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Non-linear stability in the perturbed photogravitational restricted three body problem

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Abstract. We have examined the non-linear stability of triangular equilibrium point in the perturbed photogravitational restricted three body problem. We conclude that the equilibrium point is stable in the range of linear stability except for three mass ratio.

Key Words : stability—photogravitational—RTBP

1. Introduction

The photogravitational restricted three body probelm is a modified model of the well known restricted three body problem in which the primaries are source of radiation. The model was studied by Radzievskii (1950), Chernikov (1970), Simmons *et al.* (1985), Sharma (1987) and others.

In this paper, we have investigated the effect of small perturbations in the coriolis and centrifugal forces on the non-linear stability of triangular equilibrium points in photogravitational restricted three body problem using Moser's modified version of Arnold's theorem (1961) and following the method of Deprit & Deprit (1967).

The force of radiation is taken as

$$F = F_g - F_p = F_g \left(1 - \frac{F_p}{F_g}\right) = qF_g \, ,$$

where F_g is the gravitational attraction force, F_p is the radiation pressure force and q is the mass reduction factor, q_1 and q_2 characterise the radiation effects of two primaries. For simplification, we have used $q_1 = 1 - \delta_1$ and $q_2 = 1 - \delta_2$. δ_1 and $\delta_{1,2} = 0$ i.e. $q_1 = q_2 = 1$

representing the classical case. The perturbations in the coriolis and centrifugal force are introduced by the parameters α, β respectively. We put

$$\alpha = 1 + \epsilon_1; |\epsilon_1| << 1, \beta = 1 + \epsilon_2; |\epsilon_2| << 1.$$

we assume that the radiation does not affect the mean motion of the primaries due to their large mass. We conclude that the equilibrium point is stable in the range of linear stability except for three mass ratio.

2. First order normalisation

The perturbed Lagrangian function of the problem is written as

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \alpha(x\dot{y} - \dot{x}y)\ \beta/2\left[(1-\mu)\ r_1^2 + \mu r_2^2\right] + \frac{(1-\mu)q_1}{r_1} + \frac{\mu q_2}{r_2}\ , \tag{1}$$

where

$$r_1^2 = (x+\mu)^2 + \dot{y}^2$$

$$r_2^2 = (x - 1 + \mu)^2 + y^2 \tag{2}$$

The co-ordinates of the triangular equlibrium points are

$$x = \frac{1}{2} - \mu + \frac{q_1^{2/3} - q_2^{2/3}}{2\beta^{2/3}}$$

$$y = \pm\left[\left(\frac{q_1^{2/3} + q_2^{2/3}}{2\beta^{2/3}}\right) - \left(\frac{q_1^{2/3} + q_2^{2/3}}{2\beta^{2/3}}\right)^2 - \frac{1}{4}\right]^{1/2} \tag{3}$$

as in Haque & Ishwar (1992). We shift the origin to L_4 and expanding in power series of x and y. We find that

$$L = L_0 + L_1 + L_2 + L_3 + L_4 + \ldots\ldots. \tag{4}$$

$$L_0 = \frac{1}{2}(3 + \epsilon_2) + \frac{1}{2}(1 + \epsilon_2)(\mu^2 - \mu) - \delta_1 + \mu\delta_1 - \mu\delta_2$$

$$L_1 = \frac{S}{2}(1 + \epsilon_1)\left(1 - \frac{2}{3}\delta_1 - \frac{4}{3}\mu\delta_1 + \frac{2}{3}\delta_2 + \frac{4}{3}\mu\delta_2\right)\dot{y}$$

$$-\frac{\sqrt{3}}{2}(1+\epsilon_1-\frac{4}{9}\epsilon_2-\frac{2}{9}\delta_1-\frac{2}{9}\delta_2)\,\dot{x}$$

$$L_2=\frac{1}{2}(\dot{x}^2+\dot{y}^2)+(x\dot{y}-\dot{x}y)(1+\epsilon_1)+\frac{3}{8}(1+\frac{5}{3}\epsilon_2$$
$$-\frac{2}{3}\delta_1+\frac{4}{3}\delta_2+2\mu\delta_1-2\mu\delta_2)x^2+\frac{9}{8}(1+\frac{7}{9}\epsilon_2$$
$$+\frac{2}{9}\delta_1-\frac{4}{9}\delta_2-\frac{2}{3}\mu\delta_1+\frac{2}{3}\mu\delta_2)\,y^2$$
$$+\frac{3\sqrt{3}}{4}S(1+\frac{11}{9}\epsilon_2-\frac{2}{9}\delta_1+\frac{4}{9}\delta_2-\frac{2}{3}\mu\delta_1$$
$$+\frac{2}{3}\mu\delta_2-\frac{4}{3}\mu^2\delta_1+\frac{4}{3}\mu^2\delta_2)\,xy$$

$$L_3=\frac{7}{16}S\,[1+\frac{25}{21}\epsilon_2+\frac{10}{21}\delta_1-\frac{2}{7}\delta_2+\frac{16}{21}\mu\delta_1-\frac{16}{21}\mu\delta_2$$
$$-\frac{8}{21}\mu^2\delta_1-\frac{8}{21}\mu^2\delta_2]\,x^3-\frac{\sqrt{3}}{48}[9+41\epsilon_2-26\,\delta_1+58\,\delta_2$$
$$+84\mu\delta_1-84\,\mu\delta_2x^2y-\frac{33}{16}S\,[1+\frac{15}{11}\epsilon_2+\frac{10}{33}\delta_1$$
$$+\frac{2}{33}\delta_2+\frac{8}{33}\mu\delta_1-\frac{8}{33}\mu\delta_2-\frac{8}{11}\mu^2\delta_1-\frac{8}{11}\mu^2\delta_2]\,xy^2$$
$$-\frac{\sqrt{3}}{48}[9+14\delta_1-22\delta_2+\epsilon_2-36\mu\delta_1+36\mu\delta_2]y^3$$

