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Influence of differential rotation on the wave propogation in a spherical MHD system with an application to sun

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Abstract. The dispersion relation for the MHD waves in a homogeneous medium having differencial rotation has been derived by using spherical polar coordinates. It is found that, for equatorial flow the instability of waves is influenced by the differential rotation. The waves with frequency $\omega >> V_A/r$ in the direction of increasing differential rotation will undergo damping. At the end a very brief discussion about the wave instability inside the sun has been made.

1. Introduction

A lot of work has been done on differential rotation and its application in different system by Durney (1989, see other references there). Chakraborty and Bondyopadhaya (1993) have investigated the stability of the wave propagation in the central region of the galaxy in the presence of temperature gradient. Here in this paper we also investigate the wave instability in presence of differential rotation.

2. Basic equations

We consider a single-fluid hydromagnetic system of homogeneous plasma under the influence of static magnetic field. The fluid particles are considered to be rotating differentially. We assume that the plasma is fully ionised having no viscosity, no conductivity, and no electrical resistivity. Such a system is described by the following basic equations :

The equation of motion :

$$\partial u/\partial t + (u.\nabla)u + (u.\nabla)[\Omega \times r] = -(1/\rho)\nabla p + (\mu/4\pi\rho)[(\nabla \times H) \times H] - 2[\Omega \times u] \quad (1)$$

MHD field equation : $\partial H/\partial t = \nabla \times (u \times H)$ (2)

Equation of continuity : $\partial \rho/\partial t + \nabla.(\rho u) = 0$ (3)

Heat equation : $\partial p/\partial t + (u.\nabla)p = (\gamma p/\rho)[\partial \rho/\partial t + (u.\nabla)\rho]$ (4)

Equation of state : $p = R\rho T$ (5)

where u : fluid velocity, Ω : angular velocity, p : hydrostatic pressure, μ : magnetic permeability, R : gas constant divided by the molecular weight and γ: ratio of specific heats. All the other terms have their usual meanings.

3. Assumptions and formulation of dispersion relation

To obtain the dispersion relation characterising the wave propagation, we proceed as follows : We first eliminate p from eq. (1) by using (5). Then assuming that initially the medium was at equilibrium and subsequently perturbed, we get linearized perturbed equations into spherical polar coordinates (r, θ, ϕ). Then we assume a homogeneous and uniform temperature medium where initial velocity is zero, i.e., $u_0 = 0$ and the initial magnetuc field exists only along the poloidal direction, i.e. $H_{or} = H_{0\phi} = 0$, $H_{o\theta} \neq 0$. Further it is assumed that polodial magnetic field is constant in all directions and initial rotation is independent of ϕ i.e. $\partial\Omega_0/\partial\phi = 0$. Next assuming the perturbation is of sinusoidal nature, i.e. proportional to exp $[i\{rK_r + r\theta K_\theta + (r\ \sin\theta)\phi K_\phi - \omega t\}]$, where K_r, K_θ, K_ϕ are the wave numbers along r, θ, ϕ directions respectively and ω is the wave frequency, we get the following set of equations as :

$$i\omega u'_r + 2\Omega_0 u'_\theta + \Omega_0 \sin\theta\, u'_\phi + i(V_A^2 / rH_{0\theta}) K'_\theta H'_r - (V_A^2 / H_{0\theta}) (iK'_r + 2/r) H'_\theta + (V_A^2 / r - iRT_0 K'_r) \rho'/\rho_0 - iRK'_r T' = 0 \tag{1.1}$$

$$-2\Omega_0 u'_r + i\omega u'_\theta + \Omega_0 \cos\theta\, u'_\phi + (V_A^2 / rH_{0\theta}) H'_r - i(RT_0 / r)K'_\theta \rho'/\rho_0 - i(R/r) K'_\theta T' = 0 \tag{1.2}$$

$$\sin\theta (r\partial_r\Omega_0 + 2\Omega_0) u'_r + (\sin\theta\, \partial_\theta\Omega_0 + 2\Omega_0 \cos\theta) u'_\theta - i\omega u'_\phi + i(V_A^2 /rH_{0\theta}) K'_\phi H'_\theta - (V_r^2 / rH_{0\theta}) (iK'_\theta + \cot\theta) H'_\phi + i(RT_0 / r \sin\theta) K'_\phi \rho'/\rho_0 + i(R/r \sin\theta) K'_\phi T' = 0 \tag{1.3}$$

$$(H_{0\theta}/r) (iK'_\theta + \cot\theta) u'_r + i\omega H'_r = 0 \tag{2.1}$$

$$H_{0\theta} (iK'_r + 1/r) u'_r + i(H_{0\theta} / r \sin\theta) K'_\phi u'_\phi - i\omega H'_\theta = 0 \tag{2.2}$$

$$(H_{0\theta}/r) K'_\theta u'_\phi - \omega H'_\phi = 0 \tag{2.3}$$

$$(iK'_r + 2/r) u'_r + (1/r) (iK'_\theta + \cot\theta) u'_\theta + i(1/r \sin\theta) K'_\phi u'_\phi - i\omega\rho'/\rho_0 = 0 \tag{3.1}$$

$$(iK'_r + 2/r) u'_r + (1/r) (iK'_\theta + \cot\theta) u'_\theta + i(1/r \sin\theta) K'_\phi u'_\phi - i(\omega/\gamma) (T'/T_0 + \rho'/\rho_0) = 0 \tag{4.1}$$

where $K'_r = K_r + \theta K_\theta + \phi \sin\theta\, K_\phi$, $K'_\theta = rK_\theta + r\phi \cos\theta\, K_\phi$, $K'_\phi = r \sin\theta\, K_\phi$

and $V_A = (\mu/4\pi\rho_0)^{1/2} H_{0\theta}$.

