Bull. Astr. Soc. India (1995) **23**, 165–176

Existence of periodic orbits of first kind in the photogravitational circular restricted problem of four bodies

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Received 4 August 1994; accepted 13 December 1994

Abstract. In this paper the existence of the periodic orbits of first kind in the photogravitational circular restricted problem of four bodies has been established.

Key Words : four-body problem—photogravitational field—periodic orbits

Introduction

Bhatnagar (1971) studied the existence of periodic orbits of collision in the restricted problem of four bodies. Since then the problem has been generalised by introducing different variations.

The stars are the sources of gravitational forces as well as the radiation pressures. So, in this paper we have considered all the three finite masses to be the sources of radiation pressures and have established the existence of the periodic orbits of first kind in the restricted problem of four bodies on the assumption that the three finite masses move in circular orbits about their centre of inertia forming an equilateral triangle and the infinitesimal mass moves under the photogravitational field of the three finite masses without rendering the equilateral triangular configuration of the three finite masses.

1. Equations of motion

Consider the motion of an infinitesimal mass P in the photogravitational field of the primaries P_1, P_2, P_3 of finite masses m_1, m_2, m_3 respectively in a plane on circular orbits and let the mass of P be so small that the triangular configuration is not changed.

Let C be the geometric centre of the triangular configuration $P_1P_2P_3$ and G the centre of mass of the masses m_1, m_2, m_3 situated at P_1, P_2, P_3 respectively.

Let us take the line Gx parallel to CP_1 as x-axis and the line Gy perpendicular to Gx in the sense of rotation as the y-axis. Let the co-ordinates of P be (x, y) and those of P_i, $(i = 1, 2)$ as (x_i, y_i) in the rotating system.

If q_1, q_2, q_3 be the radiation factors of the gravitational forces due to radiations, then the Hamiltonian H is given by

$$H = \frac{1}{2}(p_1^2 + p_2^2) + (p_1 y - p_2 x) - q_1\, \mu_1/r_1 - q_2\, \mu_2/r_2 - q_3\, \mu_3/r_3$$

where r_1, r_2, r_3 are the distances of P from P_1, P_2, P_3 respectively.

Let us shift the origin to C, the geometric centre of the equilateral triangle $P_1P_2P_3$ through parallel axes. Let (x_1, x_2) be the coordinates of P and $(\bar{x}_1, \bar{x}_2)$ the coordinates of the centre of mass G referred to Cx_1x_2 system. We have

$$(x_1, x_2) = (x + \bar{x}_1, y + \bar{x}_2).$$

In canonical form the equations of motion are

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial x_i}, \ (i = 1, 2) \tag{1}$$

where

$$H = \frac{1}{2}(p_1^2 + p_2^2) + [p_1(x_2 - \bar{x}_2) - p_2(x_1 - \bar{x}_1)] \\ - q_1\, \mu_1/r_1 - q_2\, \mu_2/r_2 - q_3\, \mu_3/r_3 = \mathrm{C}. \tag{2}$$

2. Regularization

For the elimination of the singularity at P_1, we shall use Levi-Civita's (1906) parabolic transformation defined by

$$S = (A + \xi_1^2 - \xi_2^2)\, p_1 + \xi_1\, \xi_2 p_2$$

so that

$$x_i = \frac{\partial S}{\partial p_i} \qquad \pi_i = \frac{\partial S}{\partial \xi_i}, \ (i = 1, 2)$$

where π_i are the momenta associated with new coordinates ξ_i, $(i = 1,2)$ and $A = 1/\sqrt{3}$.

We also introduce a new independent variable τ instead of t given by

$$dt = r_1 \,.\, d\tau, \;\; \tau = 0 \text{ when } t = 0.$$

The equations of motion (1) become

$$\frac{d\xi_i}{d\tau} = \frac{\partial K}{\partial \pi_i}, \quad \frac{d\pi_i}{d\tau} = -\frac{\partial K}{\partial \xi_i}, \; (i = 1, 2) \tag{3}$$

where K is the new Hamiltonian which is given by

$$\begin{aligned} \mathrm{K} &= r_1(H - C) \\ &= \frac{1}{8}\pi^2 + \frac{1}{2}[r_1(\pi_1\xi_2 - \pi_2\xi_1) - \mathrm{A}(\pi_1\xi_2 + \pi_2\xi_1) \\ &\quad - (\pi_1\xi_1 - \pi_2\xi_2)\bar{x}_2 + (\pi_1\xi_2 + \pi_2\xi_1)\bar{x}_1] \\ &\quad - q_1\mu_1 - q_2\mu_2 r_1/r_2 - q_3\mu_3 r_1/r_3 \\ &\quad - r_1(C_0 + C_1\mu_1 + C_2\mu_2 + C_3\mu_3) \_ O(\mu_2) \end{aligned} \tag{4}$$

$$\begin{aligned} r_1 &= \xi^2 \\ r_2^2 &= 1 + \xi^4 + 3\mathrm{A}(\xi_1^2 - \xi_2^2) - 2\xi_1\xi_2, \\ r_3^2 &= 1 + \xi^4 + 3\mathrm{A}(\xi_1^2 - \xi_2^2) + 2\xi_1\xi_2, \\ \xi^2 &= \xi_1^2 + \xi_2^2, \\ \pi^2 &= \pi_1^2 + \pi_2^2. \end{aligned} \tag{5}$$

