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Non-linear stability of a cluster of stars sharing galactic rotation

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Abstract. The non-linear stability of an ellipsoidal cluster of stars sharing galactic rotation has been studied. It is seen that the cluster is stable for all densities in the range of linear stability except for those satisfying certain equations where Arnold's theorem is not applicable.

Key Words : Arnold's theorem—stability—cluster of stars

1. Introduction

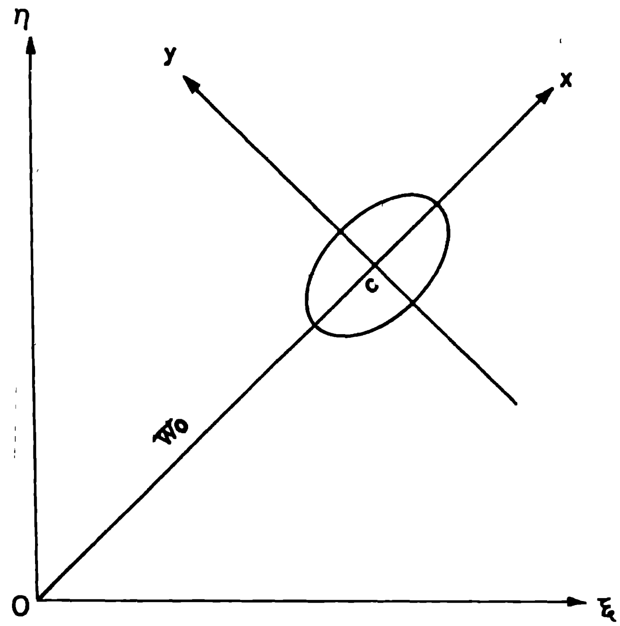
Chandrasekhar (1942) has studied the linear stability of an ellipsoidal cluster of stars sharing galactic rotation and moving in a field having both an axis and a plane of symmetry. He has approximated the smoothed out distribution in the cluster to a homogeneous ellipsoid, taking Ω, the potential energy in the form

$$\Omega = -\frac{1}{2} m\, \beta_0 + \frac{1}{2}\, (\beta_1 x^2 + \beta_2 y^2 + \beta_3 z^2),$$

where β_0, β_1, β_2 and β_3 are constants depending on the density ρ and the geometry of the ellipsoid i.e.

$$\beta_i = \pi G \rho \beta_i' \,(a{:}b{:}c),\ i = 0, 1, 2, 3$$

β_i' is a number depending on the ratios of the axes a, b and c of the ellipsoid. Further the orientation of one of the axes of the ellipsoid is taken in the radial direction. In this paper we have studied the non-linear stability of Chandrasekhar's model. First we give below very briefly the linear problem as studied by Chandrasekhar.



Let O denote the centre of the galaxy and C, the centre of gravity of the cluster, distant ϖ_0 from O. Let the centre of gravity describe a circular orbit about O with constant angular velocity w_c. If B (ϖ, ζ) denotes the general gravitational potential, the rotational and the angular velocities of C are given by

$$\theta_c^2 = \left(\varpi \frac{\partial B}{\partial \varpi}\right)_o ; \quad W_c^2 = \left(\frac{1}{\varpi} \frac{\partial B}{\partial \varpi}\right)$$

where the subscript o indicates that the quantity in parenthesis is to be evaluated at $\varpi = \varpi_0$ and $\zeta = O$. Here $0 - \xi\eta\zeta$ represents fixed frame of reference and ϖ, the distance from O. Now introduce a frame of reference C-XYZ rotating uniformly about the z-axis with the angular velocity w_c; x-axis along OC and z-axis perpendicular to the galactic plane.

The Lagrangian function L for a star of mass m of the cluster is given by

$$L = T - mB - \Omega,$$

where T is the kinetic energy, and Ω is the potential energy of the star:

$$\Omega = -G \sum_j \frac{mm_j}{r_j},$$

where m_j is the mass of any other star of the cluster at a distance r_j from the star and G is the gravitational constant.

In terms of x, y, z the Lagrangian L has the form

$$L = m\left[\frac{1}{2}\,(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - w_c\,\dot{x}y + w_c\,\dot{y}\,(\varpi_o + x) + \frac{1}{2}\,w_c^2\,\{(\varpi_o + x)^2 + y^2\}\right] - mB\,(\varpi\,, z) - \Omega,$$

where $\quad \varpi^2 = (\varpi_o + x)^2 + y^2.$

The Lagrange's equations of motion for a star of mass m are (Chandrasekhar 1942)

$$m\,(\ddot{x} - 2w_c\dot{y} - w_c^2\,(\varpi_o + x) = -m\,\frac{\varpi_o + x}{\varpi}\,\frac{\partial B}{\partial \varpi} - \frac{\partial \Omega}{\partial x},$$

$$m(\ddot{y} + 2w_c\dot{x} - w_c^2 y) = -m\,\frac{y}{\varpi}\,\frac{\partial B}{\partial \varpi} - \frac{\partial \Omega}{\partial y},$$

$$m\ddot{z} = -m\,\frac{\partial B}{\partial z} - \frac{\partial \Omega}{\partial z}.$$

