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Effects of aberration and advection on partial frequency redistribution of photon in spectral line formation

A PERAIAH

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Abstract

A formal solution of equation transfer with aberration and advection terms is developed for spectral lines taking into consideration partial redistribution functions of photon in a spherical medium which is expanding with a radial velocities of the order of V/C = 0 0167

Key words radiative transfer aberration and advection partial frequency redistribution

1 Introduction

It was shown that aberration and advection terms change the radiation field substantially even when one considers medium with coherent and isotropic scattering and stratified in plane parallel layers (Peraiah 1987a) The changes in the mean intensities and fluxes are almost directly propotional to the total optical depth of the medium The photon redistribution in the lines is always affected by the bulk motion of the gases which interacts with the radiation field Therefore it is essential to see whether the partial redistribution of photons in the line is effected by the high velocities of the absorbing and emitting media due to aberration and advection effects It would be interesting to see in general how the solution will change or how complicated it becomes when we introduce the frequency redistribution functions

2 The Redistribution function

The Redistribution functions R_I R_{II} R_{III}, R_{IV} R_V are to be included in the equation of transfer together with the aberration and advection terms The probability of emission of a photon after absorption is given by

$$R(\nu \; q \; \nu' \; q') d\nu' d\Omega' d\nu d\Omega \tag{1}$$

where ν and q are the frequency and direction of the absorbed photon and ν' and q' are the frequency and direction of the emitted photon This probability is subject to the condition that

$$\int\int\int\int R(\nu\ \mathbf{q}\ \nu'\ \mathbf{q}')d\nu' d\Omega' d\nu d\Omega = 1 \tag{2}$$

Here $d\Omega$ and $d\Omega'$ are the areal elements normal to the directions $\mathbf{q}$ and $\mathbf{q}'$ If $\phi(\nu')d\nu'$ is the probability that a photon with frequency in the interval $(\nu\ \nu + d\nu)$ is emitted in the interval $(\nu'\ \nu' + d\nu')$ then

$$4\pi \int\int R(\nu'\ \mathbf{q}'\ \nu\ \mathbf{q})d\nu d\Omega = \phi(\nu'\ \mathbf{q}') \tag{3}$$

where $\phi(\nu'\ \mathbf{q}')$ is the profile function which is again subjected to normalization condition

$$\int\int \phi(\nu'\ \mathbf{q}')d\nu' d\Omega' = 1 \tag{4}$$

There are several cases of redistribution function

Case I If we have two perfectly sharp upper and lower states the photons follow a Doppler redistribution This does not apply to any real line We call the redistribution as R_I which is given by (Mihalas 1978)

$$R_{I-AD}(x'\ \mathbf{q}'\ x\ \mathbf{q}') = \frac{g(q\ q')}{4\pi^2 sin^2 \Theta} exp\{-x'^2 - (x - xCos\Theta)^2 Cosec^2\Theta\} \tag{5}$$

where R_{I-AD} is the angle dependent redistribution function $x = \frac{\nu - \nu}{\Delta}$ Δ being some standard frequency interval Θ is the angle between the vectors q and q' x x' are the standardised frequencies before and after absorption The angle averaged function is given by

$$R_{I-A}(x, x') = \frac{1}{2} erfc|\overline{x}| \tag{6}$$

where

$$erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t}\ dt \tag{7}$$

and $\overline{x}$ is the larger value of x and x'

Case II In this case we have an atom with a perfectly sharp lower state with an upper state broadened by radiative decay This gives us a Lorentz profile This applies to resonance lines in media of low densities that collisional broadening of the upper state is negligible e g Lyman Alpha line of hydrogen in interstellar medium The angle dependent redistribution function is given by

$$R_{II-AD}(x\ \mathbf{q}\ x'\ \mathbf{q}') = \frac{g(q, q')}{4\pi^2 sin^2\Theta} exp\{\frac{(-x - x'^2)}{2} Cosec^2(\frac{\Theta}{2})\} \times H(asec\frac{\Theta}{2}\ \frac{x + x'}{2} sec\frac{\Theta}{2}) \tag{8}$$

where

$$H(a, u) = \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{e^{-y^2}}{(u - y)^2 + a^2} dy \tag{9}$$

is the Voigt function and a is the damping constant Here

$$g(\mathbf{q}\ \mathbf{q}') = \frac{1}{4\pi} \tag{10}$$

The angle averaged function is given by

$$R_{II-A}(x\ x') = \frac{1}{\pi^{\frac{3}{2}}} \int_{\frac{1}{2}|\overline{x}-\underline{x}|}^{\infty} e^{-u} \{tan^{-1}(\frac{\underline{x}+u}{a}) - tan^{-1}\frac{\overline{x}-u}{a})\}du \tag{11}$$

where $\overline{x}$ and $\underline{x}$ are the larger and smaller values of x and x'

