	aperiodic motion

When the Rayleigh number is far greater than 10^6 fully developed turbulence sets in

However when the Prandtl number is very small the transition from steady state convection to turbulence is abrupt, with the absence of above three intermediated stages Also the flow pattern is different for two and three dimensional convection though the time averaged convective heat flux is nearly the same (Weiss, 1976)

From these results it is clear that stellar convection is time dependent, aperiodic and highly turbulent A range of convective modes having vertical length scale comparable to the depth of the convection zone are excited in the convection layer In a solar type star typically 10^8 modes persist, representing eddies of size smaller than a few hundred kilometers upto the size of the thickness of the convection zone They carry almost the entire heat flux in the bulk of the convective layer At each point it is possible to divide them into energy storing, energy tansporting and energy dissipating eddies (Xiong, 1981)

Apart from the turbulent nature of the fluid motion, three more complications arise while investigating the stellar convection

1 The convection zone extends over several scale heights A fluid element having a density of, say, 0 2 gm/cc while starting from the base of the convection zone will end up having a density of only 10^{-7} gm/cc when it is stopped in the atmosphere of a solar type star Clearly the effects of compressibility cannot be completely neglected

2 The characteristic quantities like the Rayleigh number and Prandtl number are not a priori known, instead they are determined by the stellar structure Consequently the problem is one of dynamic rather than kinematics in contrast with the other problem of turbulence Hence the laboratory studies giving cell pattern or convective heat flux in terms of the Rayleigh number and Prandtl number are not very informative for the steller convection

3 The non linear advective term (see notes 3) in the differential equations governing the convective flow bring in two additional complications

(i) They control the growth rate of instability
(ii) They couple various modes introducing new length scales

On account of these constraints a full hydrodynamical calculation of the time dependent compressible convection covering a range of length scales is not feasible There have been mainly three kinds of approaches to study the convective flow a) Mixing length theories, b) Truncated modal expansion and c) Direct two dimensional or three dimensional numerical simulation Since most of the stellar convection zone models have been computed using some form of the mixing length approximation the main features of this treatment will be outlined in the next section

IV The Local Mixing-Length Theory

Mathematically the mixing length theory is a single mode approximation of the stellar convection described in terms of a characteristic length scale of the stratified fluid The scale length, in general, is a constant multiple of some local scale height (e g the pressure scale height,

$$H_p = \left| \frac{d\, r}{d \ln P} \right| \Bigg)$$

or the distance to a boundary Physically the mixing length approximation is developed along the line of the kinetic theory of gases The convection zone is supposed to be composed of turbulent eddies and the heat and momentum exchange between them is treated in a manner similar to the transport phenomena in gases Each eddy moves through a characteristic distance, L, called the mixing length with a characteristic velocity, W before it merges with the ambient medium, depositing the excess heat and momentum In the conventional theories both the size of the mean eddy and its mean free path are assumed to be the same The typical value of the mixing length for the Sun (taking $L \simeq 2H_p$) changes from ~300 km at the top of the convection zone to ~10^5 km at its base Hence the wavelength of the average mode (or the size of the representative eddy) changes by nearly two orders of magnitude Thi essentially represent the progre sive transfer of energy from larger to smaller eddies In a crude sense the mixing length approximation also takes into account the compressibility of the medium through the variation of the mixing length

The present form of the mixing length theories can be divided into two groups local and non local theories In the local theory the mean velocity of the convective eddies, the convective heat flux and similar variables are specified in terms of the physi cal variables and the mixing length at that point On the other hand, in the non local theories the derivatives of the velocity or temperature fluctuations, their correlation and the convective energy transport are determined from the local quantities The various models differ in the way the geometry of the representative eddy, the entrain ment erosion of the eddy, the efficiency of the energy transport and such details are handled The gross features of Vitense s mixing length model (Vitense, 1953, Bohm Vitense, 1958) will be outlined below

In Vitense's model a parcel of fluid or turbulent eddy travels through a distance l before it gives up its excess of heat to the surroundings The size of the eddy is also taken to be L though in her works the geometry of the cell is considered flexible As we have already noted in section II the outward moving eddy becomes progressively lighter compared to its surroundings and consequently it experiences larger buoyancy force In the model of Vitense the buoyancy force is assumed to increase linearly with the distance of flight It continuously radiates energy during its flight in view of its excess temperature over the surroundings After traversing through the distance l, all the excess energy is given up to the medium and the parcel loses its identity The energy deposited at th end of the flight represents the convective flux transported by the eddy Using a still simpler picture (which is valid except near the boundaries of the convection zone or in the highly superdiabatic layers regions where the temperature gradient is much steeper than its adiabatic value) w can derive the following expressions for the velocity and heat flux