If we suppose that the dimensions of the cluster are small compared to ϖ_o and neglect all quantities of order more than one in x, y and z, the equations of motion take the form

$$m(\ddot{x} - 2w_c\dot{y} + \alpha_1 x) = -\frac{\partial \Omega}{\partial x},$$

$$m(\ddot{y} + 2w_c\dot{x}\,, = -\frac{\partial \Omega}{\partial y},$$

$$m(\ddot{z} + \alpha_3 z) = -\frac{\partial \Omega}{\partial z},$$

where

$$\alpha_1 = \left(\frac{\partial^2 B}{\partial \varpi^2} - \frac{1}{\varpi}\,\frac{\partial B}{\partial \varpi}\right)_o; \qquad \alpha_3 = \left(\frac{\partial^2 B}{\partial z^2}\right)_o.$$

The equations of motion under the restrictions imposed by Chandrasekhar, become

$$\ddot{x} - 2w_c\dot{y} + (\alpha_1 + \beta_1)x = 0,$$

$$\ddot{y} - 2w_c\dot{x} + \beta_2 y = 0,$$

$$\ddot{z} + (\alpha_3 + \beta_3)z = 0.$$

The solution of these equations are circular functions and therefore correspond to the stable oscillations if

$$\alpha_1 + \beta_1 > 0.$$

Substituting the values of α_1 and β_1 the condition for dynamical stability is equivalent to

$$\rho > \rho^*$$

where the critical value ρ^* of the density is given by (Chandrasekhar 1942)

$$\rho^* = \frac{0.165}{\beta_1'}.$$

Also, the basic frequencies w_1, w_2, w_3 of the linear dynamical system satisfy the equations

$$w_1^2 + w_2^2 = \alpha_1 + \beta_1 + \beta_2 + 4w_c^2,$$

$$w_1^2 w_2^2 = \beta_2(\alpha_1 + \beta_1),$$

$$w_3^2 = \alpha_3 + \beta_3. \quad (1)$$

Thus Chandrasekhar has considered the stability of the cluster of stars in the linear sense. In the present study, we wish to study the nonlinear stability of the cluster. This we propose to do by applying Arnold's theorem (1963) which states that if

(i) $K_1w_1 + K_2w_2 + K_3w_3 \neq 0$ for all triplet of integers K_1, K_2, K_3 such that $|K_1| + |K_2| + |K_3| \leq 4$ where w_1, w_2, w_3 are the basic frequencies for the linear dynamical system,

(ii) determinant $D \neq 0$ where

$$D = \det.(b_{ij}), \qquad (i, j = 1,2,3,4)$$

$$b_{ij} = \left(\frac{\partial^2 H}{\partial I_i\,\partial I_j}\right)_{I_i = I_j = 0} \qquad (i, j = 1,2,3)$$

$$b_{i4} = b_{4i} = \left(\frac{\partial H}{\partial I_i}\right)_{I_i = I_j = 0} \qquad (i = 1,2,3),$$

$$b_{44} = 0,$$

$$H = w_1 I_1 + w_2 I_2 + w_3 I_3 + \frac{1}{2}\left(aI_1^2 + bI_1^2 + cI_3^2 + 2fI_2 I_3 + 2gI_3 I_1 + 2hI_1 I_2\right)$$

is the normalised Hamiltonian with I_1, I_2, I_3 as the action momenta coordinates, then the equilibrium is stable. We shall follow the procedure similar to that adopted by Bhatnagar & Hallan (1983).

2. First order normalization

Using Taylor's expansion, we find that

$$B = b_0 + b_1 x + b_2 x^2 + b_3 y^2 + b_4 z^2 + b_5 x^3 + b_6 xy^2 + b_7 xz^2 + b_8 x^4 + b_9 y^4 + b_{10} z^4 + b_{11} x^2 y^2 + b_{12} y^2 z^2 + b_{13} x^2 z^2 + \ldots,$$

where

$$b_o = (B)_o,\ b_1 = \left(\frac{\partial B}{\partial \varpi}\right)_o,\ b_2 = \frac{1}{2}\left(\alpha_1 + w_c^2\right),$$

$$b_3 = \frac{1}{2} w_c^2,\ b_4 = \frac{1}{2}\alpha_3,\ b_5 = \frac{1}{6}\left(\frac{\partial^3 B}{\partial \varpi^3}\right)_o,$$

$$b_6 = \frac{\alpha_1}{2\varpi_o},\ b_7 = \frac{1}{2}\left(\frac{\partial^3 B}{\partial \varpi \partial z^2}\right)_o,\ b_8 = \frac{1}{24}\left(\frac{\partial^4 B}{\partial \varpi^4}\right)_o,$$

$$b_9 = \frac{\alpha_1}{8\varpi_o^2},\ b_{10} = \frac{1}{24}\left(\frac{\partial^4 B}{\partial z^4}\right)_o.$$

$$b_{11} = \frac{1}{4\varpi_o}\left[\left(\frac{\partial^3 B}{\partial \varpi^3}\right)_o - \frac{2}{\varpi_o}\alpha_1\right],$$

$$b_{12} = \frac{1}{4\varpi_o}\left(\frac{\partial^3 B}{\partial \varpi\, \partial z^2}\right)_o,\ b_{13} = \frac{1}{4}\left(\frac{\partial^4 B}{\partial z^2 \partial \varpi^2}\right)_o.$$

Expanding the Lagrangian L in powers of $x, y, z, \dot{x}, \dot{y}, \dot{z}$, we can write

$$L = L_o + L_1 + L_2 + L_3 + L_4 + \ldots,$$

where

$$L_o = \frac{1}{2}\varpi_o^2 w_c^2 + \frac{1}{2}\beta_o - b_o,$$

$$L_1 = \varpi_o w_c z,$$

$$L_2 = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + w_c(x\dot{y} - y\dot{x}) + \frac{1}{2}[w_c^2 - 2b_2 - \beta_1)x^2 + (w_c^2 - 2b_3 - \beta_2)y^2 - (2b_4 + \beta_3)z^2],$$

$$L_3 = -(b_5x^3 + b_6xy^2 + b_7xz^2),$$

$$L_4 = -(b_8x^4 + b_9y^4 + b_{10}z^4 + b_{11}x^2y^2 + b_{12}y^2z^2 + b_{13}x^2z^2).$$