Case III In this case an atom will have a perfectly sharp lower state and a collisionally broadened upper state All the excited electrons are randomly distributed over the substates of the upper states before emission occurs The absorption profile is Lorentzian The damping consists of radiative and collisional rates and represents the full width of the upper state The redistribution function R_{III} is given by

$$R_{III}(\nu'\ q'\ \nu\ q) = \frac{g(q\ q')}{\pi} \int_{-\infty}^{\infty} du_1 e^{-u}\ (\frac{\delta}{\pi})\{[(\nu' - \nu_0 - Wu_1)^2 + \delta^2]^{-1}$$
$$\times \int_{-\infty}^{\infty} du_2 e^{-u_2^2} (\frac{\delta}{\pi})([\nu - W(u_1 cos\Theta + u_2 sin\Theta) - \nu_0]^2 + \delta^2)^{-1} \tag{12}$$

The same can be written in terms of dimensional units as

$$R_{III}(x'\ q'\ x\ q) = \frac{g(q'\ q)}{\pi^2}\sigma$$
$$\int_{-\infty}^{\infty} \frac{e^{-u^2} H(\sigma\ xcosec\Theta - ucot\Theta)}{(x'-u)^2 + a} du \tag{13}$$

where

$$\sigma = a\ cosec\Theta \quad \text{and} \quad a = \frac{\delta}{W} \tag{14}$$

$$W = \Delta_s = \frac{\nu_0}{c}(\frac{2kT}{m})^{\frac{1}{2}} = \nu_0(\frac{V_{Thermal}}{c}) \tag{15}$$

The angle averaged function is

$$R_{III-A}(x'\ x) = \pi^{-\frac{3}{2}} \int_0^{\infty} e^{-u^2}[tan^{-1}(\frac{x'+u}{a}) - tan^{-1}(\frac{x'-u}{a})]$$
$$\times [tan^{-1}(\frac{x+u}{a}) - tan^{-1}(\frac{x-u}{a})]du \tag{16}$$

The redistribution functions follow certain symmetric relation as follows

$$R(-x\ q, x'\ q') = R(x\ q\ x'\ q') \tag{17}$$

$$R(-x, -q, x'\ q') = R(-x\ q\ x'\ -q')$$
$$= R(x, q, x', q') \tag{18}$$

and

$$R(x\ q\ x'\ q') = R(x', q', x, q) \tag{19}$$

In a static medium all these functions hold true Therefore it is enough if we calculate one set of the functions However in a moving media there are two cases (1)observer's frame and (2)comoving frame

In observer s frame the absorption coefficient changes both in spatial as well as in angle coordinates because of Doppler shifts in frequencies due to velocities Therefore one will have to calculate these functions at every radial point and in all directions in angle In a comoving frame the absorption coefficient is constant and these functions need to be calculated only once

3 Aberration and Advection

We can write the equation of transfer with the aberration and advection and redistribution function as follows(Peraiah 1978 1987b)

$$(\beta+\mu_0)\frac{\partial U(r\ \mu_0\ x)}{\partial r}+\frac{1-\mu_0^2}{r}\{1+\mu_0\beta(1-\frac{r}{\beta}\frac{d\beta}{dr})\}\frac{\partial U(r\ \mu_0\ x)}{\partial\mu_0}$$
$$+3\{\frac{\beta}{r}(1-\mu_0^2)+\mu_0^2\frac{d\beta}{dr}\}-\frac{2(\mu_0+\beta)}{r}U(r\ \mu_0\ x)-\{\frac{v}{r}(1-\mu_0^2)$$
$$+\mu_0^2\frac{dv}{dr}\}\frac{\partial U(r\ \mu_0\ x)}{\partial x}=K_L(r)[\beta'+\phi(r\ \mu_0\ x)][S(r\ \mu_0\ x)-U(r\ \mu_0\ x)] \tag{20}$$

for $0<\mu\le 1$ and

$$(\beta-\mu_0)\frac{\partial U(r\ -\mu_0\ x)}{\partial r}-\frac{1-\mu_0^2}{r}[1-\mu_0\beta(1-\frac{r}{\beta}\frac{d\beta}{dr})]\frac{\partial U(r\ -\mu_0\ x)}{\partial\mu_0}$$
$$[\frac{v}{r}(1-\mu_0^2)+\mu_0^2\frac{dv}{dr}]\frac{\partial U(r\ -\mu_0\ x)}{\partial x}+3[\frac{\beta}{r}(1-\mu_0^2)+\mu_0^2\frac{d\beta}{dr}]$$
$$-\frac{2(\beta-\mu_0)}{r}U(r\ -\mu_0\ x)=K_L[\beta'+\phi(r\ -\mu_0\ x)][S(r\ -\mu_0\ x)-U(r\ -\mu_0\ x)] \tag{21}$$

for $-1\le\mu<0$ Here

$$\mu_0=\frac{\mu+\beta}{1-\mu\beta}\quad \beta=\frac{V}{C} \tag{22}$$

$$U(r\ \mu_0\ x)=4\pi r^2 I(r\ \mu_0\ x) \tag{23}$$

and $I(r\ \mu_0\ x)$ is the specific intensity making an angle of $cos^{-1}\mu$ with the radius vector r K_L is the absorption coefficient at the line centre Furthermore ϕ is the profile function and $\beta'=\frac{K_C}{K_L}$ where K_C is the absorption coefficient in the continuum S is the source function given by

$$S(r\ \mu_0\ x)=\frac{\phi(r\ \mu_0\ x)S_L(r\ \mu_0\ x)+\beta' S_C(r)}{\phi(r\ \mu_0\ x)+\beta'} \tag{24}$$

and

$$S(r\ -\mu_0\ x)=\frac{\phi(r\ -\mu_0\ x)S_L(r\ -\mu_0,x)+\beta' S_C(r)}{\phi(r\ -\mu_0\ x)+\beta'} \tag{25}$$

