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Some Aspects of the Interpretation of Frequency Splitting Data

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Abstract. Many dynamo and dynamical models of the convection zone require that the angular velocity be approximately constant on cylinders concentric about the rotation axis ('constant-on-cylinders'). However, inferences from the earliest helioseismology data that the internal angular velocity of the Sun is invariant across the convection zone, i.e. within the convection zone it is constant on radial cones with vertices at the center ('constant-on-cones'), have caused some concern to theoreticians.
This has led to a re-examination of these inferences and Gough and his colleagues have recently claimed that the currently available data are not inconsistent with some models for which the angular velocity is constant on cylinders within the Sun's convection zone outside the tangent cylinder.
Thus it would appear that, on the basis of current frequency splitting data, we can eliminate neither the 'constant-on-cylinders' nor the 'constant-on-cones' classes of models. In this paper, this problem is discussed and an alternative approach is proposed in which a forward method is used to 'characterize' the essential properties of a particular data set.

Key words: sun: interior – sun: oscillations – sun: rotation

1. Introduction

In several dynamical models developed during the 1980's, such as those of Gilman (1983) and of Glatzmaier (1985), and in the dynamo models of Brandenburg (e.g. Brandenberg *et al.*, 1992), the angular velocity within the convection zone is approximately constant on cylinders concentric about the rotation axis ('constant-on-cylinders'). However, from the earliest helioseismology data (Duvall *et al.* 1986, Brown and Morrow 1987, Morrow 1988, Brown *et al.* 1989, Libbrecht 1989) it was inferred that the internal angular velocity of the Sun is essentially invariant across the convection zone, i.e. it is constant on radial cones with vertices at the center ('constant-on-cones'). This view became firmly entrenched when the Big Bear Observatory (BBSO) produced it's well-known Xmas card purporting to represent the isorotation surfaces of the internal angular velocity in glorious technicolour. The conflict between these two result has been the cause of some concern.

Recently, Gough and various colleagues (Gough *et al.* 1993, Schou & Brown 1994, Sekii, Gough & Kosovichev 1995) have been at pains to rehabilitate 'constant-on-cylinders' models. Following Gough *et al.*, Schou and Brown attempted to obtain models for which the angular velocity is constant-on-cylinders in the convection zone outside the tangent cylinder but which are nevertheless *seismically equivalent* to their constant-on-cones model in the sense that they reproduced the same (or similar) theoretical a-coefficients. In order to achieve this, however, they were forced to choose somewhat unrealistic isorotation surfaces in the convection zone within the tangent cylinder.

Sekii *et al.* have carried out inversions of the BBSO data and shown that, by adopting different "penalty functions" in the regularization of these inversions, it is possible to achieve different outcomes. Indeed, Sekii (1995) has suggested that the outcome of the regularization process frequently depends upon the prejudices of the theoretician concerned. By penalizing only variations in the radial direction, they obtain a solution which is consistent with the constant-on-cones models, but by imposing an additional penalty on variations parallel to the rotation axis in the low-latitude regions of the convection zone, Sekii *et al.* obtain a solution which is more nearly constant on cylinders. Apparently, 'what you want is what you get'.

I have investigated the preparation of the BBSO Xmas card, and it appears that Libbrecht derived the angular velocity within the convection zone by taking an average of the results of inversions by Goode and by Schou, each of which minimized the radial shear, and assumed a uniform angular velocity below this region. Again it seems that 'what you want is what you get'.

Stark (1994), and Gough, Sekii, and Stark (1995) have argued that, when comparing linear combinations of indirect measurements of physical properties of systems against the corresponding quantities as derived from models of the systems, the confidence intervals of individual linear estimates must be adjusted to obtain the correct 'simultaneous coverage probality' that all such estimates lie within intervals which they call the 'simultaneous confidence intervals' appropriate to particular combinations of data. Wilson et al. 1996a), have shown that, within these intervals, it is not easy to exclude either some elementary 'constant-on-cones' or 'constant-on-cylinders' models.