Acceleration of the parcel of fluid is given by,

$$r = g\frac{\Delta\rho}{\rho} \tag{9}$$

where $\Delta\rho$ is the density excess over the surrounding medium Since the buoyancy force is assumed to be linearly proportional to the distance traversed, work done by the buoyancy force over a distance L can be computed to be,

$$U = g\frac{\Delta\rho}{\rho}\frac{L}{2}$$

Fractional density excess of the parcel of fluid after the flight is,

$$\frac{\Delta\rho}{\rho} \simeq \int\left[\frac{d\rho}{dr} \quad \left(\frac{d\rho}{dr}\right)_{ad}\right] dr$$

$$\simeq Q\left[\frac{d\ln T}{dr} \quad \left(\frac{d\ln T}{dr}\right)_{ad}\right] L \tag{10}$$

where

$$Q - \left[\frac{d\ln\rho}{d\ln T}\right]_P$$

Hence the work done is,

$$U \quad \frac{1}{2} Q g \frac{L^2}{H_p} \ (\nabla - \ \nabla_{ad}) \tag{11}$$

(Recall the definition of $\nabla = \frac{d \ln T}{d \ln P}$)

A suming that a fraction of this energy β is converted into the kinetic energy of vertical motion

$$W^2 = \ \beta Q g \frac{L}{H_p} (\nabla \quad \nabla_{ad}) \tag{12}$$

Similarly the convective heat flux is obtained from the excess energy of the eddy just before its merger with the ambient medium The expression for the convective energy flux is found to be

$$F^c \ = \ \alpha \rho W C_p L (\nabla \quad \nabla_{ad}) \frac{T}{H_p} \tag{13}$$

where C_p is the specific heat at constant pressure and α is an efficiency factor or order unity

The stellar structure can be computed by solving equations (12) and (13) together with the usual equations of stellar structure (cf notes 4) The reader is referred to the references mentioned in notes 4 for more detailed treatment of the local mixing length theory Main features of a typical solar convection zone model obtained using the local mixing length approximation is given in Table 3

Table 3

Solar Convection Zone model using the local mixing length theory ignoring the effects of turbulent pressure (from Narasimha 1983)

Modern parameters mixing length*	z + 459 km
α	1/4
β	1/8
Maximum convective velocity	3 85 km/sec
$(\nabla \quad \nabla_{ad})_{max}$	0 33
Depth of the convection zone	1 96 x 10^5 km
Physical quantities at the base of the convection zone	
Temperature	2 18 x 10^6 K
Density	0 25 gm/cc

* z is the depth from the photosphere

V Tests of the Mixing-Length Approximation

From the discussions of preceding section, it is clear that (a) turbulent eddies scanning length scales over more than four orders of magnitude in a compressible medium have been modelled in terms of a single scale length of the convection zone (b) The scale length itself is prescribed externally

Basically we are asking two kinds of questions

1 The practical question how do the models stand against the observable tests ?

2 The theoretical problem is the mixing length approximation theoretically sound or atleast self consistent ?

V 1 Observable Tests of the Convection Zone Models

The sun provides a good testing ground for the convection zone models owing to its proximity to the earth and the ideal location of its convection zone A successful model should explain the following observational results

1 The preferred length and time scales of granules and supergranules (cf Bhatnagar, Sivaraman in this Workshop, also notes 5)
2 The frequency of solar oscillations of large wavenumber (cf Antia, in this Work shop) and the observed instability band
3 The observed convective velocity profile in the solar atmosphere (Keil 1980)

We shall summarise the analyses of Antia, Chitre and Pandey (1981), Antia, Chitre and Narasimha (1983) and Narasimha and Antia (1982) pertaining to the linear convective modes excited in the solar envelope Essentially they constructed convection zone models in the mixing length appoximation and then studied the spectrum of turbulent eddies by appealing to the equations of fluid dynamics They constructed a series of the models of the solar convection zone appealing to the local mixing length formalism for a range of values of the mixing length parameters The normal mode analysis was carried out by examining the basic equations of fluid dynamics (conservation of mass, momentum and energy) and heat transport in the linear approximation (cf notes 1) The linear stability analysis for the convective modes was carried out after including the mechanical and thermal effects of the background turbulence on the mode parametrized through turbulent transport coefficients (turbulent pressure, turbulent viscosity and turbulent conductivity) To this extent the analysis should be termed a quasi linear approximation The growth rate of the convective modes as a function of the horizontal harmonic number (cf notes 1) is displayed in Figure 4 The growth rate of the modes does not show any preferred length scale when no exchange of heat or momentum takes place As the radiative and turbulent proceses are included the wavelength of the maximally growing convective mode progressively increases It is the cumulative effect of the various interactions and dissipative processes that determines the most rapidly growing modes As is evident from Figure 5, for a suitable choice of the turbulent Prandtl number (which is the ratio of the turbulent viscosity to the turbulent conductivity It is a free parameter in the analysis though in principle the mixing length model should determine this quantity) the growth rate of the modes as a function of the horizontal wavelength exhibits two peaks The existence of these peaks is an evidence for the preferred modes in the solar convection zone The observed length scales and time scales of granules and supergranules are given in Table 4a, while the horizontal wavelength and the e folding time (inverse of the growth rate) of the computed peaks of convective modes are given in Table 4b Thus for a narrow range of the parameters of the mixing length models, they were able to show that the models do explain the observed features of solar convection It turned out that the same models also give correct frequencies of the solar oscillations trapped in the convection zone From the results of these numerical computations the following inferences could be drawn

a The wavelength of the maximally growing convective mode is governed by its radiative exchange with the ambient medium as well as the momentum and heat exchange with the background eddies
b The horizontal momentum exchange is a crucial factor governing the existence of two preferred length scales
c The results are somewhat sensitive to the profile of the superadiabatic temperature gradient in the top layers of the convection zone where the degree of superadiabaticity is high