Here S_L and S_C are the source function in the line and continuum respectively We have

$$S_C(r)=\rho(r)B(x_1\ T_e(r)) \tag{26}$$

where $\rho(r)$ is an arbitrary factor and $B(x\ T\ (r))$ is the Planck function and

$$S_L(r\ \mu_0\ x)=\frac{(1-\varepsilon)}{\phi(r\ \mu_0\ x)}\int_{-\infty}^{+\infty}dx'\times\int_{-1}^{+1}R(x\ \mu_0\ r\ x'\ \mu_0'\ r)$$
$$I(x'\ \mu_0'\ r)d\mu_0'+\varepsilon B(x\ T\ (r)) \tag{27}$$

$$S_I(r, -\mu_0, x) = \frac{(1-\varepsilon)}{\phi(r, -\mu_0, x)} \int_{-\infty}^{+\infty} dx' \int_{-1}^{+1} R(x, -\mu_0, r, x', \mu_0', r)$$
$$I(x', \mu_0', r) d\mu_0' + \varepsilon B(x, T_e(r)) \tag{28}$$

where R stands for the frequency redistribution function The profile function is given by equation(3) The quantity ε is the probability per scatter that a photon will be destroyed by collisional deexcitation and is given by

$$\varepsilon = \frac{C_{21}}{C_{21} + A_{21}[1 - exp(\frac{-h\nu}{kT})]^{-1}} \tag{29}$$

we shall integrate equation (20) and (21) by expanding the specific intensity $U(r, \mu_0, x)$ as follows (Peraiah et al 1986)

$$U(r, \mu_0, x) = U_0 + U_r \xi + U_\mu \eta + U_x \chi + U_{r\mu} \eta\xi$$
$$U_{\mu_0 x}\eta\chi + U_{xr}\chi\xi + U_{r\mu x}\xi\eta\chi \tag{30}$$

where

$$\xi = \frac{r - \bar{r}}{\frac{\Delta r}{2}} \quad \bar{r} = \frac{1}{2}(r_i + r_{i-1}) \quad \Delta r = r_i - r_{i-1} \tag{30a}$$

$$\eta = \frac{\mu_0 - \overline{\mu_0}}{\frac{\Delta\mu_0}{2}} \quad \overline{\mu_0} = \frac{1}{2}(\mu_{0\,j} + \mu_{0\,j-1}) \quad \Delta\mu_0 = \mu_{0\,j} - \mu_{0\,j-1} \tag{30b}$$

$$\chi = \frac{x - \bar{x}}{\frac{\Delta x}{2}} \quad \bar{x} = \frac{1}{2}(x_k + x_{k-1}) \quad \Delta x = x_k - x_{k-1} \tag{30c}$$

This gives us

$$(U_\mu + U_{r\mu_0}\xi + U_{\mu x}\chi + U_{r\mu_0 x}\xi\chi)[1 - \mu_0\beta(1 - \frac{r}{\beta}\frac{d\beta}{dr})] \times (\frac{1-\mu_0^2}{r})\frac{2}{\Delta\mu_0}$$
$$+ (U_r U_{r\mu_0}\eta + U_{xr}\chi + U_{r\mu_0 x}\eta\chi)\frac{(\beta + \mu_0)}{\Delta r} + K_L(r)(\beta' + \phi)U(r, \mu_0, x)$$
$$-[\frac{1}{r}\{2(\mu_0 + \beta) - 3\beta(1 - \mu_0^2)\} - 3\mu_0^2\frac{d\beta}{dr}]U(r, \mu_0, x) - \frac{2}{\Delta x}[\frac{v'(r)}{r}(1 - \mu_0^2)$$
$$+ \mu_0^2\frac{dv(r)}{dr}][U_x + U_{\mu_0 x}\eta + U_{xr}\xi + U_{r\mu_0 x}\xi\eta]$$
$$= K_I\{\frac{(1-\varepsilon)}{2}\int_{-\infty}^{+\infty}\int_{-1}^{+1} R(x_1, \mu_0, x', \mu_0')U(x', \mu_0', r)dx'd\mu_0'$$
$$+ \varepsilon B(x, T_e) + \beta'\rho B(x, T_e)\} \tag{31}$$