In a new and exciting field, such ambivalence in the form of the solutions which may be attributed to a given data set is undesirable and, in this paper, a new approach to the interpretation of frequency splitting data is proposed. Rather than attempt to find a 'solution' for $\Omega(r, \theta)$ as such, we attempt to characterize the data set, first qualitatively, and then in more quantitative fashion.

2. Formulation

If a data set offers values of (say) 3 of the odd a-coefficients for each mode, then we may, in principle, attempt to determine a rotation model expressed in the form

$$\Omega(r, \theta) = \sum_{s=0}^{2} \Omega_s(r)\mu^{2s}, \tag{1}$$

where μ is the cosine of the colatitude, θ. Alternatively the rotation model may be expressed in terms of three functions $\Omega(r, \lambda_k)$ for three values of the latitude λ_k, (since for any r, three values of $\Omega_s(r)$ can be determined).

In an earlier analysis (Wilson *et al.* 1996b, Paper 1) 413 multiplets were selected from the 1539 multiplets available in the BBSO data set, and allocated ID numbers $j = 1, 413$ in order of increasing values of $\frac{\nu}{L}$ (i.e. increasing depth penetration). If the contribution function for the jth multiplet is given by

$$H_j(r) = \frac{\{\xi_{j'}^2(r) + \eta_{j'}^2(r)\}\rho(r)r^2}{\int_0^R \{\xi_{j'}^2(r) + \eta_{j'}^2(r)\}\rho(r)r^2 dr}, \qquad (2)$$

where $\xi_j(r)$ and $\eta_j(r)$ are the eigenfunctions of the velocity amplitudes for that multiplet, then

$$\bar{\Omega}_j(\lambda_k) = \frac{1}{2\pi}\int_0^1 \Omega(r, \lambda_k) H_j(r) dr, \qquad (3)$$

defines the mean value of $\Omega(r, \lambda_k)$ as averaged by the jth multiplet. Following Morrow (1988), Brown *et al.* (1989) $\bar{\Omega}_j(\lambda_k)$ may be expressed in terms of a linear combinations of the splitting coefficients, a_i^j, of the form

$$\bar{\Omega}_j(\lambda_k) = \sum_{t=0}^{2} \beta_{2t+1,k} a_{2t+1}^j, \qquad (4)$$

e.g. $\bar{\Omega}_j(0°) = a_1^j + a_3^j + a_5^j$. Using the BBSO a-coefficients for 1990, this is plotted against $\log \frac{\nu}{L}$ in Figure 1, where it can be seen that the fluctuations and the error bars are large, particularly at large values of $\frac{\nu}{L}$.

We now introduce a differential mean of $\bar{\Omega}_j(\lambda_k)$, defined by

$$\omega_j(\lambda_k) = \frac{1}{j}\{\sum_{j'=1}^{j} \bar{\Omega}_{j'}(\lambda_k)\}, \qquad (5)$$

and, using Equation 4, these may be expressed in terms of linear combinations of the observed a-coefficients of the form

$$\omega_j^{(o)}(\lambda_k) = \frac{1}{j}\{\sum_{j'=1}^{j}\sum_{t=0}^{2} \beta_{2t+1,k} a_{2t+1}^{j'}\}. \qquad (6)$$

The errors or uncertainties in the values of $\omega_j(\lambda_k)$ corresponding to the errors e_i^j in the splitting coefficients quoted by the observers are given by σ_j, where

$$\sigma_j^2 = \frac{1}{j^2}\{\sum_{j'=1}^{j}\sum_{t=0}^{N=2} \beta_{2t+1,k}^2 (e_{2t+1}^{j'})^2\}. \qquad (7)$$

A plot of $\omega_j^{(o)}(0°)$ against $\log \frac{\nu}{L}$ using the BBSO a-coefficients for 1990 is shown in Figure 2 (solid curve). The range, $\omega_j \pm \sigma_j$, which defines the 70% individual confidence intervals (i.e. the probability that the next independent determination of a particular value of ω_j falls within these intervals is 70%), is indicated by the inner dotted curves, while the outer dotted curves provide an estimate of the 95% simultaneous confidence intervals (Stark 1994) for these data (i.e. the probability that the next independent determinations of ALL the ω_j fall within these intervals is 95%).