$$L_4=-\frac{1}{128}[37+95\epsilon_2-\frac{26}{3}\delta_1+\frac{200}{3}\delta_2+\frac{226}{3}\mu\delta_1$$
$$-\frac{226}{3}\mu\delta_2]\,x^4-\frac{75}{32\sqrt{3}}S\,[1+\frac{215}{3}\epsilon_2+\frac{310}{3}\delta_1-\frac{310}{3}\delta_2$$
$$+210\mu\delta_1-210\mu\delta_2+\frac{20}{3}\mu^2\delta_1+\frac{20}{3}\mu^2\delta_2]\,x^3y$$
$$+\frac{3}{64}[41+115\epsilon_2-\frac{58}{3}\delta_1+\frac{280}{3}\delta_2+\frac{338}{3}\mu\delta_1 \tag{5}$$
$$-\frac{338}{3}\mu\delta_2]\,x^2y^2+\frac{135}{32\sqrt{3}}S\,[1+\frac{37}{27}\epsilon_2+\frac{26}{27}\delta_1$$
$$-\frac{26}{27}\delta_2+\frac{14}{9}\mu\delta_1-\frac{14}{9}\mu\delta_2-\frac{20}{27}\mu^2\delta_1-\frac{20}{27}\mu^2\delta_2]\,xy^3$$
$$+\frac{1}{128}[3-55\epsilon_2+62\delta_2-120\delta_2-182\mu\delta_1$$
$$+182\mu\delta_2]y^4$$

where $S=1-2\mu$

To the first order, Lagrange's equations of motion are

$$\ddot{x} - 2\dot{y}(1 + \epsilon_1) = \frac{3}{4}(1 + \frac{5}{3}\epsilon_2 - \frac{2}{3}\delta_1 + \frac{4}{3}\delta_2 + 2\mu\delta_1 - 2\mu\delta_2)x$$
$$+ \frac{3\sqrt{3}}{4} S\,(1 + \frac{11}{9}\epsilon_2 - \frac{2}{9}\delta_1 + \frac{4}{9}\delta_2 - \frac{2}{3}\mu\delta_1 \tag{6}$$
$$+ \frac{2}{3}\mu\delta_2 - \frac{4}{3}\mu^2\delta_1 + \frac{4}{3}\mu^2\delta^2)\,y,$$

$$\ddot{y} - 2\dot{x}(1 + \epsilon_1) = \frac{9}{4}(1 + \frac{7}{9}\epsilon_2 + \frac{2}{9}\delta_1 - \frac{4}{9}\delta_2 - \frac{2}{3}\mu\delta_1 + \frac{2}{3}\mu\delta_2)\,y$$
$$+ \frac{3\sqrt{3}}{4} S\,(1 + \frac{11}{9}\epsilon_2 - \frac{2}{9}\delta_1 + \frac{4}{9}\delta_2 - \frac{2}{3}\mu\delta_1 \tag{7}$$
$$+ \frac{2}{3}\mu\delta_2 - \frac{4}{3}\mu^2\delta_1 + \frac{4}{3}\mu^2\delta_2)\,y.$$

The characteristic equation is

$$\lambda^4 + (1 + 8\epsilon_1 - 3\epsilon_2)\lambda^2 + \frac{27}{16}(1 - S^2)$$
$$(1 + \frac{22}{9}\epsilon_2 - \frac{4}{9}\delta_1 + \frac{8}{9}\delta_2 + 8\mu\delta_1 - 8\mu\delta_2 - 40\mu^2\delta_1 + 40\mu^2\delta_2) = 0.$$

The four characteristic roots are $\pm\, i\omega_1'$, $\pm\, i\omega_2'$, where ω_1', ω_2' represent the perturbed basic frequencies. We can write

$$\omega' = w_1\,(1 + \omega\epsilon_1 + u'\epsilon_2 + u_1\delta_1 + u_2\delta_2)$$
$$\omega_2' = \omega_1\,(1 + \upsilon\epsilon_1 + \upsilon_2'\epsilon_2 + \upsilon_1\delta_1 + \upsilon_2\delta_2) \tag{9}$$

where ω_1, ω_2 represent the unperturbed basic frequencies such that

$$\omega_1^2 + \omega_2^2 = 1$$

$$\omega_2^1 \,.\, \omega_2^2 = \frac{27}{16}(1 - S^2),\ 0 < \omega_2 < \frac{1}{\sqrt{2}} < \omega_1 < 1$$

and

$$u = \frac{4}{k^2},\ \upsilon = -\frac{4}{k^2},$$

$$u' = \frac{22\omega_1^2 - 49}{18k^2},\ \upsilon' = -\left[\frac{22\omega_2^2 - 49}{18k^2}\right],$$

$$u_1 = \frac{1}{k^2}[(\frac{2}{9} - 4\mu + 20\mu^2) + (-\frac{2}{9} + 4\mu - 20\mu^2)\omega_1^2],$$

$$v_1 = \frac{1}{k^2}[(\frac{2}{9} + 4\mu - 20\mu^2)\ \omega_1^2],$$

$$u_2 = -\frac{1}{k^2}[(\frac{4}{9} - 4\mu + 20\mu^2) - (\frac{4}{9} - 4\mu + 20\mu^2)\omega_1^2],$$

$$v_2 = \frac{1}{k^2}[(\frac{4}{9} - 4\mu + 20\mu^2)\ \omega_1^2], \tag{10}$$

$$k^2 = \omega_1^2 - \omega_2^2 = 2\omega_1^2 - 1 = 1 - 2\omega_2^2\ . \tag{11}$$