$$-(U_{\mu_0} + U_{r\mu_0}\xi + U_{\mu_0 x}\chi + U_{r\mu_0 x}\xi\chi)\frac{2}{\Delta\mu_0}(\frac{1-\mu_0^2}{r}) \times [1 - \mu_0\beta(1 - \frac{r}{\beta}\frac{d\beta}{dr})]$$
$$+ \frac{2}{\Delta r}(\beta - \mu_0)(U_r + U_{r\mu_0}\eta + U_{xr}\chi + U_{r\mu_0 x}\eta\chi) + K_L(r)(\beta' + \phi)U(r, -\mu_0, x)$$
$$-\{\frac{2(\beta - \mu_0)}{r} - 3\beta(1 - \mu_0^2) - 3\mu_0^2\frac{d\beta}{dr}\}]U(r, \mu_0, x) - \frac{2}{\Delta x}[\frac{v(r)}{r}(1 - \mu_0^2)$$
$$+ \mu_0^2\frac{dv(r)}{dr}] \times (U_x U_{\mu_0 x}\eta + U_{xr}\xi + U_{r\mu_0}\xi\eta)$$
$$= K_L\{(\frac{1-\varepsilon}{2})\int_{-\infty}^{+\infty}\int_{-1}^{+1} R(x, \mu_0, x', \mu_0')U(x', \mu_0', r)dx'd\mu_0'$$
$$+ \varepsilon B(x, T_e) \times \beta'\rho B(x, T_e)\} \tag{32}$$

We shall integrate equations (31) and (32) We apply the operator

$$(\mu_{0\,j} - \mu_{0\,j-1})^{-1} \int_{0\,j-1}^{0\,j} \quad d\mu_0 \tag{33}$$

on equation (31) and obtain

$$\frac{2}{\Delta r} P_1 + \frac{2}{r\,\Delta\mu_0} P_2 P_3 + K_L P_4 - Q_1 Q_2 + \Delta\mu_0 \quad \overline{\mu_0} Q_3 Q_4 - \frac{2}{\Delta x}$$
$$[(U_x + U \quad \xi) R_1 + \frac{1}{3}(U_\mu \quad + U_{\ \mu} \quad \xi)\overline{\mu_0} \quad \Delta\mu_0 R_2] = \Sigma^+ \tag{34}$$

where

$$P_1 = (U \quad + U_{\ r}\chi)(\beta + \overline{\mu_0}) + \frac{1}{6}\Delta\mu_0 (U_{\ r} \quad \chi) \tag{35}$$

$$P_2 = \overline{\mu_0}(1 - < \mu_0^2 >)\beta R + (1 - \overline{\mu_0^2}) \tag{36}$$

$$P_3 = U_{\mu_0} + U_{\ \mu}\,\xi + U_{r}\ _x\chi + U_{\ r}\ _x\xi\chi \tag{37}$$

$$P_4 = U_0 + U_r\xi + U_x\chi + U_x \quad \chi\xi \tag{38}$$

$$Q_1 = \frac{2\overline{\mu_0} + \beta(3\overline{\mu_0^2} - 1)}{r} - 3\overline{\mu_0^2}\frac{d\beta}{dr} \tag{39}$$

$$Q_2 = U_0 + U\,\xi + U_x\chi + U_x \quad \chi\xi \tag{40}$$

$$Q_3 = \frac{1}{r}(\beta + \frac{1}{3\overline{\mu_0}}) - \frac{d\beta}{dr} \tag{41}$$

$$Q_4 = U_r \quad + U_{rr}\ \xi + U_r \quad \chi + U_{r\mu}\ _x\xi\chi \tag{42}$$

$$R_1 = \frac{v(r)}{r}(1 - \mu_0^2) + \overline{\mu_0^2}\frac{dv}{dr} \tag{43}$$

$$R_2 = \frac{dv(r)}{dr} - \frac{v(r)}{r} \tag{44}$$

$$\Sigma^+ = (\mu_{0\,j} - \mu_{0\,j-1})^{-1} \int_{\mu\ \ j\ \ 1}^{\mu\ \ j} \{[\int_{-\infty}^{+\infty} \int_{-1}^{+1} K_L(r)\frac{(1-\varepsilon)}{2}$$
$$R(x \ \ \mu_0 \ \ x' \ \ \mu_0')(U_0 + U_r\xi + U_\mu \ \ \eta + U_x\chi + U_{\ \mu_0}\xi\eta$$
$$+ U_{\mu\ x}\eta\chi + U_{xr}\chi\xi + U_{r\mu} \quad \xi\eta\chi)dx'd\mu_0'] + (\varepsilon + \rho\beta')B(x\ \ T\ )\}d'_\mu \tag{45}$$

$$R = 1 - \frac{d\ln\beta}{d\ln r} \tag{46}$$

$$< \mu_0^2 >= \frac{1}{2}(\mu_{0,j}^2 + \mu_{0,j-1}^2) \tag{47}$$

In a similar way if we apply equation (33) on (31) we obtain,

$$\frac{2}{\Delta r}P_1' - \frac{2}{\Delta\mu_0}\frac{1}{r}P_2'P_3' + K_L P_4' + Q_1'Q_2' + \frac{\Delta\mu_0 \ \mu_0}{r}Q_3'Q_4'$$
$$- \frac{2}{\Delta x}[(U_x + U_{xr}\xi)R_1' + \frac{1}{3}(U_{\mu x} + U_{r\mu_0 x}\xi)\Delta\mu_0 \ \overline{\mu_0}R_2'] = \Sigma^- \tag{48}$$