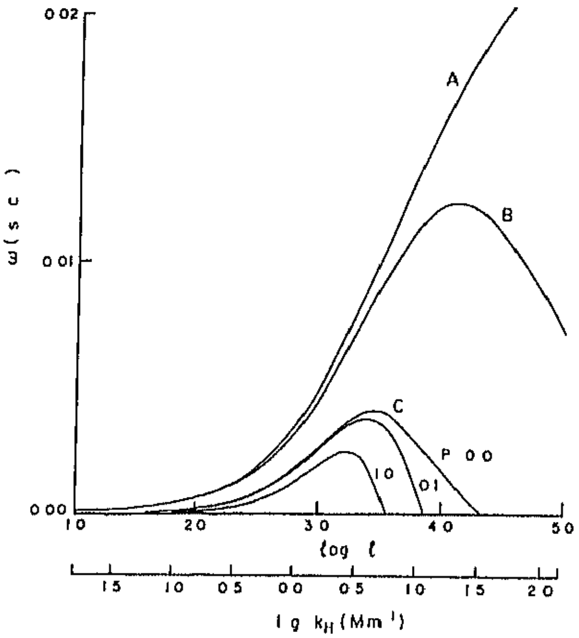

Fig.4. The growth rate of the fundamental mode excited in the solar convection zone is shown as a function of log ℓ for various cases (A) Non dissipative case (B) Radiative exchange only (C) Radiative and turbulent energy exchange but no viscosity The other two curves refer to the case of turbulent viscosity with turbulent Prandtl number σ_t = 0 1 and 1 0 respectively The horizontal wavenumber is marked at the bottom

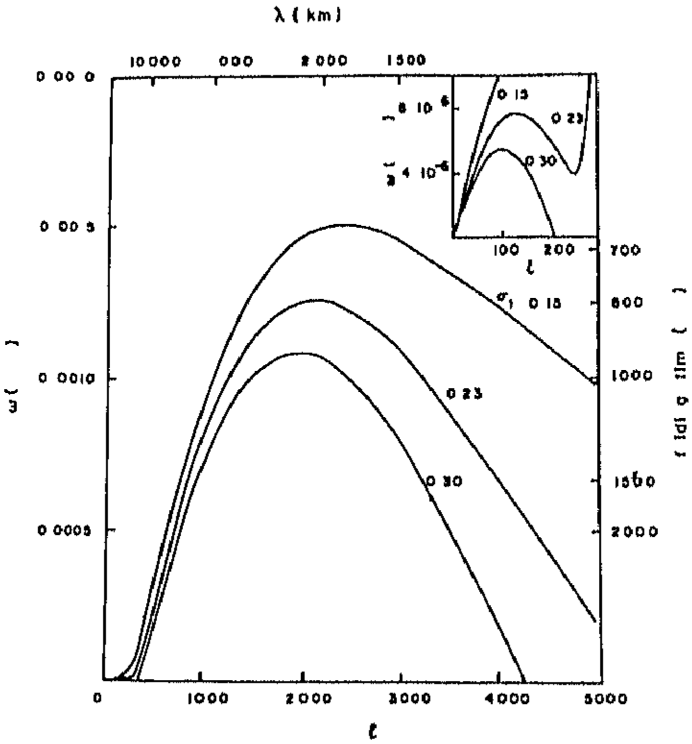

Fig 5 The growth rate of the fundamental convective mode as a function of ℓ for various values of turbulent Prandtl number σ_t for a model including the effects of turbulent pressure ($P_t = \frac{3}{4}\rho w^2$), The region in the range of ℓ from 0 to 300 is shown enlarged in the inset

Table 4a

Observed properties of the Solar Convective Cells

	Granulation	Supergranulation
Characteristic size	1050 km	32 000 km
Average life time	8 10 min	1 day

Table 4b

Properties of the maximally growing fundamental mode excited in the Solar Convection Zone Model

(Tubrulent Prandtl number = 0 23 Turbulent pressure = $\frac{3}{4}$ ρW^2)

	Primary maximum	Secondary maximum
Wavelength	2100 km	33 000 km
e folding time	13 min	36 hr

V 2 Consistency of the Mixing-Length Theory

Suppose we are given a model of the convection zone of a star constructed using the mixing length theory We can study the dynamics of convection by normal mode analysis after examining all the convective modes as described in the previous subsection Evidently the entire spectrum of the turbulent eddies responsible for the transport of energy flux can be obtained in term of the normal modes The question we would like to address the Ledoux problem (Ledoux, et al, 1961) is the following Is the convection zone model consistent with the model incorporating the dynamics of convection ? Alternatively, are the model convective flux and velocity profiles consistent with the respective profile obtained by a superposition of the contributions from individual convective modes ? Hart (1973) convincingly demonstrated that no superposition of linear adiabatic inviscid modes (i e modes obtained by neglecting the momentum as well as heat exchange with the surroundings) can even remotely reproduce the model heat flux because the amplitude of all the modes peaks in the most superadiabatic layer (the region where (∇ ∇_{ad}) is the largest) His result is not surprising because if the percolation of energy from the larger eddies to the smaller ones is neglected, all the eddies experience only the buoyancy force which is normally maximum near the most superadiabatic point However, in order to establish the hierarchy of the turbulent eddies one should take proper account of the thermal and momentum exchange between the convective eddies Narasimha and Antia (1982) who modelled this interaction through turbulent transport coefficients did find that at each depth we can indeed distinguish between the energy storing, energy carrying and energy dissipating eddies The energy dissipating eddy in one layer gradually transforms into the energy transporting eddy in another layer, as is evident from the Figure 6 In this figure the normalised convective energy flux transported by individual linear modes, given by