Following the method in Whittaker (1965), we use the canonical tranformation from the phase space (x, y, p_x, p_y) into the space of the angle (θ, θ_2) and the actions (I_1, I_2).

$$X = AT$$

where

$$X = \begin{pmatrix} x_1 \\ y \\ p_x \\ p_y \end{pmatrix}, A = (a_{ij})_{1 \le i, j \le 4}, T = \begin{pmatrix} Q_1 \\ Q_2 \\ P_1 \\ P_2 \end{pmatrix}$$

$$Q_i = \left(\frac{2I_i}{\omega_i'}\right)^{1/2} sin\ \theta_i,\ P_i = (2I_i, w_i')^{1/2} cos\ \theta_i\ (i = 1,2)$$

$$a_{ij}' = a_{ij}(1 + \alpha_{ij}\epsilon_1 + \alpha_{ij}'\epsilon_2 + \alpha_{ij}''\delta_1 + \alpha_{ij}'''\delta_2)$$

$$(i, j = 1, 2, 3, 4). \tag{12}$$

The values of a_{ij} are the same as in Deprit & Deprit (1967) and

$$\alpha_{ij} = \alpha_{ij}' = \alpha_{ij}'' = \alpha_{ij}''' = 0\ (j = 1,2)$$

$$\alpha_{13}' = -\frac{88\omega_1^2}{l_1^2 k^4}$$

$$\alpha'_{13} = \frac{1}{18 l_1^2 k^4}(-135 + 1332\omega_1^2 - 364\omega_1^4 - 176\omega_1^6]$$

$$\alpha''_{13} = \frac{1}{9 l_1^2 n_1 k^4}[81 - 270\omega_1^2 - 228\omega_1^4 + 416\omega_1^6 + 64\omega_1^8$$
$$+ \mu(-2106 + 8394\omega_1^2 - 4308\omega_1^4 + 1056\omega_1^6 - 2496\omega_1^8)$$
$$+ \mu^2(3807 - 24126\omega_1^2 + 10968\omega_1^4 + 1056\omega_1^6 - 11136\omega_1^8)]$$

$$\alpha'''_{13} = \frac{1}{9 l_1^2 n_1 k^4}[-162 + 540\omega_1^2 + 456\omega_1^4 - 832\omega_1^6 - 128\omega_1^8$$
$$+ \mu(2106 - 8394\omega_1^2 + 4308\omega_1^2 - 1056\omega_1^6 + 2496\omega_1^8)$$
$$+ \mu^2(-3807 + 24126\omega_1^2 - 10968\omega_1^4 + 1056\omega_1^6 - 11136\omega_1^8)]$$

$$\alpha_{21} = \frac{1}{l_1^2 k^4}[-63 + 24\omega_1^2 + 20\omega_1^4 + 16\omega_1^6],$$

$$\alpha'_{21} = \frac{1}{18 l_1^2 k^4}[621 - 324\omega_1^2 - 76\omega_1^4 - 176\omega_1^6],$$

$$\alpha''_{21} = \frac{1}{9 l_1^2 n_1 k^4}[-81 + 54\omega_1^2 + 60\omega_1^4 - 160\omega_1^6 + 64\omega_1^8 + \mu(648 + 402\omega_1^2$$
$$+ 900\omega_1^4 - 3648\omega_1^6 + 2880\omega_1^8) + \mu^2(-5265 - 6516\omega_1^2$$
$$+ 6120\omega_1^4 + 16032\omega_1^6 - 10368\omega_1^8)],$$

$$\alpha'''_{21} = \frac{1}{9 l_1^2 n_1 k^4}[-162 - 108\omega_1^2 - 120\omega_1^4 + 320\omega_1^6 - 128\omega_1^8$$
$$+ \mu(-648 - 402\omega_1^2 - 900\omega_1^4 + 3648\omega_1^6 - 2880\omega_1^8)$$
$$+ \mu^2(5265 + 6516\omega_1^2 - 6120\omega_1^4 - 16032\omega_1^6 + 10368\omega_1^8)]$$

$$\alpha_{23} = \frac{-8w_1^2(7 + 8\omega_1^2)}{l_1^2 k^4}$$

$$\alpha'_{23} = \frac{-63 + 740\omega_1^2 + 532\omega_1^4 - 176\omega_1^6}{18 l_1^2 k^4},$$

$$\alpha''_{23} = \frac{1}{9 l_1^2 n_1 k^4}[-27 + 231\omega_1^2 - 100\omega_1^4 - 224\omega_1^6 + 64\omega_1^8$$
$$+ \mu(-1458 + 3522\omega_1^2 + 6636\omega_1^4 - 102072\omega_1^6 + 192\omega_1^8)$$
$$+ \mu^2(4131 - 16302\omega_1^2 + 2783\omega_1^4 + 3177\omega_1^6 + 384\omega_1^8)],$$

$$\alpha_{23}''' = \frac{1}{9l_1^2 n_1 k^4}[54 - 462\omega_1^2 + 200\omega_1^4 + 448\omega_1^6 - 128\omega_1^8 + \mu(1458 - 3522\omega_1^2 - 6636\omega_1^4 + 102072\omega_1^6) - 192\omega_1^8 + \mu^2(-4131 + 16302\omega_1^2 - 2783\omega_1^4 - 3177\omega_1^6 - 384\omega_1^8)],$$