Similar to Bhatnagar (1971), we suppose that $\mu_3 = \mu$, $\mu_2 = \alpha\mu$ so that $\mu_1 = 1 - \mu(1 + \alpha)$, where α is a significant constant.

We have

$$\bar{x}_1 = (2 - 3\alpha\mu - 3\mu)\ \mathrm{A}/2,$$

$$\bar{x}_2 = (\alpha - 1)\mu/2.$$

The Hamiltonian (4) can be put into the form

$$K = K_0 + \mu K_1,$$

where

$$K_0 = \frac{1}{8}\pi^2 + \frac{1}{2}r_1(\pi_2\xi_1 - \pi_1\xi_2 - 2C_0') - 1 + I_i = -\epsilon < 0 \tag{6}$$

$$K_1 = -\frac{1}{4}(\alpha - 1)(\pi_1\xi_1 - \pi_2\xi_2) - \frac{3}{4}A(\alpha + 1)(\pi_1\xi_2 + \pi_2\xi_1)$$
$$+ (1 - I_1)(\alpha + 1) - (1 - I_2)\,\alpha\, r_1/r_2 - (1 - I_3)\, r_1/r_3 + r_1(\alpha C_1' + C_2') \tag{7}$$
$$C_0' = C_0 + C_1,\ C_1' = C_1 - C_2,\ C_2' = C_1 - C_3.$$
$$q_i = 1 - I_i,\ |I_i| << 1.$$

Clearly, $I_1 = I_2 = I_3 = 0$ when the effect of the radiation pressures are neglected.

Now, K_0 has almost the same expression as in Bhatnagar (1971) and with him we shall assume that K_0 is negative which puts a restriction on the range of I_I.

In order to solve the Hamiltonian Jacobi equation associated with K_0, let

$$\pi_i = \frac{\partial w}{\partial \xi_i},\ (i = 1, 2) \tag{8}$$

and $\alpha = 1 - I_1 - \epsilon > 0.$

It follows that

$$\frac{1}{8}\left[\left(\frac{\partial W}{\partial \xi_1}\right)^2 + \left(\frac{\partial W}{\partial \xi_2}\right)^2\right] + \frac{1}{2}\xi^2\left[\xi_2\frac{\partial W}{\partial \xi_1} - \xi_1\frac{\partial W}{\partial \xi_2} - 2C_0\right] = \alpha$$

which by introducing polar co-ordinates

$$\xi,\ \phi$$

with the help of the relations

$$\xi_1 = \xi\cos\phi,\ \xi_2 = \xi\sin\phi,$$

becomes

$$\frac{1}{8}\left[\left(\frac{\partial W}{\partial \xi}\right)^2 + \frac{1}{\xi^2}\left(\frac{\partial W}{\partial \phi}\right)^2\right] + \frac{\xi^2}{2}\left(-\frac{\partial W}{\partial \phi} - 2C_0\right) = \alpha \tag{9}$$

This may be satisfied by

$$W = U(\xi) + 2G\phi,$$

where G is an arbitrary parameter. Hence

$$\left(\frac{dU}{dz}\right)^2 = -[2(G + C_0)/z^2]\,F(z) \tag{10}$$

where

$$z = \xi^2$$

and

$$F(z) = [G^2/2(G + C_0)] - [\alpha/(G + C_0)]z - z^2 \tag{11}$$

Hence, we have

$$U(z, G, \alpha) = [-2(G + C_0)]^{1/2} \int_{z_1}^{z} F(\zeta)\,\frac{d\zeta}{\zeta}$$

where z_1 is the smaller of the two roots of $F(z) = 0$.