$$F_{\ell}^{c} = a_{\ell}^{2} < \rho_{0} T_{0} v_{\ell} s_{\ell} + P_{\ell} v_{\ell} > \qquad (14)$$

is plotted against the logarithm of the equilibrium pressure Here a_ℓ is the normalisation constant, ρ_0 and T_0 are the density and temperature while v_ℓ, s_ℓ and P_ℓ are respectively the radial velocity, entropy and pressure eigenfunctions corresponding to the normal mode of horizontal harmonic number ℓ Since the modes peak at different layers, with each mode transporting flux mainly over a region of thickness comparable to the local mixing length, it is clear that, by choosing suitable real constants a_ℓ we can always get agreement between the model convective flux and the superposed flux In addition to this, Narasimha and Antia were also able to show that the superposed r m s radial velocity

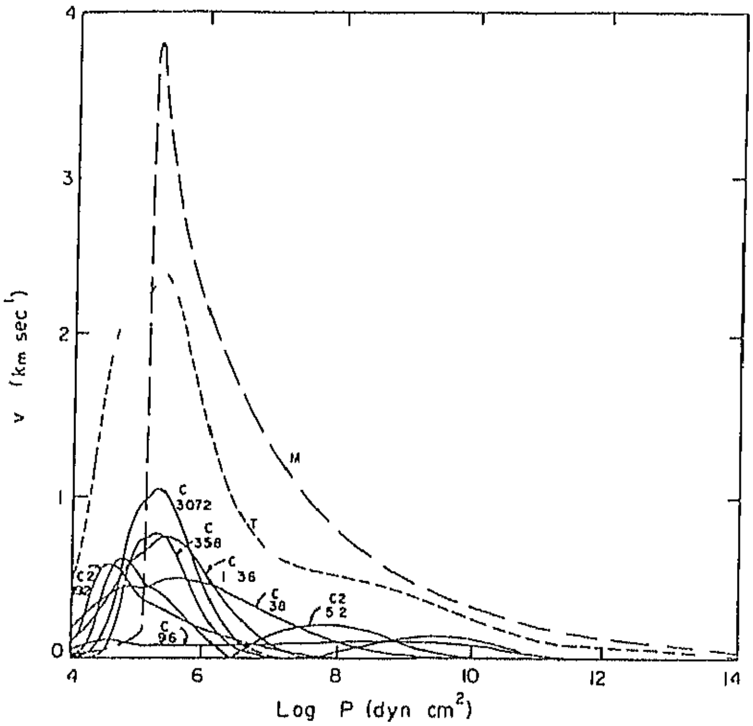

Fig.6. The energy transported by individual convective modes in the solar envelope model is plotted against logarithm of the pressure The curves are labelled by the value of horizontal harmonic number ℓ The dot dashed curve shows the model convective flux as given by the mixing length theory

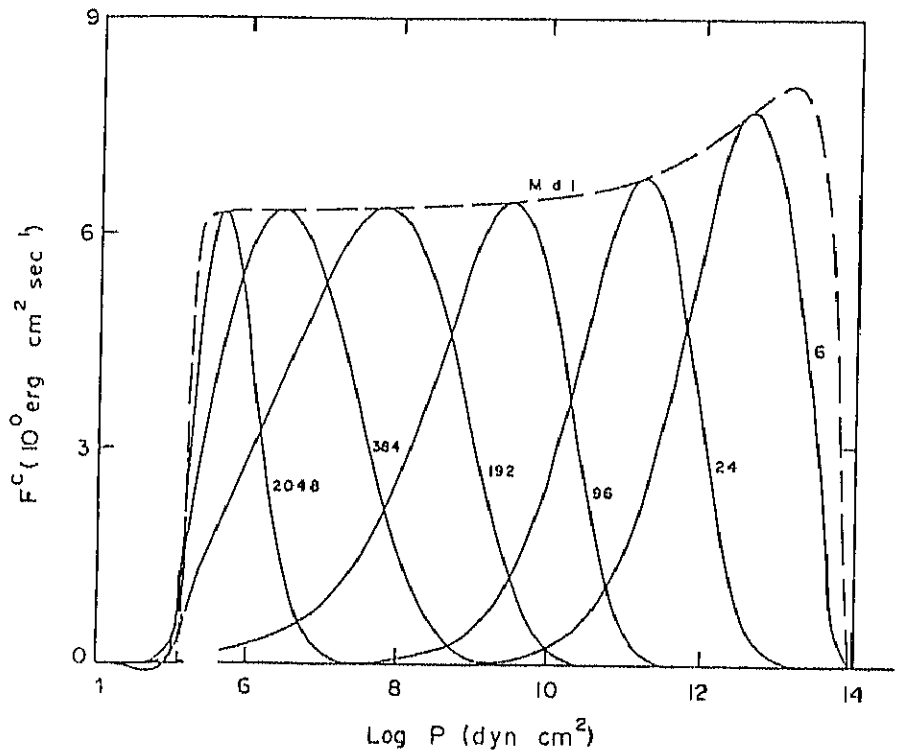

Fig.7 The contribution to vertical velocity v_r by individual modes is plotted against logarithm of the pressure The curves are labelled by the values of ℓ and the mode identification (C1 fundamental mode, C2 first overtone) The dashed curve shows the superposed vertical velocity while the dot dashed line represents the rms velocity yielded by the mixing length theory