3. Comparison of models with data

Because the averaging process defined by Equation (3) weights the data from the surface layers more heavily, both the scatter of the data points and the error bars at large values of $\frac{\nu}{L}$ are considerably reduced. For a given angular velocity distribution, $\Omega(r, \lambda_k)$, Equations (3) and (5) may be combined to give a theoretical ω_j-distribution

$$\omega_j^{(th)}(\lambda_k) = \frac{1}{2\pi}\int_0^1 \Omega(r, \lambda_k)\Pi_j(r)dr, \tag{8}$$

where

$$\Pi_j(r) = \frac{1}{j}\sum_{j'=1}^{j} H_{j'}(r), \tag{9}$$

and, for given $\Omega(r, \lambda_k)$, this may be compared directly with the 'observed' distribution, $\omega_j^{(o)}(\lambda_k)$.

The equatorial component of a model, m_1, representing the 'constant-on-cones' model considered by Schou and Brown (1994) is

$$\frac{\Omega(r, 0^\circ)}{2\pi} = 461.4, \quad 1 > r > .7, \quad = 443, \quad r \leq 0.7. \tag{10}$$

Similarly, the equatorial component of a 'constant-on-cylinders' model, m_2, is given by

$$\frac{\Omega(r, 0^\circ)}{2\pi} = 466.2 - 49(1 - r^2) - 84(1 - r^2)^2, \quad 0.7 < r < 1.0, \tag{11}$$

$$\frac{\Omega(r, 0^\circ)}{2\pi} = 550, \quad 0.6 < r < 0.7, \quad = 420, \quad r \leq 0.6, \tag{12}$$

(all units in nHz), and these simple models are illustrated in Figure 3. Using Equations (8) & (9), theoretical ω_j-distributions may be compared with the observed distribution shown in Figure 2, where it can be seen that the theoretical distribution for m_1 lies just within the simultaneous confidence intervals of the observed distribution, while that for m_2 intersects the observed distribution, but lies outside the simultaneous confidence intervals for some range of values of $\frac{\nu}{L}$.

Following a suggestion by Stark (private communication), we define a ***blow-up*** factor, f, for a model as the smallest factor such that, if the 1σ confidence intervals (i.e. $[\omega_j \pm \sigma_j]$) are expanded to $[\omega_j \pm f\sigma_j]$, the theoretical distribution for that solution would lie on or within the expanded confidence intervals. From Figure 3 it is easy to estimate that, for m_1, the f-value is of order 4, while for m_2 it is ~ 10. Of course, similar comparisons must be made at two other latitudes (see paper 1) and the largest f-values for the set of three may be taken to provide an estimate of the compatibility of the model with the data set. Thus we may say that, as measured by their f-values, the 'constant-on-cylinders' model provides a poorer fit to the BBSO data set for 1990 than the 'constant-on-cones' model by almost an order-of magnitude.

4. The 'Characterization' of a Data Set.

Apart from testing particular models, it is fair to ask whether a given data set provides any specific information about the internal angular velocity. In particular, we would like to know how the radial velocity gradient varies with depth, whether there are angular velocity maxima and minima and, in particular, whether there are regions of strong radial shears. Thus we seek the essential characteristics of a data set.

Because multiplets with increasing values of $\frac{\nu}{L}$ probe deeper into the solar interior, a qualitative picture of the Sun's internal angular velocity can be inferred from the 'observed' $\omega_j^{(o)}(\lambda_k)$-distribution shown in Figure 2. The initial increase in the observed distribution implies that the angular velocity increases initially below the surface. However, the abrupt change in gradient near $\frac{\nu}{L} = 30\ \mu$Hz ($r \approx 0.93$), so that it becomes zero and then negative for $\frac{\nu}{L} > 60\ \mu$Hz ($r \approx 0.85$), can be explained only if the regions differentially probed by the modes in these ranges have lower angular velocities than those above them. Thus we obtain a qualitative interpretation of the BBSO data set which is untainted by the prejudices of an inverter.