$$\alpha_{31} = \frac{-27 - \omega_1^2 + 44\omega_1^4}{l_1^2 k^4 m_1},$$

$$\alpha_{31}' = \frac{1755 + 1656\omega_1^2 - 5924\omega_1^4 + 864\omega_1^6 + 704\omega_1^8}{18 l_1^2 k^4 m_1},$$

$$\alpha_{31}'' = \frac{1}{9k^4 l_1^2 m_1 n_1}[-243 + 1098\omega_1^2 + 3332\omega_1^4 - 2806\omega_1^6 + 320\omega_1^8 - 256\omega_1^{10} + \mu(-4644 + 30162\omega_1^2 - 39036\omega_1^4 + 27792\omega_1^6 - 15936\omega_1^8 - 768\omega_1^{10}) + \mu^2(-11097 - 40890\omega_1^2 + 25104\omega_1^4 + 20736\omega_1^6 + 12952\omega_1^8 - 1536\omega_1^{10})],$$

$$\alpha_{31}''' = \frac{1}{9k^4 l_1^2 m_1 n_1}[486 - 2196\omega_1^2 - 6664\omega_1^4 - 5612\omega_1^6 - 640\omega_1^8 - 512\omega_1^{10} + \mu(4644 - 30162\omega_1^2 + 39036\omega_1^4 - 27792\omega_1^6) + 15936\omega_1^8 + 768\omega_1^{10}) + \mu^2(11097 + 40890\omega_1^2 - 25104\omega_1^4 - 20736\omega_1^6 - 12595\omega_1^8 + 1536\omega_1^{10})],$$

$$\alpha_{33} = 1 + \alpha_{23},\ \alpha_{33}'' = \alpha_{33}'',$$

$$\alpha_{33}' = \alpha_{23}' = \alpha_{33}''' = \alpha_{23}''',$$

$$\alpha_{41} = \alpha_{23}' + 2u,\ \alpha_{41}' = \alpha_{23}' + 2u',$$

$$\alpha_{41}'' = \alpha_{23}'' + 2u_1,\ \alpha_{41}''' = \alpha_{23}''' + 2u_2,$$

$$\alpha_{43} = \frac{81 - 540\omega_1^2 - 492\omega_1^4 + 64\omega_1^6 - 64\omega_1^8}{k^4 l_1^2 n_1}$$

$$\alpha_{43}' = \frac{-1215 + 6480\omega_1^2 + 4644\omega_1^4 - 2432\omega_1^6 + 704\omega_1^8}{18k^4 l_1^2 n_1},$$

$$\alpha_{43}'' = \frac{1}{9l_1^2 m_1 k^4 n_1}[2025 - 7362\omega_1^2 + 1807\omega_1^4 + 1776\omega_1^6 + 1216\omega_1^8 + 2240\omega_1^{10} + \mu(-1895 + 61938\omega_1^2 - 11484\omega_1^4 - 14928\omega_1^6 + 10944\omega_1^8 - 33024\omega_1^{10}) + \mu^2(5832 - 51030\omega_1^2 - 630576\omega_1^4 + 1033728\omega_1^6 + 566784\omega_1^8 + 127488\omega_1^{10})],$$

$$\alpha'''_{43} = \frac{1}{9l_1^2 m_1 k^4 n_1}[-4050 + 14724\omega_1^2 - 3614\omega_1^4 - 3554\omega_1^6 - 2432\omega_1^8 - 4480\omega_1^{10} + \mu(1895 - 61398\omega_1^2 + 11484\omega_1^4 + 14928\omega_1^6 - 10944\omega_1^8 + 33024\omega_1^{10}) + \mu^2(-5832 + 51030\omega_1^2 + 630576\omega_1^4 - 1033728\omega_1^6 - 566784\omega_1^8 + 127488\omega_1^{10})], \quad (13)$$

$$l_i^2 = 9 + 4\omega_i^2,\ m_i = 1 + 4\omega_i^2,\ n_i = [9 - 4\omega_2^2].$$

The values of α_{ij}, α'_{ij}, α''_{ij}, α'''_{ij} for $j = 2, 4$ can be obtained from those for $j = 1, 3$ respectively by replacing ω_1 by ω_2, l_1 by l_2, m_1 by m_2, n_1 by n_2 wherever they occur, keeping k unchanged.

The transformation changes the second order part of the Hamiltonian into the normal form.

$$H_2 = \omega'_1 I_1 - \omega'_2 I_2. \quad (14)$$

The general solution of the corresponding equations of motion are

$$I_i = \text{constatnt } (i = 1, 2)$$

$$\theta_1 = \omega'_1 t + \text{constant}$$

$$\theta_2 = \omega'_2 t + \text{constant} \quad (15)$$

3. Second order normalisation of the Hamiltonian

Here we have to perform Birkhoff's normalisation. For this, we expand (x, y) in double d'Alembert series

$$x = \sum_{n\geq 1} B_n^{1,0}(\theta_1, \theta_2, I_1, I_2)$$

$$y = \sum_{n\geq 1} B_n^{0,1}(\theta_1, \theta_2, I_1, I_2)$$

Here $B_n^{1,0}$ and $B_n^{0,1}$ are the homogenious components of degree n in $I_1^{1/2}$, $I_2^{1/2}$ and are of the form

$$\sum_{0\leq m\leq n} I_1^{1/2(n-m)}\, I_2^{1/2m} \sum_{(i,j)} [C_{n-m,m,i,j}\cos(i\theta_1 + j\theta_2) + S_{n-m,m,i,j}\sin(i\theta_1 + j\theta_2)].$$