Also, expanding the Hamiltonian function H in power series of x, y, z, p_x, p_y and p_z, we can write

$$H = H_o + H_1 + H_2 + H_3 + H_4 + \ldots$$

where

$$H_o = b_o - \frac{1}{2}\beta_o,$$

$$H_1 = -b_7 w_c \varpi_o + b_1 x,$$

$$H_2 = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + \varpi_c(yp_x - xp_y) + \frac{1}{2}[(2b_2 + \beta_1))x^2 + (2b_3 + \beta_2)y^2 + (2b_4 + \beta_3)z^2],$$

$$H_3 = -L_3,$$

$$H_4 = -L_4.$$

....
....

The Canonical equations of motion are

$$\dot{x} = P_x + w_c y,$$

$$\dot{y} = P_y - w_c\,(\varpi_o + x)$$

$$\dot{z} = P_z,$$

$$\dot{P}_x = -\,[b_1 + (2b_2 + \beta_1)x - w_c P_y + 3b_5 x^2 + b_6 y^2 + b_7 z^2 + 4b_8 x^3 + 2b_{11} xy^2 + 2b_{13} xz^2],$$

$$\dot{P}_y = -\,[(2b_3 + \beta_2)y + w_c P_x + 2b_6 xy + 4b_9 y^3 + 2b_{11} x^2 y + b_{12} yz^2],$$

$$\dot{P}_z = -\,[(2b_4 + \beta_3)z + 2b_7 xz + 4b_{10} z^3 + b_{12} y^2 z + 2b_{13} x^2 z].$$

Equilibrium point in the phase space x, y, z, P_x, P_y, P_z is $(0, 0, 0, w_c, \varpi_o, 0)$. Shifting the origin to the equilibrium point, we have

$$H = H_o + H_1 + H_2 + H_3 + H_4 + \ldots$$

where

$$H_o = b_o - \frac{1}{2}\beta_o - \frac{1}{2}\varpi_o^2\, w_c^2,$$

$$H_1 = 0,$$

$$H_2 = \frac{1}{2}(P_x^2 + P_y^2 + P_z^2) + \varpi_c\,(yP_x - xP_y) + \frac{1}{2}[(2b_2 + \beta_1)x^2 + (2b_3 + \beta_2)y^2 + (2b_4 + \beta_3)z^2],$$

$$H_3 = -\,L_3,$$

$$H_4 = -\,L_4,$$

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Following the method given in Whittaker (1965), we use the canonical transformation from the phase space (x, y, z, P_x, P_y, P_z) into the phase space $(\phi_1, \phi_2, \phi_3, I_1, I_2, I_3)$ of the angles ϕ_1, ϕ_2, ϕ_3 and the action momenta I_1, I_2, I_3 given by

$$\mathrm{X} = \mathrm{AT}, \tag{2}$$

where

$$X = \begin{bmatrix} x \\ y \\ z \\ P_x \\ P_y \\ P_z \end{bmatrix}, \qquad A = (a_{ij})_{1 \le i,j \le 6}, \qquad T = \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix},$$

$$Q_i = (2I_i/w_i)^{1/2} \sin\phi_i, \; P_i = \left(2I_i w_i\right)^{1/2} \cos\phi_i \; (i = 1, 2, 3),$$

$$a_{1i} = a_{5i} = 0 \; (i = 1, 2, 3, 6), \; a_{2i} = a_{4i} = 0 \; (i = 3, 4, 5, 6),$$

$$a_{3i} = a_{6i} = 0 \; (i = 1, 2, 4, 5), a_{33} = a_{66} = 1,$$

$$a_{14} = 4h_1 w_1 w_c, \; a_{15} = 4h_2 w_2 w_c.$$

$$a_{21} = 2h_1 w_1 (p - w_1^2), \; a_{22} = 2h_2 w_2 (p - w_2^2),$$

$$a_{41} = 2h_1 w_1 w_c (p + w_1^2), a_{42} = -2h_2 w_2 w_c (p + w_2^2),$$

$$a_{54} = 2h_1 w_1 (p + 2w_c^2 - w_1^2), \; a_{55} = 2h_2 w_2 (p + 2w_c^2 - w_2^2),$$
$$h_i^2 = [4w_i^2 (w_i^4 - 2pw_i^2 - q)]^{-1} \; (i = 1, 2),$$

$$p = \alpha_1 + \beta_1, \; q = -p(p + 4w_c^2).$$

The transformation changes the second order part in $I_i^{1/2}$ $(i = 1, 2, 3)$ of the Hamiltonian into the normal form

$$H_2 = w_1 I_1 + w_2 I_2 + w_3 I_3.$$

The general solution of the corresponding equations of motion are I_i = constant, ϕ_i, $= w_i t$ + constant $(i = 1, 2, 3)$.