However it is desirable to quantify this interpretation and, again, the concept of an f-value for a model in relation to a data set proves to be of value. Equations (6), (8), and (9) may be combined to give

$$\frac{1}{j}\{\sum_{j'=1}^{j}\sum_{t=0}^{2}\beta_{2t+1,k}a_{2t+1}^{j'}\} \pm \sigma_j = \frac{1}{2\pi}\int_0^1 \Omega(r,\lambda_k)\Pi_j(r)dr, \tag{13}$$

and, in Paper 1, a method was described for deriving step-wise continuous functions $\Omega(r, \lambda_k)$ from this equation and determining their f-values by comparing their theoretical ω-distributions with the observed distribution. We can, in this way, find models for which $f \leq 1$.

The equatorial component of one such solution, m_3, is shown in Figure 3, and it can be seen that there are maxima at $r = 0.94$, 0.82, and 0.48, with corresponding minima at 0.87 and 0.57. In Figure 2, and in the enlargement shown in Figure 4, it can be seen that the theoretical ω-distribution for m_3 follows the observed distribution very closely, i.e. $f < 1$, but it must be asked whether the radial fluctuations in $\Omega(r, \lambda_k)$, which are required to produce this close fit, are essential to the data or whether a simpler model can fit within the 1σ error bars equally well. Solution m_4 (also shown in Fig.3) is derived from solution m_3 by a 'trial and error' elimination of the fluctuations, subject to the requirement that $f = 1$. Only the maximum at $r = 0.94$ must be retained and, below this level, the solution declines monotonically. In particular, there is no clear evidence of a shear zone at the base of the convection zone, although there is some suggestion of a shear near $r = 0.6$. If the maximum at $r = 0.94$ is eliminated, however, as it is in model m_1, the f-value increases to ~ 4. We infer that only this maximum is essential in order to satisfy the $f = 1$ requirement, and therefore that it is a real feature of the data set.

5. Conclusion

By associating with any model and data set an f-value as defined above, we can 'characterize' a given data set by the features of the simplest model with f-values ≈ 1.

Of course, it will be noticed that this procedure has its parallel in the regularizations which impose a penalty on radial variations. However, while the choice of the form of the 'smoothing' terms, and of the smoothing parameters, are essentially arbitrary in the regularized inversion process, the smoothing procedure used here is directly related to the uncertainties associated with the data set.

Further, although it may not be possible to reject other models on the basis of a given data set, this approach provides that they may be ranked in order of their compatibility with the data. Since the f-values of the 'constant-on-cylinders' model considered here is of order 10, while that of the 'constant-on-cones' model is ~ 4, we may infer that the former is less compatible with the BBSO data than the latter, and in this way, we have a procedure for assessing models based on physical considerations against data.