The double summation over the indices i and j is such that (i) i runs over those integers in the interval $0 \leq i \leq n - m$ that have the same parity as $n - m$ and (ii) j runs over those integers in

The double summation over the indices i and j is such that (i) i runs over those integers in the interval $0 \le i \le n - m$ that have the same parity as $n - m$ and (ii) j runs over those integers in the interval $-m \le j \le m$ that have the same parity as m, and I_1, I_2 are to be taken as constants of integration, θ_1, θ_2 are to be determined as linear functions of time such that

$$\frac{d\theta_1}{dt} = \dot{\theta}_1 = \omega_1' + \sum_{n\ge1} f_{2n}(I_1, I_2),$$

$$\frac{d\theta_2}{dt} = \dot{\theta}_2 = -\omega_2' = \sum_{n\ge1} g_{2n}(I_1, I_2), \tag{16}$$

where f_{2n} and g_{2n} are of the form

$$f_{2n} = \sum_{0\le m\le n} f'_{2(n-m),2m} I_1^{n-m} I_2^m,$$

$$g_{2n} = \sum_{0\le m\le n} g_{2(n-m),2m} I_1^{n-m} I_2^m, \tag{17}$$

Now $B_1^{1,0}$ and $B_1^{0,1}$ are the components of the first order in $I_1^{1/2}$ and $I_2^{1/2}$ and these components are the values of x and y given by equations (12). The second order components $B_2^{1,0}$ and $B_2^{0,1}$ are the solutions of the partial differential equations

$$\Delta_1\Delta_2 B_n^{1,0} = \phi_2$$

$$\Delta_1\Delta_2 B_2^{0,1} = \psi_2 \tag{18}$$

$$\Delta_i = D_2 + \omega_i'^2 (i = 1,2)$$

$$\phi_2 = [D^2 - \frac{1}{4}(9 + 7\epsilon_2 + 2\delta_1 - 4\delta_2 - 6\mu\delta_1 + 6\mu\delta_2]X_2$$
$$+ [2(1 + \epsilon_1)D + \frac{3\sqrt{3}}{4}S(1 + \frac{11}{9}\epsilon_2 - \frac{2}{9}\delta_1 + \frac{4}{9}\delta_2 - \frac{2}{3}\mu\delta_1$$
$$+ \frac{2}{3}\mu\delta_2 - \frac{4}{3}\mu^2\delta_1 + \frac{4}{3}\mu^2\delta_2)]Y_2,$$

$$\psi_2 = [D^2 - \frac{1}{4}(3 + 5\epsilon_2 - 2\delta_1 + 4\delta_2 + 6\mu\delta_1 - 6\mu\delta_2)]Y_2$$
$$- [2(1 + \epsilon_1)D - \frac{3\sqrt{3}}{4}S(1 + \frac{11}{9}\epsilon_2 - \frac{2}{9}\delta_1 + \frac{4}{9}\delta_2 - \frac{2}{3}\mu\delta_1$$

$D = \omega_1'\left(\frac{\partial}{\partial\theta_1}\right) - \omega_2'\left(\frac{\partial}{\partial\theta_2}\right)$ and X_2, Y_2 are obtained by substituting respectibely in $\frac{\partial L_3}{\partial x}$, $\frac{\partial L_3}{\partial y}$ the first order compnents for x and y.

The second order components $B_n^{1,0}$ and $B_n^{0,1}$ are as follows :

$$B_2^{1,0} = r_1'I_1r_2'I_2 + r_3'I_1\cos 2\theta_1 + r_4'I_2\cos 2\theta_2 + r_5'I_1\sin 2\theta_1 + r_6'I_2\sin 2\theta_2 + r_7'I_1^{1/2}I_2^{1/2}\cos(\theta_1 + \theta_2) + r_8'I_1^{1/2}I_2^{1/2}\cos(\theta_1 - \theta_2) + r_9'I_1^{1/2}I_2^{1/2}\sin(\theta_1 + \theta_2) + r_{10}'I_1^{1/2}I_2^{1/2}\sin(\theta_1 - \theta_2), \quad (20)$$

$$B_2^{0,1} = s_1'I_1 + s_2'I_2 + s_3'I_1\cos 2\theta_1 + s_4'I_2\cos 2\theta_2 + s_5'I_1\sin 2\theta_1 + s_6'I_2\sin 2\theta_2 + s_7'I_1^{1/2}I_2^{1/2}\cos(\theta_1 + \theta_2) + s_8'I_1^{1/2}I_2^{1/2}\cos(\theta_1 - \theta_2) + s_9'I_1^{1/2}I_2^{1/2}\sin(\theta_1 + \theta_2) + s_{10}'I_1^{1/2}I_2^{1/2}\sin(\theta_1 - \theta_2), \quad (21)$$

where

$$r_i' = r_i\,(1 + \alpha_i\epsilon_i + \alpha_i'\epsilon_2 + \alpha_i''\delta_1 + \alpha_i'''\delta_2),$$

$$s_i' = s_i\,(1 + \beta_i\epsilon_1 + \beta_i'\epsilon_2 + \beta_i''\delta_1 + \beta_i'''\delta_2),\ (i = 1, 2, 3, \ldots\ldots, 10)$$