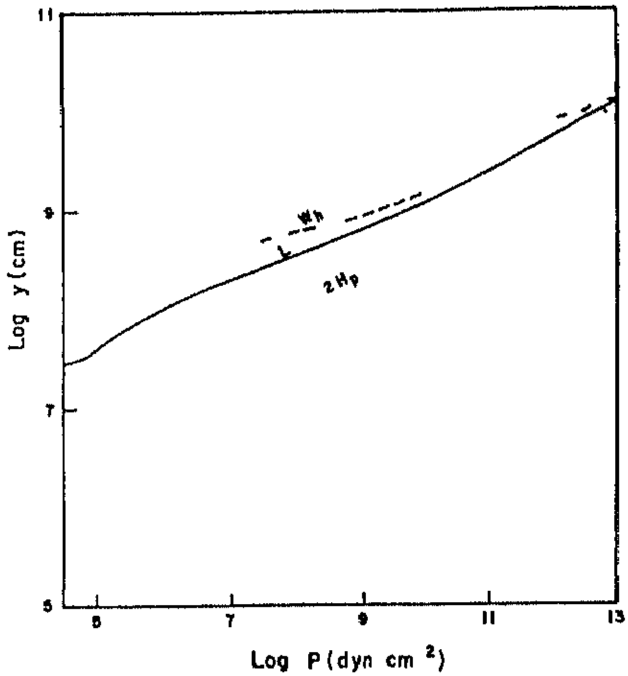

Fig.8. The equivalent width of the dominant convective mode at a given depth W_h (dashed curve), the mixing length (continuous line) and the curve $y = 2\,H_p$ (dotted curve) are displayed as functions of logarithm of the pressure for a mixing length model of the solar envelope

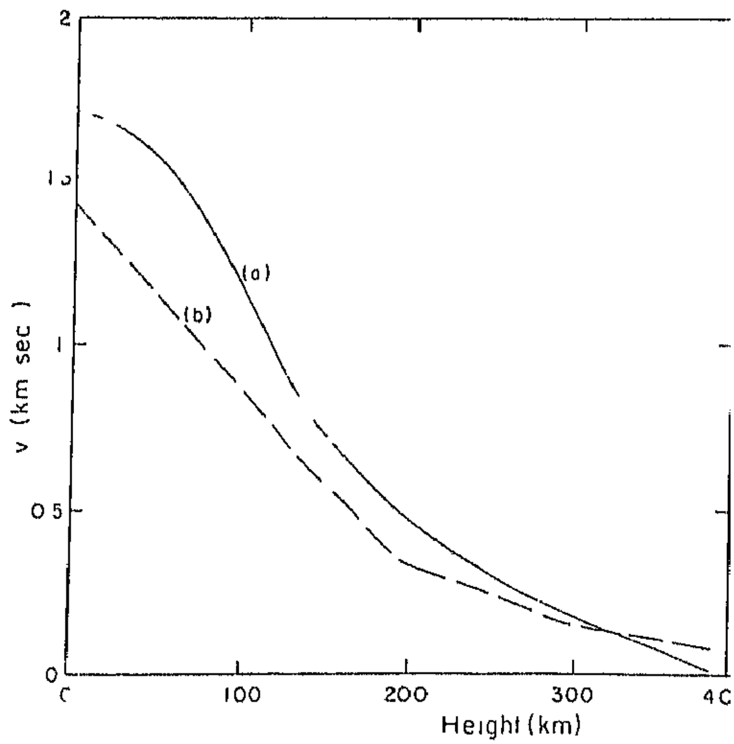

Fig.9 The continuous line shows the vertical velocity, v_r, contributed by modes with $\ell \gtrsim 1100$ plotted against height, while the dot dashed line is the observed granular vertical velocity in the solar atmosphere as given by Keil (1980)

agreed with the model convective velocity very well for most part of the convection zone (Figure 7) Thus it was convincingly demonstrated that the model convective flux computed by appealing to the mixing length theory is consistent with the superposition of energy transport by individual convective modes, provided, the mixing length was chosen suitably The conclusion was found to be valid even for a ymptotic red giant stars where the stratification of the convection zone is qualitatively different from that of the sun in many respects (Antia, Chitre and Narasimha 1984) Some additional results of the investigation were the following

a The mixing length at a point can be identified with the equivalent width of the convective flux profile of the dominant convective mode at that depth The agreement between the mixing length and the equivalent width of the convective mode is good throughout the convection zone (Figure 8) Thus the physical basis for the mixing length theory is vindicated through a hydrodynamical computation

b The superposed convective velocity component in the vertical direction is in good agreement with the observed value in the solar atmosphere (Figure 9)

The analysis also gives an indication of some of the drawbacks of the local mixing length theory