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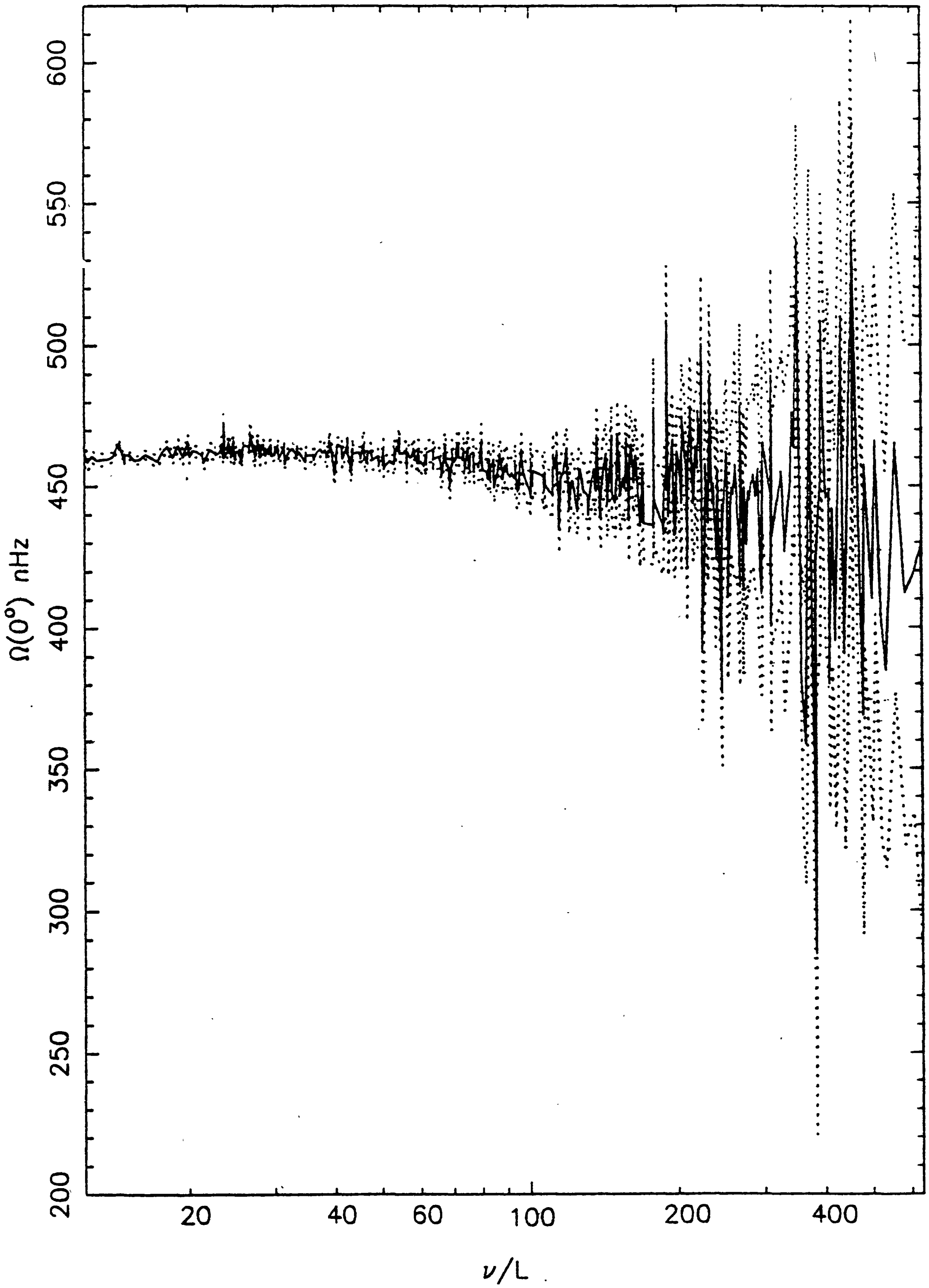

Figure 1. The observed Ω-distribution for $\lambda_k = 0°$ ($=a_1^j + a_3^j + a_5^j$) for 1990 is plotted against $\log\frac{\nu}{L}$ (solid line), while the corresponding errors derived from those quoted by observers are indicated by the dotted lines.