$$r_1 = -\frac{33}{8k^2w_1}S,$$

$$r_3 = \frac{(27 + 321\omega_1^2 - 76\omega_1^4)}{8k^2\omega_1z_1I_1^2},$$

$$r_5 = \frac{(18 - 53\omega_1^2 + 44\omega_1^4)}{k^2z_1\sqrt{3}.I_1^2}$$

$$r_7 = \frac{3(-36 + 229l - 72l^2)S}{4k^2I_1I_2\,(5l - 2)\sqrt{l}}$$

$$r_9 = \frac{(15 + 3l - 44l^2)(\omega_1 - \omega_2)\sqrt{3}}{k^2 I_1 I_2 (5l - 2)\sqrt{l}},$$

$$s_1 = \frac{(9 + 8w_1^2)\sqrt{3}}{24k^2 w_1^2},$$

$$s_3 = \frac{\sqrt{3}(729 - 2733\omega_1^2 + 144\omega_1^4 - 736\omega_1^6)}{72k^2 I_1^2 w_1 z_1}.$$

$$s_5 = \frac{(24 - 59\omega_1^2 S)}{k^2 I_1^2 z_1}$$

$$s_7 = \frac{3(-180 + 26l - 160l^2 + 144l^3)}{12k^2 I_1 I_2 (5l - 2)\sqrt{l}},$$

$$s_9 = \frac{3(\omega_1 - \omega_2)(9 + 7l)}{k^2 I_1 I_2 (5l - 2)\sqrt{l}} S,$$

$$1 = \omega_1 . \omega_2,\ z_i = 1 - 5\omega_i^2,\ S = 1 - 2\mu\ (i = 1, 2)$$

The values of r_i, s_i for i = 2, 4, 6 can be found respectively from those for i = 1, 3, 5 by replacing ω_1 by $-\omega_2$, I_1 by I_2, k^2 by $-k^2$, z_1 by z_2 wherever they occur and the values of r_i, s_i i = 8, 10 can be obtained respectively from those for i = 7, 9 by replacing ω_2 by $-\omega_2$ keeping $\omega_1, k^2, k^4, I_1, I_2, I_1^2, I_2^2, \sqrt{\omega_1 \omega_2}$ unchanged wherever they occur.

Thus we have examined following Deprit & Deprit (1967) that by the transformation $x = B_1^{1,0} + B_2^{1,0}$, $y = B_1^{0,1} + B_2^{0,1}$, the third order part H_3 of the Hamilonian in $I_1^{1/2}$, $I_2^{1/2}$ is zero.

4. Second order co-efficients in the frequencies

The third order components $B_3^{1,0}$ and $B_3^{0,1}$ in te co-ordinates x and y and the second order polynomials f_2 and g_2 in the frequencies $\dot{\theta}_1$ and $\dot{\theta}_2$ satisfy the partial differential equations, as in Deprit & Deprit (1967),

$$\Delta_1 \Delta_2 B_3^{1,0} = \phi_3 - 2f_2 P - 2g_2 Q$$

$$\Delta_1 \Delta_2 B_3^{0,1} = \psi_3 - 2f_2 U - 2g_2 V \tag{23}$$

where

$$\phi_3 = [D^2 - \frac{1}{4}(9 + 7\epsilon_2 + 2\delta_1 - 4\delta_2 - 6\mu\delta_1 + 6\mu\delta_2)]X_3$$
$$+ [2(1 + \epsilon_1)D + \frac{3\sqrt{3}}{4}) S(1 + \frac{11}{9}\epsilon_2 - \frac{2}{9}\delta_1 + \frac{4}{9}\delta_2 - \frac{2}{3}\mu\delta_1)$$
$$+ \frac{2}{3}\mu\delta_2 - \frac{4}{3}\mu^2\delta_1 + \frac{4}{3}\mu^2\delta_2)]Y_3,$$

$$\psi_3 = [D^2 - \frac{1}{4}(3 + 5\epsilon_2 - 2\delta_1 + 4\delta_2 + 6\mu\delta_1 - 6\mu\delta_2)]Y_3$$
$$- [2(1 + \epsilon_1)D - \frac{3\sqrt{3}}{2} S(1 + \frac{11}{9}\epsilon_2 - \frac{2}{9}\delta_1 + \frac{4}{9}\delta_2 - \frac{2}{3}\mu\delta_1$$
$$+ \frac{2}{3}\mu\delta_2 - \frac{4}{3}\mu^2\delta_1 + \frac{4}{3}\mu^2\delta_2)]X_3,$$

$$P = \frac{\partial}{\partial\theta_1}[\{\omega_1'^2 \frac{\partial_2}{\partial\theta_1^2} - \frac{1}{4}(9 + 7\epsilon_1 + 2\delta_1 - 4\delta_2 - 6\mu\delta_1 + 6\mu\delta_2\}$$
$$\{\omega_1' \frac{\partial B_1^{1,0}}{\partial\theta_1} - (1 + \epsilon_1)B_1^{0,1}\} + \{2(1 + \epsilon_1)\omega_1' \frac{\partial}{\partial\theta_1}$$
$$+ \frac{3\sqrt{3}}{4} S\,(1 + \frac{11}{9}\epsilon_2 - \frac{2}{9}\delta_1 + \frac{4}{9}\delta_2 - \frac{2}{3}\mu\delta_1 + \frac{2}{3}\mu\delta_2)$$
$$- \frac{4}{3}\mu^2\delta_1 + \frac{4}{3}\mu^2\delta_2)\}\ \{\omega_1' \frac{\partial B_1^{0,1}}{\partial\theta_1} - (1 + \epsilon_1)B_1^{1,0}\}],$$