where

$$P_1' = (U_r + U_{r}\chi)(\beta - \overline{\mu_0}) - \frac{1}{3}\Delta\mu_0 U_{\mu_0} + U_{r\mu}) \tag{49}$$

$$P_2' = (1 - \overline{\mu_0^2}) - \overline{\mu_0}(1 - < \mu_0^2 >)\beta R \tag{50}$$

$$P_3' = U_\mu + U_{r\mu_0}\xi + U_{\mu_0 x}\chi + U_{r\mu_0}\xi\chi \tag{51}$$

$$P_4' = U_0 + U_r\xi + U \chi + U_{xr}\chi\xi \tag{52}$$

$$Q_1' = \frac{\overline{2\mu_0} + \beta - 3\beta\overline{\mu_0^2}}{r} + 3\overline{\mu_0^2}\frac{d\beta}{dr} \tag{53}$$

$$Q_2' = U_0 + U_r\xi + U \chi + U_x \chi\xi \tag{54}$$

$$Q_3' = r\frac{d\beta}{dr} - \beta + \frac{1}{3\overline{\mu_0}} \tag{55}$$

$$Q_4' = U_{\mu_0} + U_{r\mu_0}\xi + U_{\mu_0}\chi + U_{r\mu_0 x}\xi\chi \tag{56}$$

$$R_1' = (1 - \overline{\mu_0^2})\frac{v(r)}{r} - \overline{\mu_0^2}\frac{dv(r)}{dr} \tag{57}$$

$$R_2' = \frac{dv(r)}{dr} - \frac{v(r)}{r} \tag{58}$$

$$\Sigma^- = (\mu_{0j} - \mu_{0j-1})^{-1}\int_{0j-1}^{0j}\{[\int_{-\infty}^{+\infty}\int_{-1}^{+1} K_L(r)(\frac{1-\varepsilon}{2})R(x - \mu_0 x', \mu_0')$$
$$(U_0 + U_r\xi + U_\mu \eta + U_x\chi + U_{r\mu_0}\xi\eta + U_{\mu_0}\eta\chi + U_{xr}\chi\xi$$
$$+ U_{r\mu_0 x}\xi\eta\chi)dx'd\mu_0'] + (\varepsilon + \rho\beta')B(x T_e)\}d\mu_0' \tag{59}$$

Now we shall apply the operator

$$\Gamma = \frac{1}{V}\int_{r-1}^{r} 4\pi r^2 dr \tag{60}$$

on equation (34) and (48) This will give us,

$$\frac{2}{\Delta r}a + (U_{\mu} \ + U_{\mu \ x}\chi)b + (U_{r\mu} \ + U_{r\mu} \ \chi)G_c + K_L d$$
$$\{3\beta\overline{\mu_0^2} + 2\overline{\mu_0} - 1\}e + 3\overline{\mu_0^2}\frac{d\beta}{dr}f - \Delta\mu_0\overline{\mu_0}g - \frac{2}{\Delta x}h = \Gamma' \tag{61}$$

where

$$a = \frac{1}{6}\Delta\mu_0(U_{r\mu} \ + U_{r\mu_0} \ ) + (U \ + U_x \ \chi)(\beta + \overline{\mu}) \tag{62}$$

$$b = \Delta A(\frac{1-\overline{\mu_0^2}}{\Delta\mu_0}) + G(\frac{1}{2}\frac{\Delta A}{V} - \frac{1}{\beta}\frac{d\beta}{dr}) \tag{63}$$

$$c = \frac{2}{\Delta r}\frac{1-\overline{\mu_0^2}}{\Delta\mu_0}(2 - \frac{\overline{r}\Delta A}{V}) - \frac{1}{6}\frac{\Delta A}{\overline{A}}\frac{1}{\beta}\frac{d\beta}{dr} + \frac{2}{\Delta r} - \frac{1}{\Delta r}\frac{\Delta A}{V} \tag{64}$$

$$d = \frac{1}{6}\frac{\Delta A}{\overline{A}}(U_r + U_{xr}\chi) + U_0 + U_x\chi \tag{65}$$

$$e = \frac{U_r \ + U_{rx}\chi}{\Delta r}(2 - \frac{\overline{r}\Delta A}{V}) + \frac{1}{2}(U_0 + U_x\chi)\frac{\Delta A}{\overline{A}} \tag{66}$$

$$f = (U_r + U_{xr}\chi)\frac{1}{6}\frac{\Delta A}{\overline{A}} + (U_0 + U_x\chi) \tag{67}$$

$$g = [(U_{\ \mu_0} + U_{r\mu \ x}\chi)(\frac{2}{\Delta r} - \frac{\overline{r}}{\Delta r}\frac{\Delta A}{V}) + \frac{1}{2}\frac{\Delta A}{V}(U_{\mu_0} + U_{\mu_0 x}\chi)]$$
$$(\beta + \frac{1}{3\overline{\mu_0}}) - [\frac{1}{6}\frac{\Delta A}{\overline{A}}(U_{r\mu_0} + U_{r\mu \ x}\chi) + (U_{\mu_0} + U_{\mu \ x}\chi)]\frac{d\beta}{dr} \tag{68}$$