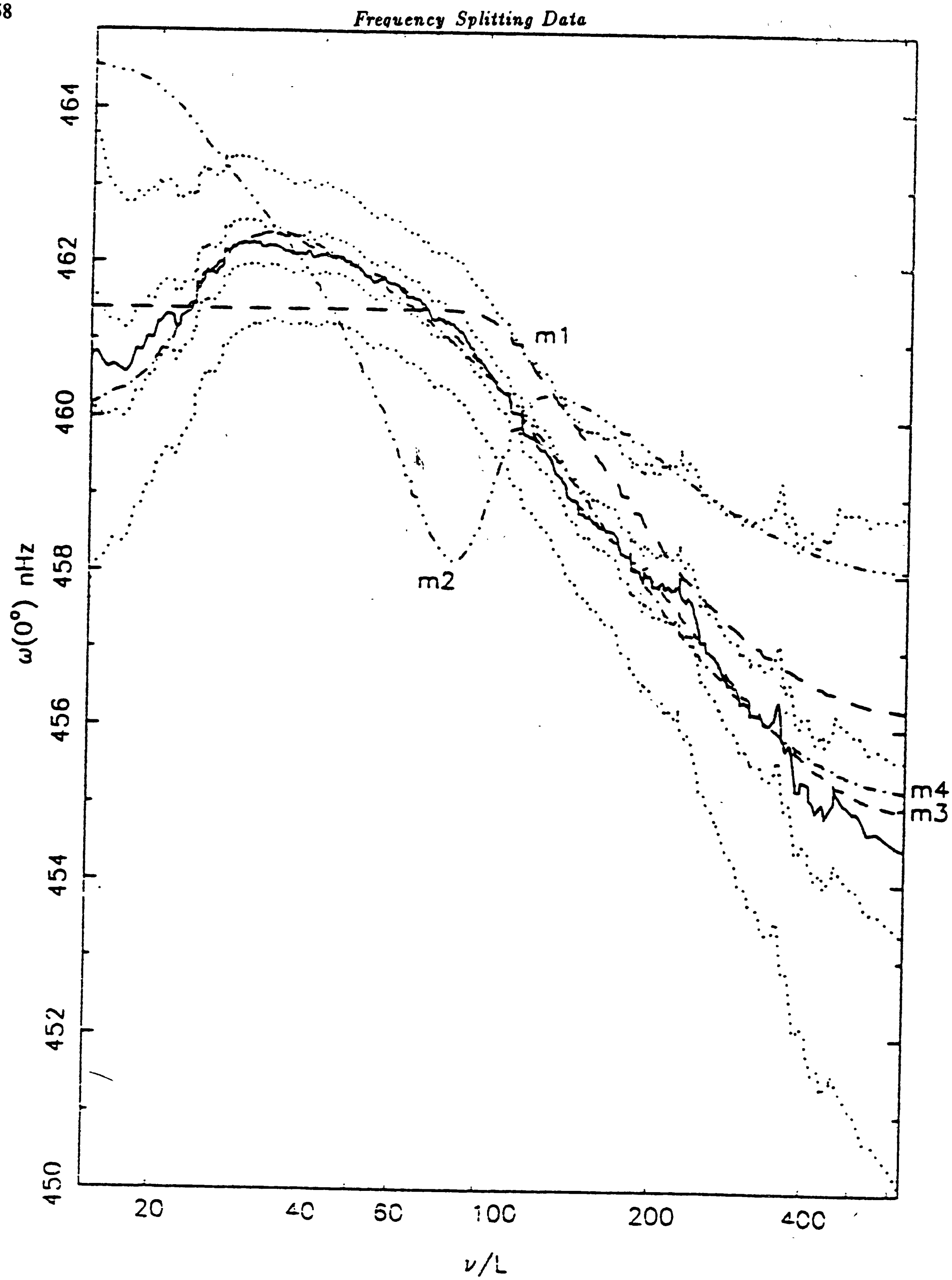

Figure 2. The observed (1990) ω-distribution for $\lambda_k = 0°$ is plotted against $\log\frac{\nu}{L}$ for 1990 (solid line), while representative values of $\frac{\nu}{L}$ are shown on the horizontal axis. The corresponding 70% individual confidence intervals $[\omega_j \pm \sigma_j]$ are represented by the (inner) dotted curves while the 95% simultaneous confidence intervals $[\omega_j \pm 3.84\sigma_j]$ are indicated by the (outer) dotted curves.

The theoretical ω-distributions for the equatorial components of models m_1, m_2, m_3, and m_4 are also shown.