3. Second order normalization

We wish to perform Birkhoff's normalization for which the coordinates (x, y, z) are to be expanded in double D'Alembert's series :

$$x = \sum_{n\geq 1} B_n^{1,0,0}, \qquad y = \sum_{n\geq 1} B_n^{0,1,0}, \qquad z = \sum_{n\geq 1} B_n^{0,0,1},$$

where the homogeneous components $B_n^{1,0,0}$, $B_n^{0,1,0}$, $B_n^{0,0,1}$, of degree n are of the form

$$\sum_{0\leq l,m\leq n} I_1^{(1/2)(n-l-m)}, I_2^{(1/2)l}\, I_3^{(1/2)m} \sum_{(i,j,k)} [C_{n-l-m,l,m,i,j,k}$$

$$\times \cos(i\phi_1 + j\phi_2 + k\phi_3) + S_{n-l-m,l,m,i,j,k}$$

$$\times \sin(i\phi_1 + j\phi_2 + k\phi_3)].$$

The double summation over the indices i, j and k is such that (a) i runs over those integers in the interval $0 \leq i \leq n\text{-}l\text{-}m$ that have the same parity as $n\text{–}l\text{–}m$, (b) j runs over those integers in the interval $-l \leq j \leq l$ that have the same parity as l, (c) k runs over those integers in the interval $-m \leq k \leq m$ that have the same parity as m. I_1, I_2 and I_3 are to be regarded as constants of integration and ϕ_1, ϕ_2 and ϕ_3 are to be determined as linear functions of time such that

$$\dot{\phi}_1 = w_1 + \sum_{n\geq 1} f_{2n}(I_1, I_2, I_3),$$

$$\dot{\phi}_2 = w_2 + \sum_{n\geq 1} g_{2n}(I_1, I_2, I_3),$$

$$\dot{\phi}_3 = w_3 + \sum_{n\geq 1} h_{2n}(I_1, I_2, I_3), \qquad (3)$$

where f_{2n}, g_{2n}, h_{2n} are of the form

$$f_{2n} = \sum_{0\leq l,m\leq n} f_{2n-2l-2m,2l,2m}\, I_1^{n-l-m}\, I_2^{l}\, I_3^{m},$$

$$g_{2n} = \sum_{0\leq l,m\leq n} g_{2n-2l-2m,2l,2m}\, I_1^{n-l-m}\, I_2^{l}\, I_3^{m},$$

$$h_{2n} = \sum_{0\leq l,m\leq n} h_{2n-2l-2m,2l,2m}\, I_1^{n-l-m}\, I_2^{l}\, I_3^{m}.$$

As in Bhatnagar & Hallan (1983), the first order components $B_1^{1,0,0}$, $B_1^{0,1,0}$ and $B_1^{0,0,1}$ are the values of x, y and z given by equations (2). $B_2^{1,0,0}$, $B_2^{0,1,0}$ and $B_2^{0,0,1}$ are the solutions of the partial differential equations

$$\Delta_1\Delta_2\, B_2^{1,0,0} = \Phi_2,$$

$$\Delta_1\Delta_2\, B_2^{0,1,0} = \Psi_2,$$

$$\Delta_3\, B_2^{0,0,1} = Z_2, \tag{4}$$

where

$$\Delta_i = D^2 + w_i^2 \quad (i = 1, 2, 3),$$

$$\Phi_2 = (D^2 + \beta_2)X_2 + 2w_c D Y_2,$$

$$\Psi_2 = (D^2 + p)Y_2 - 2w_c D X_2,$$

$$D = \left(w_1 \frac{\partial}{\partial\phi_1} + w_2 \frac{\partial}{\partial\phi_2} + w_3 \frac{\partial}{\partial\phi_3}\right) \text{ and } X_2, Y_2,$$

Z_2 are obtained from $\frac{\partial L_3}{\partial x}, \frac{\partial L_3}{\partial y}, \frac{\partial L_3}{\partial z}$ respectively by substituting the first order components for x, y, z.

Equation (4) can be solved for $B_2^{1,0,0}$, $B_2^{0,1,0}$ and $B_2^{0,0,1}$ by using the formula

$$\frac{1}{\Delta_1\Delta_2}\begin{bmatrix}\cos(l\phi_1 + m\phi_2 + n\phi_3)\\ \text{or}\\ \sin(l\phi_1 + m\phi_2 + n\phi_3)\end{bmatrix} = \frac{1}{\Delta_{l,m,n}}\begin{bmatrix}\cos(\perp\phi_1 + m\phi_2 + n\phi_3)\\ \text{or}\\ \sin(\perp\phi_1 + m\phi_2 + n\phi_3)\end{bmatrix},$$

$$\frac{1}{\Delta_3}\begin{bmatrix}\cos(\perp\phi_1 + m\phi_2 + n\phi_3)\\ \text{or}\\ \sin(\perp\phi_1 + m\phi_2 + n\phi_3)\end{bmatrix} =$$

$$\frac{1}{w_3^2 - (\perp w_1 + mw_2 + nw_3)^2}\begin{bmatrix}\cos(\perp\phi_1 + m\phi_2 + n\phi_3)\\ \text{or}\\ \sin(\perp\phi_1 + m\phi_2 + n\phi_3)\end{bmatrix}$$

where

$$\Delta_{l,m,n} = [w_1^2 - (\perp w_1 + mw_2 + nw_3)^2]\,[w_2^2 - (\perp w_1 + mw_2 + nw_3)^2].$$