$$Q = \frac{\partial}{\partial\theta_2} w_1'^2 \frac{\partial_2}{\partial\theta_2^2} - \frac{1}{4}(9 + 7\epsilon_2 + 2\delta_1 - 4\delta_2 - 6\mu\delta_1 + 6\mu\delta_2)$$
$$\{\omega_2' \frac{\partial B_1^{1,0}}{\partial\theta_2} - (1 + \epsilon_1)B_1^{0,1}) + (-2(1 + \epsilon_1)\omega_2' \frac{\partial}{\partial\theta_2}$$
$$+ \frac{3\sqrt{3}}{4} S\,(1 + \frac{11}{9}\epsilon_2 - \frac{2}{9}\delta_1 + \frac{4}{9}\delta_2 - \frac{2}{3}\mu\delta_1 + \frac{2}{3}\mu\delta_2)$$
$$- \frac{4}{3}\mu^2\delta_1 + \frac{4}{3}\mu^2\delta_2))\,(-\omega_2' \frac{\partial B_2^{0,1}}{\partial\theta_1} + (1 + \epsilon_1)B_1^{1,0}))\ ,$$

$$U = \frac{\partial}{\partial\theta_1}[\{\omega_1'^2 \frac{\partial^2}{\partial\theta_1^2} - \frac{1}{4}(3 + 5\epsilon_2 - 2\delta_1 + 4\delta_2 + 6\mu\delta_1 - 6\mu\delta_2)\}$$
$$\{\omega_1' \frac{\partial B_1^{0,1}}{\partial\theta_1} + (1 + \epsilon_1)B_1^{1,0}\} - \{2(1 + \epsilon) \frac{\partial}{\partial\theta_1}$$

$$- \frac{3\sqrt{3}}{4} S \left(1 + \frac{11}{9}\epsilon_2 - \frac{2}{9}\delta_1 + \frac{4}{9}\delta_2 - \frac{2}{3}\mu\delta_1 + \frac{2}{3}\mu\delta_2\right)$$

$$- \frac{4}{3}\mu^2\delta_1 + \frac{4}{3}\mu^2\delta_2)\} \{\omega_1' \frac{\partial B_1^{1,0}}{\partial\theta_1} + (1+\epsilon_1) B_1^{0,1}\}],$$

$$V = \frac{\partial}{\partial\theta_2}[\{\omega_2'^2 \frac{\partial^2}{\partial\theta_2^2} - \frac{1}{4}(3 + 5\epsilon_2 - 2\delta_1 + 4\delta_2 + 6\mu\delta_1 - 6\mu\delta_2)$$

$$\{-\omega_2' \frac{\partial B_1^{0,1}}{\partial\theta_2} + (1+\epsilon_1)B_1^{1,0}) - (2(1+\epsilon)\omega_1' \frac{\partial}{\partial\theta_1})$$

$$- \frac{3\sqrt{3}}{4} S \left(1 + \frac{11}{9}\epsilon_2 - \frac{2}{9}\delta_1 + \frac{4}{9}\delta_2 - \frac{2}{3}\mu\delta_1 + \frac{2}{3}\mu\delta_2\right.$$

$$- \frac{4}{3}\mu^2\delta_1 + \frac{4}{3}\mu^2\delta_2)\} \{\omega_2' \frac{\partial B_1^{1,0}}{\partial\theta_2} - (1+\epsilon_1) B_1^{0,1}\}] \tag{24}$$

and X_3, Y_3 are the homogeneous components of order three obtained by substituting $x = B_1^{1,0} + B_2^{1,0}$ and $y = B_1^{0,1} + B_2^{0,1}$ in $\frac{\partial}{\partial x}(L_3 + L_4)$ and $\frac{\partial}{\partial y}(L_3 + L_4)$ respectively. The components $B_3^{1,0}$ and $B_3^{0,1}$ are not required to be found out. We find the coefficients of $\cos\theta_1$, $\sin\theta_1$, $\cos\theta_2$ and $\sin\theta_2$ in the right hand side of the equations (23). They are the critical terms. We eliminate these terms by choosing properly the co-efficients in the polynomials

$$f_2 = f_{2,0}' I_1 + f_{0,2}' I_2 \tag{25}$$

$$g_2 = g_{2,0}' I_1 + g_{0,2}' I_2.$$

We find that

$$A = f_{2,0}' = f_{2,0}[1 + (\rho_1 - \rho_3)\epsilon_1 + (\rho_1' - \rho_3')\epsilon_2 + (\rho_1'' - \rho_3'')\delta_1 + (\rho_1''' - \rho_3''')\delta_2],$$

$$B = f_{0,2}' = f_{0,2}[1 + (\rho_2 - \rho_3)\epsilon_1 + (\rho_2' - \rho_3')\epsilon_2 + (\rho_2'' - \rho_3'')\delta_1 + (\rho_2''' - \rho_3''')\delta_2],$$

$$C = g_{0,2}' = g_{0,2}[1 + (\rho_4 - \rho_5)\epsilon_1 + (\rho_4' - \rho_5')\epsilon_2 + (\rho_4'' - \rho_5'')\delta_1 + (\rho_4''' - \rho_5''')\delta_2],$$

where

$$f_{2,0} = \frac{\omega_1}{72(1 - 2\omega_1^2)(1 - 5\omega_1^2)}[81 - 696\omega_1^2 + 124\omega_1^4],$$

$$f_{0,2} = \frac{\omega_1\omega_2(43 + 64\omega_1^2\omega_2^2)}{6(1 - 2\omega_1^2)(1 - 2\omega_2^2)(1 - 5\omega_1^2)(1 - 5\omega_2^2)}$$

$$g_{0,2} = \frac{\omega_1^2(81 - 696\omega_1^2 + 124\omega_1^4)}{72(1 - 2\omega_2^2)(1 - 5\omega_2^2)} \quad (26)$$

$\rho_i, \rho_i', \rho_i'', \rho_i'''$, $(i = 1, 2, 3, 4, 5)$ can be obtained by mathematical manipulations as found earlier by other research workers.