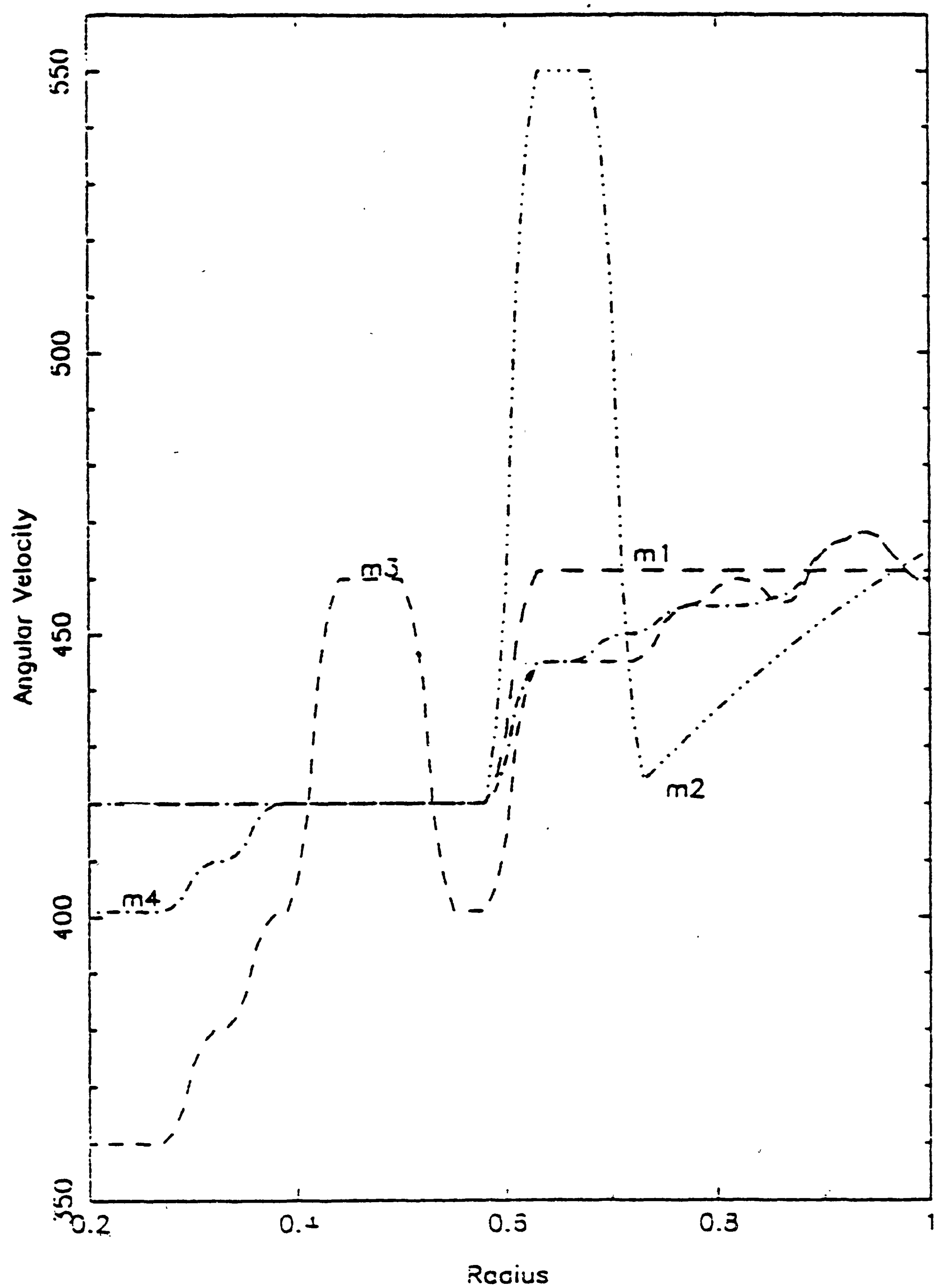

Figure 3. The equatorial components of models m_1, m_2, m_3, and m_4 are plotted against fractional radius.