The second order components $B_2^{1,0,0}$, $B_2^{0,1,0}$ and $B_2^{0,0,1}$ are as follows :

$$B_2^{1,0,0} = \frac{A_1 I_1 + A_2 I_2 + A_3 I_3}{w_1^2 w_2^2} - \frac{B_1 I_1 \cos 2\phi_1}{3w_1^2 (w_1^2 - 4w_1^2)} - \frac{B_2 I_2 \cos 2\phi_2}{3w_2^2 (w_1^2 - 4w_2^2)}$$

$$+ \frac{B_3 I_3 \cos 2\phi_3}{(w_1^2 - 4w_2^2)(w_2^2 - 4w_3^2)} + \frac{C_1 (I_1 I_2)^{1/2} \cos(\phi_1 + \phi_2)}{w_1 w_2 (w_2 + 2w_1)(w_1 + 2w_2)}$$

$$+ \frac{C_2 (I_1 I_2)^{1/2} \cos(\phi_1 - \phi_2)}{w_1 w_2 (2w_1 - w_2)(2w_2 - w_1)},$$

$$B_2^{0,1,0} = -\frac{B_1' I_1 \sin 2\phi_1}{3w_1^2 (w_2^2 - 4w_1^2)} - \frac{B_2' I_2 \sin 2\phi_2}{3w_2^2 (w_1^2 - 4w_2^2)}$$

$$+ \frac{B_3' I_3 \sin 2\phi_3}{(w_1^2 - 4w_2^2)(w_2^2 - 4w_3^2)} + \frac{C_1' (I_1 I_2)^{1/2} \sin(\phi_1 + \phi_2)}{w_1 w_2 (w_2 + 2w_1)(w_1 + 2w_2)}$$

$$+ \frac{C_2' (I_1 I_2)^{1/2} \sin(\phi_1 - \phi_2)}{w_1 w_2 (2w_1 - w_2)(2w_2 - w_1)},$$

$$B_2^{0,0,1} = -\frac{D_1 (I_1 I_3)^{1/2} \sin(\phi_1 + \phi_3)}{w_1 (w_1 + 2w_3)} + \frac{D_1 (I_1 I_3)^{1/2} \sin(\phi_1 - \phi_3)}{w_1 (w_1 - 2w_3)}$$

$$-\frac{D_2 (I_2 I_3)^{1/2} \sin(\phi_2 + \phi_3)}{w_2 (w_2 + 2w_3)} + \frac{D_2 (I_2 I_3)^{1/2} \sin(\phi_2 - \phi_3)}{w_2 (w_2 - 2w_3)}$$

where

$$A_i = -4\beta_2 w_i h_i^2 \, [12 b_5 + w_c^2 w_i^2 + b_6 (w_i^2 - p^2)], \qquad (i = 1, 2)$$

$$B_i = -4 w_i h_i^2 \, [12 b_5 \, w_c^2 w_i^2 (\beta_2 - 4w_i^2) - b_6 (w_i^2 - p)^2 \, (\beta_2 - 4w_i^2) - 16 b_6^2 w_c^2 w_i^2 \, (w_i^2 - p)] \qquad (i = 1, 2),$$

$$A_3 = -\frac{b_7 \beta_2}{w_3},$$

$$B_3 = \frac{b_7 (\beta_2 - 4w_3^2)}{w_3},$$

$$C_1 = -8h_1h_2(w_1w_2)^{1/2}[12b_5w_c^2 w_1w_2\{\beta_2 - (w_1+w_2)^2\} - b_6(w_1^2-p)(w_2^2-p)\{\beta_2 - (w_1+w_2)^2\} - 4b_6w_c^2(w_1w_2-p)(w_1+w_2)^2],$$

$$C_2 = -8h_1h_2(w_1w_2)^{1/2}[12b_5w_c^2 w_1w_2\{\beta_2 - (w_1-w_2)^2\} + b_6(w_1^2-p)(w_2^2-p)\{\beta_2 - (w_1-w_2)\} - 4b_6w_c^2(w_1w_2-p)(w_1-w_2)^2],$$

$$B_i' = -16w_c h_i^2 w_i^2[12b_5w_i^2w_c^2 - b_6(w_i^2-p)^2 - b_6(w_i^2-p)(\beta_2 - 4w_i^2)], \qquad (i = 1, 2),$$

$$B_3' = 4w_c b_7,$$

$$C_1' = -16w_c h_1h_2(w_1w_2)^{1/2}[12b_5w_1w_2(w_1+w_2)w_c^2 - b_6(w_1^2-p)(w_2^2-p)(w_1+w_2) - b_6(w_1w_2-p)(w_1+w_2)\{\beta_2 - (w_1+w_2)^2\}],$$

$$C_2' = -16w_c h_1h_2(w_1w_2)^{1/2}[12b_5w_1w_2(w_1-w_2)w_c^2 + b_6(w_1^2-p)(w_2^2-p)(w_1-w_2) + b_6(w_1+w_2+p)(w_1-w_2)\{\beta_2 - (w_1-w_2)^2\}],$$

$$D_i = -8b_7w_c h_i w_i\left(\frac{w_i}{w_3}\right)^{1/2}, \qquad (i = 1, 2).$$

4. Second order coefficients in the frequencies

Proceeding as in Bhatnagar and Hallan (1983), the third order components $B_3^{1,0,0}$, $B_3^{0,1,0}$ and $B_3^{0,0,1}$ in the coordinates x, y and z and the second order polynominals f_2, g_2 and h_2' in the frequencies ϕ_1, ϕ_2 and ϕ_3 satisfy the partial differential equations