Since the first condition of Moser's theorem is applicable, Birkhoff's normalization up to order three can be obtained. First condition of Moser's theorem is satisfied in the interval $0 < \mu < \mu_c$, if the mass ratio does not take the critical values

$$\mu_1' = \mu_1 + 0.3986277...\epsilon_1 - 0.2103868...\epsilon_2 + 0.00681895...\delta_1 - 0.017892...\delta_2,$$

$$\mu_2' = \mu_2 + 0.219260387...\epsilon_1 - 0.1157207...\epsilon_2 + 0.0047089406...\delta_1 - 0.0107995...\delta_2,$$

where

$$\mu_1 = 0.02429389...$$

$$\mu_2 = 0.013516016...$$

If $\epsilon_1 = \epsilon_2 = \delta_1 = \delta_2 = 0$, then $\mu_1' = \mu_1$, $\mu_2' = \mu_2$ and these values agree with those found by Deprit & Deprit (1967).

5. Stability

The normalized Hamiltoninan up to fourth order can be written as

$$\mathrm{H} = \omega_1' I_1 - \omega_1' I_2 + \frac{1}{2}(AI_1^2 + 2BI_1I_2 + CI_2^2) \quad (27)$$

The determinant D occuring in second condition of Moser's theorem is

$$\mathrm{D} = \frac{-1}{72(1 - 4\omega_1^2\omega_2^2)(4 - 25\omega_1^2\omega_2^2)} \times [9(36 - 541\omega_1^2\omega_2^2 + 644\omega_1^4\omega_2^4) + M\epsilon_1 + M'\epsilon_2 + M''\delta_1 + M'''\delta_2] \quad (28)$$

where

$$M = -\omega_1^4(81 - 696\omega_1^2 + 124\omega_1^4)(\rho_1 - \rho_3 + 2v)z_2 - 24\omega_1^2\omega_2^2$$
$$\times (43 + 64\omega_1^2)(\rho_2 - \rho_3 + u + v) - \omega_1^4(81 - 696\omega_2^2 + 124\omega_2^4)$$
$$\times (\rho_4 - \rho_5 + 2u)z_1 \quad (29)$$

M′, M″, M‴ can be obtained from M by replacing ρ_i by $\rho'_i, \rho''_i, \rho'''_i$ (i = 1, 2, 3, 4, 5) respectively. Second condition of Moser's theorem is satisfied i.e. D ≠ 0 if in the interval $0 < \mu < \mu_c$, mass ratio does not take the value

$$\mu'_3 = \mu_3[1 + R\epsilon_1 + R'\epsilon_2 + R''\delta_1 + R'''\delta_2]$$

where $\mu_3 = 0.0109136...$

$$R = -\frac{1}{11592\rho\sigma}\begin{bmatrix}(M)\omega_1^2 = \frac{1}{2}(1 + \sigma') \\ \omega_2^2 = \frac{1}{2}(1 - \sigma)\end{bmatrix}$$

$$R' = -\frac{1}{11592\rho\sigma}\begin{bmatrix}(M')\omega_1^2 = \frac{1}{2}(1 + \sigma') \\ \omega_2^2 = \frac{1}{2}(1 - \sigma)\end{bmatrix}$$

$$R'' = -\frac{1}{11592\rho\sigma}\begin{bmatrix}(M'')\omega_1^2 = \frac{1}{2}(1 + \sigma') \\ \omega_2^2 = \frac{1}{2}(1 - \sigma)\end{bmatrix}$$

$$R''' = -\frac{1}{11592\rho\sigma}\begin{bmatrix}(M''')\omega_1^2 = \frac{1}{2}(1 + \sigma') \\ \omega_2^2 = \frac{1}{2}(1 - \sigma')\end{bmatrix}$$

Therefore, we find that both the conditions of Moser's theorem are satisfied in the interval $0 < \mu < \mu_e$ and hence the equilibrium points are stable except for three mass ratios μ'_1, μ'_2, μ'_3 at which Moser's theorem is not applicable.

6. Conclusion

It is established that, in non-linear sense, the triangular point of the photogravitational restricted three body problem under perturbed coriolis and centrifugal forces, is stable in the range of linear stability except for three mass ratios.

$$\mu_1' = 0.02429389... + 0.3986277...\epsilon_1 - 0.2103868...\epsilon_2 + 0.00681895...\delta_1 - 0.017892...\delta_2$$

$$\mu_2' = 0.013516016... + 0.219260387...\epsilon_1 - 0.1157207\ \epsilon_2 + 0.004708946...\delta_1 - 0.0107995...\delta_2$$

$$\mu_3' = 0.0109136...\ (1 + R\epsilon_1 + R'\epsilon_2 + R''\delta_1 + R'''\delta_2)$$

at which Moser's theorem does not apply.

The result of Deprit & Deprit (1967) and Bhatnagar & Hallan (1983) can be obtained by putting $\epsilon_1 = \epsilon_2 = 0$ and $\delta_1 = \delta_2 = 0$ respectively which establishes the veracity of our conclusion.

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