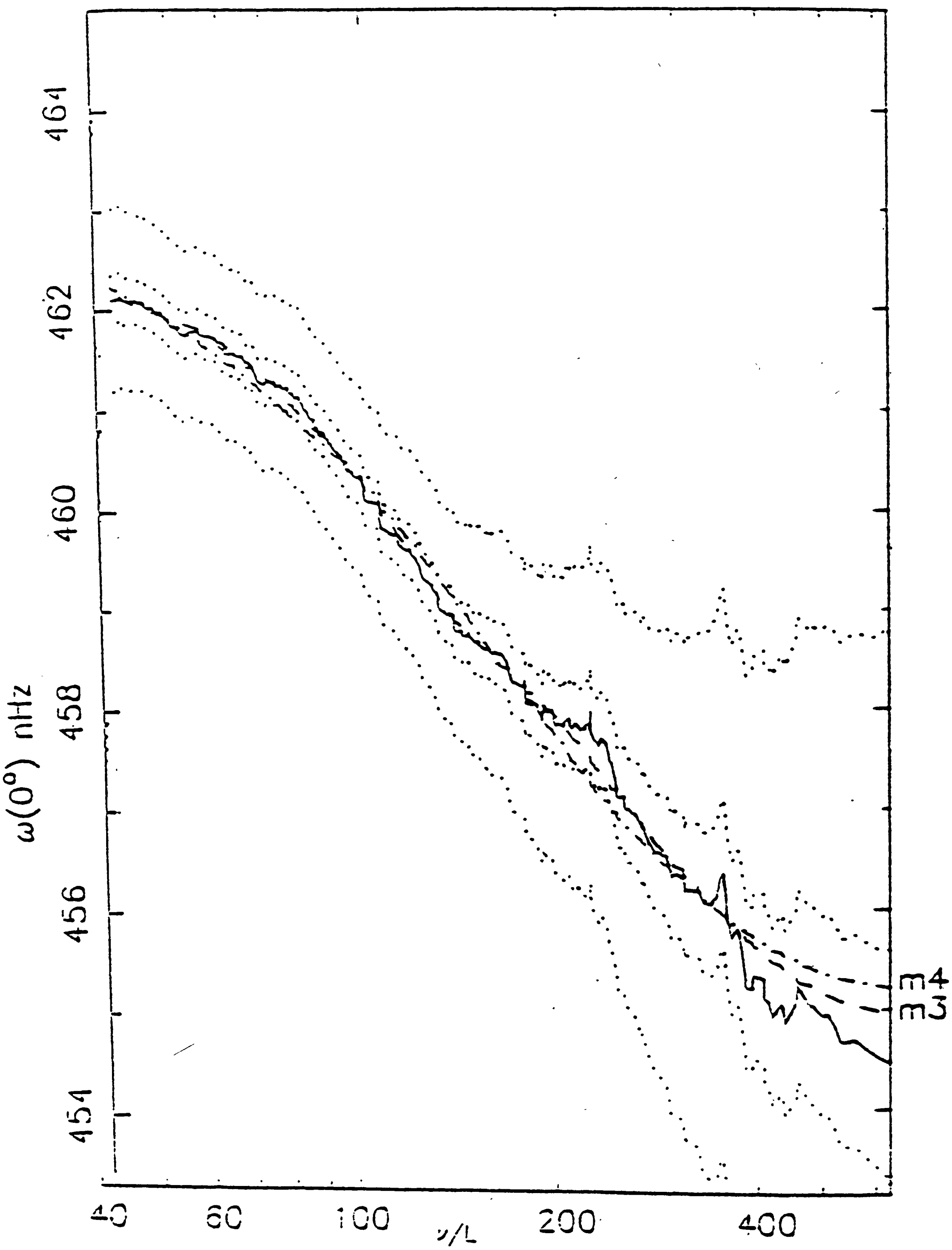

Figure 4. An enlargement of Fig. 2 highlights the theoretical ω-distributions for the equatorial components of models m_3 and m_4.