$$h = (1-\overline{\mu_0^2})V(r)\{\frac{1}{2}\frac{\Delta A}{V}U_x + \frac{U_{xr}}{\Delta r}(2 - \frac{\overline{r}\Delta A}{A})\} + \overline{\mu_0^2}\frac{dv(r)}{dr}(U_x + \frac{1}{6}\frac{\Delta A}{\overline{A}}U_{xr})$$
$$+ \frac{1}{3}\overline{\mu_0} \ \ \Delta\mu_0\{\frac{dv(r)}{dr}(U_{\mu_0} \ + \frac{1}{6}\frac{\Delta A}{\overline{A}}U_{r\mu_0} \ )\} - v(r)\frac{1}{3}\overline{\mu_0} \ \ \Delta\mu_0$$
$$\{\frac{1}{2}\frac{\Delta A}{V}U_{\mu \ x} + (\frac{2}{\Delta r} - \frac{\overline{r}A}{\Delta r \ V})U_{\ \mu_0 x}\} \tag{69}$$

$$\Gamma' = \Gamma\Sigma^{+} \tag{70}$$

Similarly application Γ on (48) will give us

$$\frac{2}{\Delta r}a_1 + G(U_{\mu_0} + U_{\mu_0} \ \chi)b_1 + (U_{r\mu} \ + U_{r\mu \ x}\chi)c_1 + K_L d_1 + (U_0 + U_x\chi)$$
$$e_1 + (U_r + U_{xr}\chi)f_1 + (U_{\mu} \ + U_{\mu_0} \ \chi)g_1 + (U_{r\mu_0 x}\chi)h_1 + \frac{2}{\Delta x}i_1 = \Gamma' \tag{71}$$

where

$$a_1 = (\beta - \overline{\mu_0})(U_r + U_{xr}\chi) - \frac{1}{6}\Delta\mu_0(U_{r\mu_0} + U_{r\mu_0 x}\chi) \tag{72}$$

$$b_1 = (\frac{1}{2}\frac{\Delta A}{V} - \frac{1}{\beta}\frac{d\beta}{dr})G - \frac{\Delta A}{V}\frac{1-\mu_0^2}{\Delta\mu_0} \quad (73)$$

$$c_1 = (P - \frac{1}{6\beta}\frac{\Delta A}{\overline{A}}\frac{d\beta}{dr})G - \frac{2}{\Delta\mu_0}(1-\overline{\mu_0^2})P \quad (74)$$

$$d_1 = \frac{1}{6}\frac{\Delta A}{V}(U \;+ U \;\;\chi) + (U_0 + U_x\chi) \quad (75)$$

$$e_1 = \frac{1}{2}\frac{\Delta A}{V}(2\overline{\mu_0} + \beta - 3\overline{\mu_0^2}) + 3\overline{\mu_0^2}\frac{d\beta}{dr} \quad (76)$$

$$f_1 = P(2\overline{\mu_0} + \beta - 3\beta\mu_0^2) + \frac{1}{2}\overline{\mu_0^2}\frac{d\beta}{dr}\frac{\Delta A}{\overline{A}} \quad (77)$$

$$g_1 = \Delta\mu_0 \;\; \overline{\mu_0}[\frac{d\beta}{dr} + \frac{1}{2}\frac{\Delta A}{V}(\frac{1}{3\overline{\mu_0}} - \beta)] \quad (78)$$

$$h_1 = \Delta\mu_0 \;\; \overline{\mu_0}[\frac{1}{6}\frac{\Delta A}{\overline{A}}\frac{d\beta}{dr} + P(\frac{1}{3\overline{\mu_0}} - \beta)] \quad (79)$$

$$i_1 = V(r)(1-\overline{\mu_0^2})(\frac{1}{2}\frac{\Delta A}{V}U \;+ PU \;\;) + \overline{\mu_0^2}\frac{dv(r)}{dr}(U_x + \frac{1}{6}\frac{\Delta A}{\overline{A}}U_x\;) + \frac{1}{3}\Delta\mu_0 \;\; \overline{\mu_0}$$
$$\{\frac{dv(r)}{dr}(U_r \;\; + \frac{1}{6}\frac{\Delta A}{\overline{A}}U_{\;\mu_0}\;)\} - \frac{1}{3}\Delta\mu_0 \;\; \overline{\mu_0}V(r)(\frac{1}{2}\frac{\Delta A}{V}U_{\mu\;x} + PU_{\;r\;x}) \quad (80)$$

where

$$P = \frac{2}{\Delta r} - \frac{\overline{r}}{V}\frac{\Delta A}{\Delta r} \quad (81)$$