1 There is substantial overshooting of the eddies into the overlying atmosphere resulting in large superposed convective velocity But the local theory cannot treat the overshooting with any degree of reliability The differences between the model velocity profile and the superposed profile in the highly superadiabatic zone of the solar convection zone and the overlying atmosphere is apparent in Figure 7

The overshooting is also important for mixing between chemically inhomogeneous layers during the evolution of a star The importance of mixing in governing the stellar evolution has been stressed by many researchers

2 The normal mode analysis indicates a large horizontal velocity in the boundary between the convective and radiative region in the photosphere The formation of this narrow boundary layer is an artefact of the local mixing length theory, and is also a consequence of a continuity equation

Thus we find that an investigation of the consistency of the local mixing length theory naturally leads us to the consideration of a non local theory of convection

VI Non-Local Theories

A major limitation of the local mixing length approximation is that the velocity of the convective eddy and its temperature difference over the surroundings depend only on the local properties of the medium In a non local theory, effort is made to obtain the acceleration of the eddy, rather than its velocity, in terms of the temperature fluctuation and the local variables The temperature fluctuation of the convective element, in turn is governed by its dynamics Consequently one obtains differential equations, instead of algebraic relations, governing the temperature excess and velo ity of an out going parcel of fluid, which directly leads to the phenomenon of overshooting due to the inertia of the eddy In essence, the various non local approaches can be divided into two groups : (a) Convective eddy or fluid element approach (b) Fluid dynamical approach Here one or more "mixing lengths" are introduced to express the third order correlation coefficients occuring in the governing equations (like, for instance $[v_i(\partial/\partial x_i)v_j v^j]$ in terms of lower order terms

We shall avoid any specific non local theory (cf notes 6 for references) but only give an idea of the above two groups In the convective eddy approach the spirit of the local mixing length theory is retained The non locality appears in the form of the finite size of the eddy due to which the convective flux or the velocity at a depth z, are sampled over all eddies centred at (z L/2) to (z + L/2) Alternatively the flux transport from all the fluid elements originating from various points and dissolving at the depth z is evaluated by taking weighted average of the contribution from fluid elements formed at different depths The integral obtained in this way can be converted into a second order differential equation for the convective flux (Spiegel, 1963) of the form

$$\frac{d^2F_c}{ds^2} = F_c \simeq F_{c\ local} \tag{15}$$

where s is the radial coordinate in dimensionless form, and $F_{c\ local}$ is the convective flux in the corresponding local mixing length formalism A profile of the convective flux obtained using the non local formulation of Spiegel (Antia and Chitre private communication) is displayed in Figure 10 Note that when convection is very efficient and ∇_{ad} is nearly constant in the convection zone, the flux profile differs from its local counterpart only near the boundaries The convective velocity profile (displayed in Figure 11) shows a similar behaviour Thus it is clear that in the convective eddy approach the non local model differs from the local one mainly at the boundaries and in the adjacent outside layers where a significant amount of penetration takes place

In the fluid dynamical approach, the role of the mixing length is roughly incorporated in the computation of various diffusivities The formalism is useful if one can express the various correlation scale lengths in terms of a single length scale of the convection zone Xiong (1981) who has given an extensive formalism of non local theory uses results from isotropic turbulence to relate the various scale lengths The analogy between the resultant system of sixteenth order differential equation and the governing equations of fluid dynamics is easy to recognize However the equations are a bit cumbersome to handle in the computations of stellar evolution Also, here the mixing in the chemically inhomogeneous layer outside the convection zone is governed by scale lengths obtained from the theory of isotropic turbulence which is not valid in these layer Nevertheless the fluid dynamical approach of non local theory, using the mixing length determined by the self consistency requirement (discussed in the previous section) should be considered as a definitive progress in the treatment of stellar convection

Does this mean that the important problems of stellar convection are solved by the non local theory? No To name a few of the uncomfortable features

1 The concept of a single scale length (or its constant multiple) to describe the turbulent convection appears to be a simplistic view

2 The time dependency of the stellar convection becomes important when interaction with other physical processes are to be analysed The treatment of time dependent convection is not at all in a satisfactory state in the conventional mixing length theories

3 The effects of compressibility are important specially when convection is near the sonic limit but they are filtred out in the usual approximations

Obviously we need a more rigorous treatment Even though such theories may not help computation of realistic stellar convection zone models they should nevertheless help us understand various features of stellar convection (like the role of pressure fluctuation or the interaction with magnetic field) The main results of some of such attempts will be summarised in the next section

VII 1 Truncated Model Analysis

With the availability of powerful computers two methods are becoming popular in the numerical study of non linear convection finite difference scheme and truncated model analysis In section V we have already discussed the normal mode analysis, where