$$\Delta_1\Delta_2B_3^{1,0,0} = \Phi_3 - 2f_2P_1' - 2g_2P_2' - 2h_2'P_3',$$

$$\Delta_1\Delta_2B_3^{0,1,0} = \Psi_3 - 2f_2U_1 - 2g_2U_2 - 2h_2'U_3,$$

$$\Delta_3B_3^{0,0,1} = Z_3 - 2f_2M_1 - 2g_2M_2 - 2h_2'M_3, \qquad (5)$$

where

$$\Phi_3 = (D^2 + \beta_2)X_3 + 2w_cDY_3,$$

$$\Psi_3 = (D^2 + P)Y_3 - 2w_cDX_3,$$

$$P_i' = (D^2 + \beta_2)\left[w_i \frac{\partial^2 B_1^{1,0,0}}{\partial\phi_i^2} - w_c \frac{\partial B_1^{0,1,0}}{\partial\phi_i}\right]$$

$$+ 2w_c D\left[w_i \frac{\partial^2 B_1^{0,1,0}}{\partial\phi_i^2} + w_c \frac{\partial B_1^{1,0,0}}{\partial\phi_i}\right], \text{(i = 1, 2, 3)},$$

$$U_i = (D^2 + p)\left[w_i \frac{\partial^2 B_1^{0,1,0}}{\partial\phi_i^2} + w_c \frac{\partial B_1^{1,0,0}}{\partial\phi_i}\right]$$

$$-2w_c D\left[w_i \frac{\partial^2 B_1^{1,0,0}}{\partial\phi_i^2} - w_c \frac{\partial B_1^{0,1,0}}{\partial\phi_i}\right], \text{ (i = 1, 2, 3)},$$

$$M_i = w_i \frac{\partial^2 B_1^{0,0,1}}{\partial\phi_i^2} \; (i = 1, 2, 3),$$

and X_3, Y_3, Z_3 are the homogeneous components of order 3 obtained respectively from $\frac{\partial}{\partial x}(L_3 + L_4)$, $\frac{\partial}{\partial y}(L_3 + L_4)$, $\frac{\partial}{\partial z}(L_3 + L_4)$ by substituting

$$x = B_1^{1,0,0} + B_2^{1,0,0}, \; y = B_1^{0,1,0} + B_2^{0,1,0} \; z = B_1^{0,0,1} + B_2^{0,0,1}.$$

The components $B_3^{1,0,0}$, $B_3^{0,1,0}$ and $B_3^{0,0,1}$ are not required to be found out. We find the coefficients of $\cos\phi_i$, $\sin\phi_i$ $(i = 1, 2, 3)$ on the right hand side of equations (5). They are the critical terms as $\Delta_{1,0,0} = \Delta_{0,1,0} = \Delta_{0,0,1} = 0$.

We eliminate these terms by choosing properly the coefficients in the polynomials

$$f_2 = f_{2,0,0}\, I_1 + f_{0,2,0}\, I_2 + f_{0,0,2}\, I_3,$$

$$g_2 = g_{2,0,0}\, I_1 + g_{0,2,0}\, I_2 + g_{0,0,2}\, I_3,$$

$$h_2' = h_{2,0,0}\, I_1 + h_{0,2,0}\, I_2 + h_{0,0,2}\, I_3.$$

We find that

$$f_{2,0,0} = \frac{H_1'}{2H_7'}; \; f_{0,2,0} = g_{2,0,0} = \frac{H_2'}{2H_7'};$$

$$f_{0,0,2} = h_{2,0,0} = \frac{H_3'}{2H_7'};\; g_{0,2,0} = \frac{H_4'}{2H_8'};$$

$$g_{0,0,2} = h_{0,2,0} = \frac{H_5'}{2H_8'};\; h_{0,0,2} = -\frac{H_6'}{2\sqrt{2w_3}};$$

where

$$H_1' = -[-(w_1^2 + \beta_1)(3b_5\lambda_1 + b_6\lambda_2 + 4b_8\lambda_3 + 2b_{11}\lambda_4) + 2w_c w_1 (2b_6\lambda_5 + 4b_9\lambda_6 + 2b_{11}\lambda_7)],$$

$$H_2' = -[(w_1^2 + \beta)(3b_5\mu_1 + b_6\mu_2 + 4b_8 u_3 + 2b_{11}\mu_4) + 2w_c w_1 (2b_6\mu_5 + 4b_9\mu_6 + 2b_{11}\mu_7)],$$

$$H_3' = -[-(w_1^2 + \beta_2)(3b_5\upsilon_1 + b_7\upsilon_2 + 2b_{13}\upsilon_3) + 2w_c w_1 (2b_6\upsilon_4 + 2b_{12}\upsilon_5)],$$

$$H_4' = -[-(w_2^2 + \beta_2)(3b_5\lambda_8 + b_6\lambda_9 + 4b_8\lambda_{10} + 2b_{11}\lambda_{11}) + 2w_c w_2 (2b_6\lambda_{12} + 4b_9\lambda_{13} + 2b_{11}\lambda_{14})],$$

$$H_5' = -[-(w_2^2 + \beta_2)(3b_5\upsilon_6 + b_7\upsilon_7 + 2b_{13}\upsilon_8) + 2w_c w_2 (2b_6\upsilon_9 + 2b_{12}\upsilon_{10})]$$

$$H_6' = -(2b_7\lambda_{15} + 4b_{10}\lambda_{16}),$$

$$H_{i+6}' = 2\sqrt{2}\, w_c h_i w_i^{1/2} [-2w_i^2 (-w_i^2 + \beta_2 + 2w_c^2) + (w_i^2 - p)(w_i^2 + \beta_2)], \qquad i = (1,2)$$

$$\lambda_1 = \frac{4\sqrt{2} w_c h_1}{\sqrt{w_1}} \left[\frac{2A_1}{w_2^2} - \frac{B_1}{3(w_2^2 - 4w_1^2)}\right],$$