We shall apply the operator Z given by

$$Z = \frac{1}{\Delta x}\int_x \quad dx \quad (82)$$

on equation (61) and (71) we obtain

$$[(1-\overline{\mu_0^2})V(r)\{\frac{1}{2}\frac{\Delta A}{V}U_x + \frac{1}{\Delta r}(2 - \frac{\overline{r}\;\;\Delta A}{V})U_x\;\} + \overline{\mu_0^2}\frac{dv(r)}{dr}(U \;+ \frac{1}{6}\frac{\Delta A}{\overline{A}}U \;\;)$$
$$+ \frac{1}{3}\Delta\mu_0 \;\; \overline{\mu_0^2}\{\frac{dv(r)}{dr}(U_{r\,0x} + \frac{1}{6}\frac{\Delta A}{\overline{A}}U_{r\mu_0 x})\} - \frac{1}{3}V(r)\overline{\mu_0} \;\; \Delta\mu_0$$
$$\frac{1}{2}\frac{\Delta A}{V}U_{r\;\;x} + \frac{1}{\Delta r}(2 - \frac{\overline{r}\;\;\Delta A}{V})U_{\;r\;x}\}\frac{2}{\Delta x} + \Delta\mu_0\overline{\mu_0}[(\frac{1}{3\overline{\mu_0}} + \beta)$$
$$\{\frac{1}{2}\frac{\Delta A}{V}U_r \;\; + \frac{1}{\Delta r}(2 - \frac{\overline{r}\Delta A}{V})U_{r\mu_0}\} - \frac{d\beta}{dr}(U_{\mu_0} + \frac{1}{6}\frac{\Delta A}{\overline{A}}U_{r\mu}\;)] - K_L(U_0 + \frac{1}{6}\frac{\Delta A}{\overline{A}}U_r)$$
$$+ \{2\overline{\mu_0} + (3\overline{\mu_0^2} - 1)\beta\}\{\frac{1}{2}\frac{\Delta A}{V}U_0 + \frac{1}{\Delta r}(2 - \frac{\overline{r}\Delta A}{V})U_r\} - 3\overline{\mu_0^2}\frac{d\beta}{dr}(U_0 + \frac{1}{6}\frac{\Delta A}{A}U_r)$$
$$= U_{\;\mu_0}\{(\frac{2}{\Delta r} - \frac{\overline{r}}{\Delta r}\frac{\Delta A}{V})G - \frac{1}{6}\frac{\Delta A}{\overline{A}}\frac{G}{\beta}\frac{d\beta}{dr} + \frac{2}{\Delta r\;\;\Delta\mu_0}(1-\overline{\mu_0^2})(2 - \frac{\overline{r}\Delta A}{V})\}$$
$$+ U_\mu\;\{\frac{1}{2}\frac{\Delta A}{V}\frac{G}{\beta}\frac{d\beta}{dr} + \frac{\Delta A}{V}(\frac{1-\mu_0^2}{\Delta\mu_0})\} + \frac{2}{\Delta r}\{(\beta + \overline{\mu_0})U_r + \frac{1}{6}\Delta\mu_0 \;\; U_{r\mu}\;\} - \Gamma_2 \quad (83)$$

where

$$\Gamma_2 = Z\Gamma' \tag{84}$$

and

$$[\frac{1}{3}\overline{\mu_0} \quad \mu_0\{\frac{dv(r)}{dr}(U_{\mu\ x} + \frac{1}{6}\frac{A}{\overline{A}}U_{\ \mu\ x}) - V(r)(PU_{\ \mu_0} + \frac{1}{2}\frac{A}{V}U_{\ \ x})\}$$
$$+\{(1-\overline{\mu_0^2})V(r)(\frac{1}{2}\frac{A}{V}U_x + PU \ \ ) + \mu_0^2\frac{dv(r)}{dr}(U_x + \frac{1}{6}\frac{A}{\overline{A}}U_x\ )\}]$$
$$\frac{2}{x} - K_L(U_0 + \frac{1}{6}\frac{A}{\overline{A}}U_r + U_0Y_1 + U_rY_2 + U_{\ \ }Y_3 + U_{\ \ }Y_4)$$
$$= U_\mu\ (\frac{1}{2}\frac{A}{V}G - \frac{G}{\beta}\frac{d\beta}{dr} - \frac{A}{V}\frac{1-\mu_0^2}{\mu_0} + \frac{2}{r}\{(\beta - \overline{\mu_0})U \ - \frac{1}{6}\ \mu_0 U_{\ \ }\}$$
$$+U_{\ \ }\{GP - \frac{1}{6}\frac{A}{\overline{A}}\frac{G}{\beta}\frac{d\beta}{dr} - \frac{2}{\mu\ \ _0}(1-\overline{\mu_0^2})P\} - \Gamma_3 \tag{85}$$

where

$$\Gamma_3 = Z\Gamma_1' \tag{86}$$

$$Y_1 = \{2\overline{\mu_0} + \beta(1 - 3\overline{\mu_0^2}) + 3\overline{\mu_0^2}\frac{d\beta}{dr}\}\frac{1}{2}\frac{A}{V} \tag{87}$$

$$Y_2 = \frac{1}{2}\overline{\mu_0^2}\frac{d\beta}{dr}\frac{A}{\overline{A}} + P\{2\overline{\mu_0} + \beta(1 - 3\overline{\mu_0^2})\} \tag{88}$$

$$Y_3 = \{\frac{1}{2}\frac{A}{V\ 3\overline{\mu_0}}(1 - 3\overline{\mu_0}\beta) + \frac{d\beta}{dr}\}\mu\ \ _0\ \overline{\mu_0} \tag{89}$$