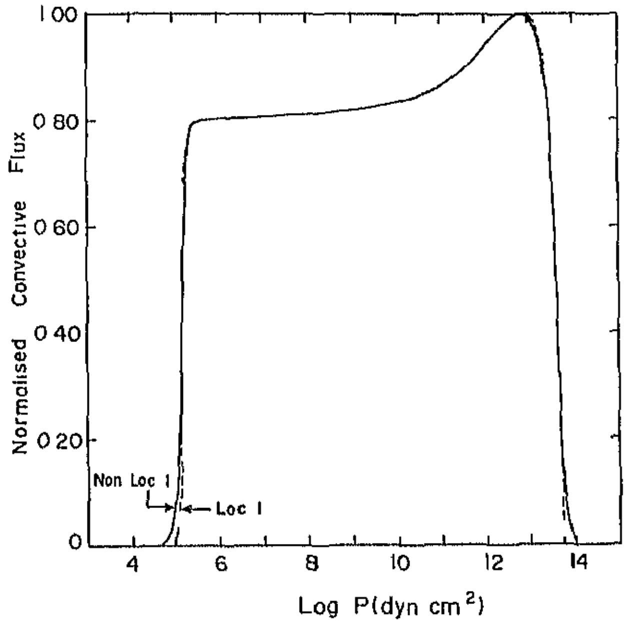

Fig 10 The convective flux computed using a non local mixing length theory (continuous curve) and its counterpart in the local theory (dashed line) are plotted against logarithm of the pressure

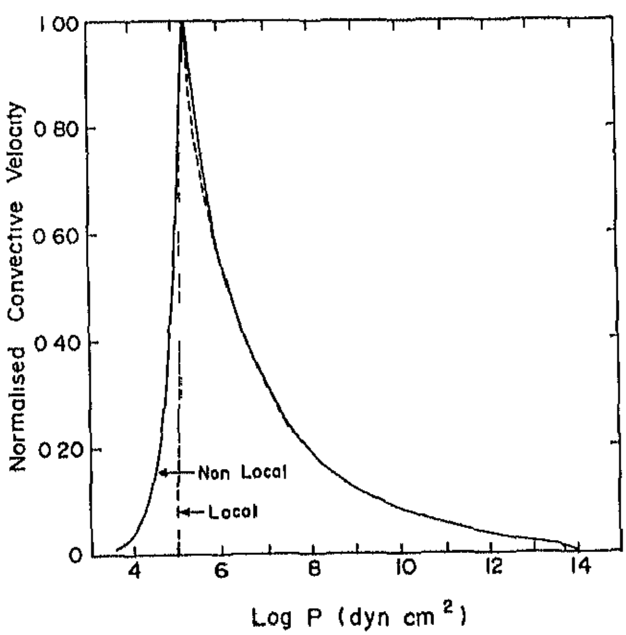

Fig.11 The convective velocity obtained using the non local mixing length theory (continuous curve) and the corresponding local theory (dashed line) are plotted against logarithm of the pres ure

the fluctuating component of the flow was assumed to be small and the governing equations are linearised. In the model analysis a similar expansion in terms of normal modes is carried out, but the amplitude of the modes is not supposed to be small. The non-linear terms consequently need to be retained, which physically represent the formation of new scales of flow due to the interaction between turbulent eddies. However, a few approximations are necessary to manage the large scale numerical computations; the chief among them are :

a. Only a small number of modes are manageable. Hence there is a cut-off in the smallest horizontal wavelength of the modes.

b. The mechanical and thermal effects of the motion of smaller length-scales have to be parametrized in terms of turbulent transport coefficients as discussed in section V.

c. The full effects of compressibility of the fluid are not taken into account.

We shall summarise a few of the results from the works of Toomre et al. (1976), Marcus (1980), and Massaguer et al. (1984).

i) Even in the outer convection zone of A-type stars where energy is largely tansported by radiation, the penetrative convection is found to be important.

ii) The penetration is an outcome of large pressure fluctuations in the superadiabatic layers.

iii) When the Rayleigh number is greater than 10^{10} and Prandtl number ~ 1, the small scale flow (the "inertial range") is nearly isotropic though the large scale motion is time-dependent and shows no spatial symmetries.

iv) Intermittent bursts in the convective flux were found, which cascade energy from the largest scale of convective motion to the inertial range.

It should be noted that these investigations are still far from the type adopted for solar-type convection zones. The turbulent motion involving more than a few million modes is represented by less than a few hundred modes. Nevertheless these studies have helped to gain a better understanding of a few important aspects like the penetration and time-dependency of the flow.

VII.2. Finite-Difference Methods

The finite-difference method of analysing convective flow becomes a powerful tool at low Rayleigh numbers. As we have already noted, when the Rayleigh number increases from R_{cr} to more than hundred times the critical value the convection developes from a laminar pattern to time-dependent turbulent flow. A finite-difference scheme is well-suited to the analysis of the new scales that emerge during this transition. Graham (1976) and Unno and Urata (1987) numerically studied the flow pattern in an idealised geometry for a range of Rayleigh number and Prandtl number beyond the normal laboratory limits. Graham (1976) considered a convection zone that extended over several scale heights and included the effects of compressibility though most of his work was restricted to two dimensional flow. His main conclusions were that :

a) In two-dimension, a steady flow exists though the time taken to attain steady state increases as the Prandtl number is lowered.

b) The only length-scale the convective flow samples appears to be the depth of the convection zone, even when the convection zone extends over several pressure scale-heights.

c) The upward and downward moving eddies are asymmetric, with the heavier downward moving eddies having larger velocity. This result, though surprising, appears to be consistent with the observed solar granulation where a rising broad column of gas is surrounded by narrower but rapidly sinking ring of cold gas.

d) The three dimensional flow shows two dimensional flow pattern when the horizontal extent is comparable to the depth. But when the width is increased the two flows are qualitatively different, with the three dimensional motion becoming time dependent. Unno and Urata find that for certain values of the Rayleigh number small regions in the convection zone become subadiabatic for short time scales before again turning into a superadiabatic flow. Though surprisingly, both Graham, and Unno and Urata find that when the convection zone allows modes of several wavelengths the flow does <u>not</u> develop so as to maximise the convective flux transport.