From eqs. (3.1) and (4.1), we have

$$T' = (\gamma - 1) T_0\rho'/\rho_0. \tag{4.2}$$

Now we assume that the waves are propagating along the azimuthal direction only, i.e. $K_r = K_\theta = 0$. We now consider the propagation on the equatorial plane only i.e. $\theta = \pi/2$. Then $K_r^{'} = \phi K_\phi$, $K_\theta^{'} = 0$, $K_\phi^{'} = rK_\phi$.

Substituting the above values of $K_r^{'}$, $K_\theta^{'}$, and $K_\phi^{'}$ in the set of eqns. (1.1) to (3.1) and using (4.2) in (1.1), (1.2) and (1.3), we get the dispersion relation as

$$\alpha_1 K^2 + \alpha_2 K + \alpha_3 = 0 \text{ by writing } K_\phi = K \qquad (6)$$

where $\alpha_1 = (V_A^2 + C_S^2)\,[(1 + \phi^2)\omega^2 - i(r\partial_r\Omega_0 + \Omega_0)\,\omega\phi - 4\Omega_0^2] - (V_A^2 / r^2)\,(V_A^2 + C_S^2)$

$$\alpha_2 = (\omega / r)\,[-2i(V_A^2 + C_S^2)\omega\phi - V_A^2\,(r\partial_r\Omega_0\,\Omega_0) + 2C_S^2\Omega_0]$$

$$\alpha_3 = \omega^2\,[-\omega^2 + \Omega_0\,(6\Omega_0 + r\partial_r\Omega_0)]$$

and $Cs = (\gamma RT_0)^{1/2}$

4. Discussion

It is clear from the above dispersion relation that differential rotation has definite influence on instability. If we assume, $\omega >> V_A / r$ and $\omega\phi >> V_A/r$, then

$$\propto_1 = (V_A^2 + C_S^2)\,[(1 + \phi^2)\omega^2 - i(r\partial_r\Omega_0 + \Omega_0)\,\omega\phi - 4\Omega_0^2]$$

Also for sufficiently large r, $\alpha_2/\alpha_1 \to 0$ as $\alpha_2/\alpha_1 \sim 1/r$

So from (6) we get

$$K^2 = \frac{\omega^2\,[\omega^2 - \Omega_0\,(6\Omega_0 + r\partial_r\Omega_0)}{(V_A^2 + C_S^2)\,[(1 + \phi^2)\omega^2 - i(r\partial_r\Omega_0 + \Omega_0)\,\omega\phi + 4\Omega_0^2]} \qquad (7)$$

Relation (7) can also be written as $K^2 = A + iB$ (8)

where $$A = \frac{\omega^2\,[\omega^2 - \Omega_0\,(6\Omega_0 + r\partial_r\Omega_0)]\,[(1 + \phi^2)\omega^2 + 4\Omega_0^2]}{(V_A^2 + C_S^2)\,[\{(1 + \phi^2)\omega^2 + 4\Omega_0^2\}^2 + \omega^2\phi^2\,(r\partial_r\Omega_0 + \Omega_0)^2]}$$

$$B = \frac{\omega\phi(r\partial_r\Omega_0 + \Omega_0)}{(V_A^2 + C_S^2)\,[\{(1 + \phi^2)\omega^2 + 4\Omega_0^2\}^2 + \omega^2\phi^2\,(r\partial_r\Omega_0 + \Omega_0)^2]}$$

Observations :

i) It can be seen easily from (7) that if $\omega\phi >> (r\partial_r\Omega_0 + \Omega_0)$ and $\omega^2 > (6\Omega_0 + r\partial_r\Omega_0)\,\Omega_0$ then stable mode can exist.

In that case phase velocity,

$$\text{Vph} = \left[\frac{(V_A^2 + C_S^2)\,\{(1 + \phi^2)\,\omega^2 + 4\Omega_0^2\}}{\omega^2 - \Omega_0\,(6\Omega_0 + r\partial_r\Omega_0)}\right]^{1/2}$$

This shows that phase velocity increases due to the existence of positive differntial rotation.

ii) It is known that the wave will be damped or amplified according as B > 0 or B < 0. Clearly $\partial_r\Omega_0 > 0$ then B > 0. Thus the wave propagating in the direction of increasing differential rotation may be damped provided its frequency w >> V_A/r. For amplification, however, we must have $\partial_r\Omega_0 < 0$ and $|\partial_r\Omega_0| > \Omega_0 / r$.

The above theory can be applied inside the sun (particularly in the convective region) to study the instability of the MHD waves. The differential rotation inside the sun, required for calculating B in (8), can be estimated from the observed data (Duvall, 1984). Hence, which of the waves with frequency $\omega >> V_A / r$ will undergo damping or amplification can be identified. For example, at the region 0.45 $R_\odot < r < 0.75 R_\odot$, differential rotation $\partial_r\Omega_0 > 0$. So, if we consider the region r = 0.65 R_O and if the magnetic field is taken to be 250 Gauss (say), then the waves with frequency $\omega >> V_A / r\ 10^{-9}\ \text{sec}^{-1}$ will undergo damping. There are also regions inside the sun where $\delta_r\Omega_0 < 0$. Naturally there the waves may be amplified. However, the detailed numerical study will be communicated elsewhere.

References

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