$$Y_4 = \frac{1}{3}\mu\ \ _0\{\frac{1}{2}\overline{\mu_0}\frac{d\beta}{dr}\frac{A}{\overline{A}} + P(1 - 3\beta\overline{\mu_0})\} \tag{90}$$

collecting the interpolation coefficients in equation (83) and (85) we can write these two equations as

$$\mathbf{L}M = \Gamma_4 \tag{91}$$

$$\mathbf{L} = \begin{bmatrix} l_1 & l_2 & l_3 & l_4 & l_5 & l_6 & l_7 & l_8 \\ l_1 & l_2 & l_3 & l_4 & l_5 & l_6 & l_7 & l_8 \end{bmatrix} \tag{92}$$

$$\mathbf{M} = [U_rU_\mu\ U_{\ \mu_0}U_0U_xU_{xr}U_{\mu_0 x}U_{r\mu\ x}]^T \tag{93}$$

$$l_1 = \frac{r}{2}\frac{A}{\overline{A}}\overline{\mu_0^2}\frac{dg}{dr} - P\ \ r[g(3\overline{\mu_0^2} - 1) + 2\overline{\mu_0}] + 2(g + \overline{\mu_0}) + \frac{r}{6}\frac{A}{\overline{A}}K_I \tag{94}$$

$$l_2 = \frac{A\quad r}{V}(\frac{1-\overline{\mu_0^2}}{\mu_0}) + G\quad r(\frac{1}{2}\frac{A}{V} - \frac{1}{g}\frac{dg}{dr})$$
$$-\ \ r\ \ \mu\ \ _0\overline{\mu_0}\{\frac{3g\overline{\mu_0}+1}{6\overline{\mu_0}}\frac{A}{V} - \frac{dg}{dr}\} \tag{95}$$

$$l_3 = \frac{1}{3}\Delta\mu_0 + \Delta r \quad CP - \frac{1}{6}\frac{\Delta r \quad \Delta A}{\overline{A}}\frac{G}{g}\frac{dg}{dr} + 2P \quad \frac{\Delta r}{\Delta\mu_0}(1-\overline{\mu_0^2})$$
$$- \Delta r \quad \overline{\mu_0}\Delta\mu_0\{\frac{P}{3\overline{\mu_0}}(1+3g\mu_0) - \frac{1}{6}\frac{\Delta A}{\overline{A}}\frac{dg}{dr}\} \quad (96)$$

$$l_4 = \Delta r \quad K_x - \frac{1}{2}\frac{\Delta r \Delta A}{V}\{2\overline{\mu_0} + g(3\overline{\mu_0^2} - 1)\} + 3\overline{\mu_0^2}\frac{dg}{dr} \quad (97)$$

$$l_1' = 2(g - \overline{\mu_0}) + \Delta r(\frac{1}{6}\frac{\Delta A}{\overline{A}}K_L + Y_2) \quad (98)$$

$$l_2' = \Delta r(x_1 + Y_3) \quad (99)$$

$$l_3' = \Delta r(Y_4 + x_2) - \frac{1}{3}\frac{\Delta\mu_0}{\Delta r} \quad (100)$$

$$l_4' = r(K_L = Y_1) \quad (101)$$

$$l_5 = 2\frac{\Delta r}{\Delta x}\{\frac{1}{2}(1-\overline{\mu_0^2})V(r)\frac{\Delta a}{V} + \overline{\mu_0^2}\frac{dv(r)}{dr}\} \quad (102)$$

$$l_6 = 2\frac{\Delta r}{\Delta x}\{(1-\overline{\mu_0^2})PV(r) + \frac{1}{6}\frac{dv(r)}{dr}\overline{\mu_0^2}\frac{\Delta A}{\overline{A}}\} \quad (103)$$

$$l_7 = 2\frac{\Delta r}{\Delta x}\{\frac{1}{3}\overline{\mu_0}\Delta\mu_0[\frac{dv(r)}{dr} - \frac{1}{2}\frac{\Delta A}{V}V(r)]\} \quad (104)$$

$$l_8 = 2\frac{\Delta r}{\Delta x}\{\frac{1}{18}\overline{\mu_0} \quad \Delta\mu_0\frac{dv(r)}{dr}\frac{\Delta A}{\overline{A}} - V(r)P\} \quad (105)$$

$$g = \frac{V}{C} \quad (106)$$

where C is the velocity of light

$$x_1 = \frac{G}{2}\frac{\Delta A}{V} - G\frac{1}{\beta}\frac{d\beta}{dr} - \frac{\Delta A}{V}\frac{1-\overline{\mu_0^2}}{\Delta\mu_0} \quad (107)$$

$$x_2 = PG - 2P\frac{1-\overline{\mu_0^2}}{\Delta\mu_0} - \frac{1}{6}\frac{\Delta A}{\overline{A}}\frac{G}{\beta}\frac{d\beta}{dr} \quad (108)$$

From equation (91) we can derive the transmission and reflection operators for the partial frequency redistribution function in the line by following the procedure set in Peraiah(1987b)

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