$$\lambda_2 = \frac{2\sqrt{2}\, h_1 (w_1^2 - p) B_1'}{3w_1^{3/2} (w_2^2 - 4w_1^2)},\; \lambda_3 = 96\sqrt{2}\, w_c^3 h_1^3 w_1^{9/2},$$

$$\lambda_4 = 8\sqrt{2}\, w_c h_1^3 w_1^{5/2} (w_1^2 - p)^2,$$

$$\lambda_5 = -\frac{2\sqrt{2}\, w_c h_1 w_1^{3/2} B_1'}{3w_1^2 (w_2^2 - 4w_1^2)} - \sqrt{2}\, h_1 w_1^{1/2} (w_1^2 - p)$$

$$\left[\frac{2A_1}{w_1^2 w_2^2} + \frac{B_1}{3w_1^2 (w_2^2 - 4w_1^2)}\right],$$

$$\lambda_6 = -12\sqrt{2}\, h_1^3 w_1^{3/2} (w_1^2 - p)^3, \quad \lambda_7 = -16\sqrt{2}\, w_c^2 h_1^3 w_1^{7/2} (w_1^2 - p),$$

$$\lambda_8 = \frac{4\sqrt{2}\, w_c h_2}{\sqrt{w_2}} \left[\frac{2A_2}{w_1^2} - \frac{B_2}{3(w_1^2 - 4w_2^2)}\right],$$

$$\lambda_9 = \frac{2\sqrt{2h_2}\, (w_2^2 - p) B_2'}{3(w_2)^{3/2} (w_1^2 - 4w_2^2)}, \quad \lambda_{10} = 96\sqrt{2}\, w_c^3 h_2^3 w_2^{9/2}$$

$$\lambda_{11} = 8\sqrt{2}\, w_c h_2^3 w_2^{5/2} (w_2 - p)^2,$$

$$\lambda_{12} = -\frac{2\sqrt{2}\, w_c h_2 B_2'}{3w_2^{1/2} (w_1^2 - 4w_2^2)} - \frac{\sqrt{2}\, h_2 (w_2^2 - p)}{w_2^{3/2}} \left[\frac{2A_2}{w_1^2} + \frac{B_2}{3(w_1^2 - 4w_2^2)}\right],$$

$$\lambda_{13} = -12\sqrt{2}\, h_2^3 w_2^{3/2} (w_2^2 - p)^3, \quad \lambda_{14} = -16\sqrt{2}\, w_c^2 h_2^3 w_2^{7/2} (w_2^2 - p),$$

$$\lambda_{15} = \left(\frac{2}{w_3}\right)^{1/2} \left[\frac{A_3}{w_1^2 w_2^2} - \frac{B_3}{2(w_1^2 - 4w_3^2)(w_2^2 - 4w_3^2)}\right],$$

$$\lambda_{16} = \frac{3}{\sqrt{2}\, (w_3)^{3/2}},$$

$$\mu_1 = \frac{4\sqrt{2}\, w_c}{w_1 w_2} \left[\frac{2h_1 A_2 w_1^{1/2}}{w_2} + h_2 w_2^{3/2} \left\{\frac{C_1}{(w_2 + 2w_1)(w_1 + 2w_2)} + \frac{C_2}{(2w_1 - w_2)(2w_2 - w_1)}\right\}\right],$$

$$\mu_2 = -\frac{2\sqrt{2}\, h_2 (w_2^2 - p)}{w_1 \sqrt{w_2}} \left[\frac{C_1'}{(w_2 + 2w_1)(w_1 + 2w_2)} - \frac{C_1'}{(2w_1 - w_2)(2w_2 - w_1)}\right],$$

$$\mu_3 = 192\sqrt{2}\, w_c^2\, h_1\, h_2^2\, w_1^{3/2}\, w_2^3\, ;\ \mu_4 = 16\sqrt{2}\, w_c h_1 h_2^2\, w_1^{3/2}\, w_2\, (w_2^2 - p)^2,$$

$$\mu_5 = \frac{2\sqrt{2}\, w_c h_2 w_2^{3/2}}{w_1 w_2}, \left[\frac{C_1'}{(w_2 + 2w_1)(w_1 + 2w_2)} + \frac{C_2'}{(2w_1 - w_2)(2w_2 - w_1)}\right]$$

$$- \frac{2\sqrt{2}\, A_2 h_1\, (w_1^2 - p)}{w_1^{3/2}\, w_2^2},$$

$$\mu_6 = -24\sqrt{2}\, h_1 h_2^2\, w_1^{1/2}\, w_2 (w_1^2 - p)\, (w_2^2 - p)^2,$$

$$\mu_7 = -32\sqrt{2}\, w_c^2\, h_1 h_2^2\, w_1^{1/2}\, w_2^3\, (w_1^2 - p),$$

$$\upsilon_1 = \frac{8\sqrt{2}\, w_c h_1 A_3}{w_1^{1/2}\, w_2^2},\ \upsilon_2 = -\left(\frac{2}{w_3}\right)^{1/2} \frac{D_1}{w_1} \left(\frac{1}{w_1 + 2w_3} + \frac{1}{w_1 - 2w_3}\right),$$

$$\upsilon_3 = \frac{4\sqrt{2}\, w_c h_1 w_1^{3/2}}{w_3},\ \ \upsilon_4 = -\frac{2\sqrt{2}\, h_1\, A_3\, (w_1^2 - p)}{w_1^{3/2}\, w_2^2},$$