Though the numerical computations have been carried out in situations that are far from realistic stellar conditions, they have already given a few surprising results. Already attempts have been made to study two dimensional time dependent convection dring carbon and helium burning by finite difference technique. With the advent of more powerful computers and progress in the theory of turbulence, the finite difference method is likely to get more prominence.

VIII Summary and Future Outlook

Convection can develope in stars mainly due to two processes. When the nuclear energy generation rate is large enough, the core or nuclear burning shell becomes convectively unstable. When the hydrogen or helium in the outer envelope undergoes partial ionization, the opacity increases and adiabatic temperature gradient becomes flatter. This results in the occurrence of an outer convective envelope. The Schwarzschild criterion appears to be the adequate condition for the stability against convection.

The stellar convection is characterised by large Rayleigh number, low Prandtl number and high Reynolds number. The consequent flow pattern is three dimensional, time dependent and turbulent showing no spatial symmetry. In view of these complications, until recently the only practical method for the computation of stellar convection zones has been the local mixing length approximation. It is a model of the heat and momentum transport by eddies or parcels of fluids similar to the transport phenomena in the kinetic theory of gases. Its extention to the non local theory gets over some of the shortcomings of the model, though the mixing length model still remains too simplistic when the inertia of the individual eddies becomes important. The truncated model analysis and finite difference schemes have increased our understanding of the phenomenon of stellar convection though so far they have had limited success in the computations of stellar structure.

In the near future the non local mixing length theory would appear to be the best available for the construction of stellar convection zone models, at least upto the end of core helium burning. It is therefore highly desirable to construct as reliable a formulation as possible. A theory, with the mixing length and other parameters determined from the consistency requirement should be trustworthy at least for its overall gross description of the stellar convection. However, when the interaction between convection and other physical processes like pulsation needs to be investigated, the time dependency becomes very important.

With the availability of powerful computers we expect the finite difference schemes to be increasingly more popular in the computation of stellar convection zones. While certainly it is a big leap foreward from the conventional local mixing length formalism, we should be aware of the qualitative difference between two and three dimensional convection. Possibly we still need to undertake a simplified approach for isolating and analysing the underlying physics. In this respect we think a study of non linear convection in a polytropic fluid would be worthwhile an attempt.

Notes

The perturbations in a system can be investigated in terms of a complete set of eigenfunctions, called normal modes (cf Chandrasekhar 1961) The spherical harmonics Y_ℓ^m form a complete set in the (θ,ϕ) surface for a star which has spherical symmetry in the steady state Each value of ℓ (called horizontal harmonic number) correspond to a mode of horizontal wavelength $\lambda = [2\pi r_*/\sqrt{\ell(\ell+1)}]$, where r_* is the radius of the star The convective and oscillatory modes excited in the sun can be studied in terms of the spherical harmonics at least in the linear regime It is useful to expand the variables in terms of a complete set of eigenfunctions only in the (θ,ϕ) direction but not along the radius This is due to the fact that the steady state has no dependence on $(\theta\,\phi)$ Hence the linearized equations separate into different spherical harmonics which can be studied individually

2 Lebovitz (1965 a,b) derived the condition for stability against convection by normal mode analysis starting from the Lagrangian for adiabatic motion of the fluid and appealing to the variational principle Chandrasekhar (1965) showed that the condition is applicable even to relativistic fluid

3 Consider the equation of conservation of momentum,

$$\rho \frac{\partial v}{\partial t} + \rho(\underset{\sim}{v}\ \underset{\sim}{\nabla})\underset{\sim}{v} = \quad \underset{\sim}{\nabla} P \quad \rho \underset{\sim}{g} + \underset{\sim}{\nabla}\varphi \tag{16}$$

where φ is the stess tensor (cf Unno et al , 1979) The non linear term $(\underset{\sim}{v}\underset{\sim}{\nabla})\underset{\sim}{v}$ denotes the advection Physically it represents the change in momentum at a point due to the inflow of fluid from other regions as against the change brought about directly by the variation of local conditions

4 The text book 'Principles of Stellar Structure', (Cox and Giuli, 1968) gives a very readable account of some of the physical ideas and mathematical treatments presented in this article A good description of the local mixing length theory can be found in Chapter 14 The equations of stellar structure and an outline of the computation of models are given in Chapters 20 and 21

Gough (1976) presents an elaborate and reasonably upto date description of the local mixing length formalism in his review article Spiegel (1971, 1972), Gough (1977) and Xiong (1978) are worth studying for a feeling of the non linear convection

5 Beckers (1981) gives a fairly detailed description of the different types of observed velocity fields in the solar atmosphere and also outlines the technique involved in the analysis

6 Gough (1976) gives a brief account of the non local mixing length formulation An idea of the range of the physical and mathematical approximations invoked can be derived from the following sample of references Spiegel (1963), Ulrich (1976) Maeder (1975), Kuhfuss (1986), Xiong (1981)

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