$$\upsilon_5 = -\frac{2\sqrt{2}\, h_1\, (w_1)^{1/2}\, (w_1^2 - p)}{w_3},$$

$$\upsilon_6 = \frac{8\sqrt{2}\, w_c h_2 A_3}{w_1^2\, w_2^{1/2}},$$

$$\upsilon_7 = -\left(\frac{2}{w_3}\right)^{1/2} \frac{D_2}{w_2} \left(\frac{1}{w_2 + 2w_3} + \frac{1}{w_2 - 2w_3}\right),$$

$$\upsilon_8 = \frac{4\sqrt{2}\, w_o h_2 w_2^{3/2}}{w_3},\ \ \ \upsilon_9 = -\frac{2\sqrt{2}\, h_2 A_3\, (w_2^2 - p)}{w_1^2\, w_2^{3/2}},$$

$$\upsilon_{10} = -\frac{2\sqrt{2}\, h_2 w_2^{1/2}\, (w_2^2 - p)}{w_3}$$

If the normalised Hamiltonian is written as

$$H = w_1 I_1 + w_2 I_2 + w_3 I_3 + \frac{1}{2}(aI_1^2 + bI_2^2 + cI_3^2 + 2fI_2 I_3 + 2gI_3 I_1 + 2hI_1 I_2)$$

then from Hamilton's equations of motion,

$$\phi_i = \frac{\partial H}{\partial I_i} \quad (i = 1, 2, 3)$$

and the equations (3), we find that

$$a = f_{2,0,0}\,;\; b = g_{0,2,0}\,;\; c = h_{0,0,2}\,;\; f = g_{0,0,2} = h_{0,2,0}\,;$$

$$g = f_{0,0,2} = h_{2,0,0};\; h = f_{0,2,0} = g_{2,0,0}\,.$$

5. Stability

Now we apply Arnold's theorem (1963) to discuss the non-linear stability of the cluster of stars. We assume that $w_1 > w_2 > w_3$. It may be mentioned that the correct ordering of the frequencies w_1, w_2 and w_3 requires an analysis of the equations (1). More generally, the issue depends on the geometry of the ellipsoidal form of the cluster and the ordering of the constants β_1, β_2 and β_3. We have made the assumption $w_1 > w_2 > w_3$ just to illustrate the procedure to be adopted for deciding the stability of the cluster.

The condition (i) of Arnold theorem is satisfied provided the basic frequencies do not satisfy the equations

(i)	$w_1 = 2w_2$	(vi)	$w_1 = 3w_3$
(ii)	$w_1 = 3w_2$	(vii)	$-w_1 + 2w_2 - w_3 = 0$
(iii)	$w_2 = 2w_3$	(viii)	$-w_1 + 2w_2 + w_3 = 0$
(iv)	$w_2 = 3w_3$	(ix)	$-w_1 + w_2 + 2w_3 = 0$
(v)	$w_1 = 2w_3$	(x)	$w_1 - w_2 - w_3 = 0$

Taking any of the conditions (i) to (x) and eliminating w_1, w_2, w_3 from that equation and the equations (1), the eliminant will be an equation in ρ. Let us call these equations $f_i(\rho) = 0$, $i = 1, 2$ ...... 10. Thus the condition (i) of Arnold's theorem is satisfied for all $\rho > \rho^*$ except for those which satisfy the equations $f_i\,(\rho) = 0$, $i = 1, 2,$ .... 10. As an illustration, let us consider the case $w_1 = 2w_2$. Eliminating w_1, w_2, w_3 from the equation $w_1 = 2w_2$ and the equations (1), we get

$$4[\alpha_1 + \pi G\rho(\beta_1' + \beta_2^1) + 4w_c^2]^2 = 25\pi G\rho\beta_2'\,(a_1 + \pi G\rho\beta_1'),$$

which is an equation in ρ.

For the condition (ii) of Arnold theorem, the normalized Hamiltonian up to 4th order is

$$H = w_1 I_1 + w_2 I_2 + w_3 I_3 + \frac{1}{2}\,(a I_1^2 + b I_2^2 + c I_3^2 + 2f I_2 I_3 + 2g I_3 I_1 + 2h I_1 I_2).$$

The determinant D occuring in the condition is

$$D = -\left[A' w_1^2 + B' w_2^2 + C' w_3^2 + 2F^1 w_2 w_3 + 2G' w_3 w_1 + 2H' w_1 w_2\right]$$

where A', B', C', F', G', H' are the co-factors of a, b, c, f, g, h respectively in the determinant

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

$D \neq 0$ if the density of the cluster does not satisfy the equation obtained by eliminating w_1, w_2, w_3 from the equations

$$A' w_1^2 + B' w_2^2 + C' w_3^2 + 2F' w_2 w_3 + 2G' w_3 w_1 + 2h' w_1 w_2 = 0,$$

$$w_1^2 + w_2^2 = \alpha_1 + \pi G \rho\,(\beta_1' + \beta_2') + 4 w_c^2,$$

$$w_1^2 \; w_2^2 = \pi G \rho \beta_2'\,(\alpha_1 + \pi G \rho \beta_1'),$$

$$w_3^2 = \alpha_3 + \pi G \rho \beta_3'.$$

Thus the ellipsoidal cluster of stars sharing galactic rotation is generally stable in the non-linear sense for all the densities $\rho > \rho^*$, which is the range of linear stability. However, for those densities which satisfy the equations $f_i(\rho) = 0$, $i = 1, 2, \ldots 10$ and the equations (6) where Arnold's theorem is not applicable, we cannot take any decision.

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