Let the parameters a, e, and l be defined by

$$z_1 = a(1 - e),\ z_2 = a(1 + e)$$

$$z = z_1 \cos^2 l/2 + z_2 \sin^2 l/2 \tag{12}$$

$$= a(1 - e\cos l).$$

The equations of motion associated with K_0 are

$$\xi_i' = \frac{\partial K_0}{\partial \pi_i} = \frac{1}{4}\pi_i + \frac{1}{2}\xi^2\xi_j(-1)^j,\ (i = 1, 2;\ j = 1, 2;\ j \neq i)$$

and from these it follows that

$$\sum_{i=1}^{2} \xi_i\,\xi_i' = \xi\,\xi' = \frac{1}{4}\sum_{i=1}^{2} \xi_i\,\pi_i.$$

On the other hand,

$$\sum_{i=1}^{2} \xi_i \, \pi_i = \xi \frac{\partial W}{\partial \xi} = \xi \frac{dU}{d\xi} = 2z \frac{dU}{dz},$$

so that

$$\frac{dz}{d\tau} = [-2(G + C_0)]^{1/2} \, [F(z)]^{1/2},$$

from which we obtain

$$\int_{z_1}^{z} \frac{d\zeta}{[F(\zeta)]^{1/2}} = [-2(G + C_0)]^{1/2} \, (\tau - \tau_0) \tag{13}$$

where

$$z = z_1 \text{ at } \tau = \tau_0.$$

Let us introduce L by the relation

$$\alpha = L \, [-2(G + C_0)]^{1/2} > 0$$

so that

$$a = \frac{L}{[-2(G + C_0)]^{1/2}} > 0$$

$$e = [1 - G^2/L^2]^{1/2} \tag{14}$$

Using equation (12), it is found that

$$F(z) = a^2 e^2 \sin^2 l,$$

and equation (13) becomes

$$l = [-2(G + C_0)]^{1/2} \, (\tau - \tau_0). \tag{15}$$

Furthermore

$$\frac{\partial W}{\partial L} = \frac{dU}{dL} = \int_{z_1}^{z} \frac{d\zeta}{[F(\zeta)]^{1/2}} = l \tag{16}$$

$$\frac{\partial W}{\partial G} = 2\phi + \frac{dU}{\partial G} = 2\phi + 2\frac{(L^2 - G^2)^{1/2}}{G + C_0} \sin l - \mathrm{f} = \mathrm{g} \tag{17}$$

where

$$f = (1 - e^2)^{1/2} \int \frac{dl}{(1 - e\cos l)} . \tag{18}$$

Equations (16) & (17) establish the canonical set (l, L; g, G).

Since

$$K_0 = \alpha - 1 + I_1,$$

it follows that

$$K_0 = L[-2(G + C_0)]^{1/2} - 1 + I_1,$$

and therefore, for the problem generated by this Hamiltonian (regularised two-body problem in rotating co-ordinates) one has

$$\frac{dL}{d\tau} = -\frac{\partial K_0}{\partial_l} = 0,$$

$$\frac{dG}{d\tau} = -\frac{\partial K_0}{\partial g} = 0, \tag{19}$$

$$\frac{dl}{d\tau} = \frac{\partial K_0}{\partial L} = [-2(G + C_0)]^{1/2} = \text{constant}$$

$$\frac{dg}{d\tau} = \frac{\partial K_0}{\partial G} = -L/[-2(G + C_0)]^{1/2} = \text{constant.}$$

The argument ϕ is obtained from equation (17). It is given by

$$\phi = \frac{1}{2}(f + g) - [(L^2 - G^2)^{1/2}/(G + C_0)] \sin l. \tag{20}$$

The variables ξ_i and π_i ($i = 1, 2$) are then expressed by the canonical elements. In fact

$$\xi_1 = \xi \cos\phi = \pm z^{1/2} \cos\phi$$

$$\xi_2 = \xi \sin\phi = \pm z^{1/2} \sin\phi \tag{21}$$

and

$$\pi_1 = \frac{\partial W}{\partial \xi_1} = \cos\phi \frac{\partial W}{\partial \xi} - \frac{\sin\phi}{\xi} \frac{\partial W}{\partial \phi}$$

$$\pi_2 = \frac{\partial W}{\partial \xi_2} = \sin\phi \frac{\partial W}{\partial \xi} - \frac{\cos\phi}{\xi} \frac{\partial W}{\partial \phi}. \tag{22}$$

Since

$$\frac{\partial W}{\partial \xi} = \frac{dU}{d\xi} = 2\xi \frac{dU}{dz} = 2\xi [-2(G + C_0)]^{1/2} \frac{[F(z)]^{1/2}}{z},$$

$$\frac{\partial W}{\partial \phi} = 2G,$$

it follows that

$$\frac{\partial W}{\partial \xi} = 2e\, L\sin l/[\pm\{\dot{a}(1 - e\cos l)\}^{1/2}],$$

and therefore

$$\pm \xi_1 = [a(1 - e\cos l)]^{1/2} \cos\phi$$

$$\pm \xi_2 = [a(1 - e\cos l)]^{1/2} \sin\phi \tag{23}$$

$$\pm \pi_1 = \frac{2eL\sin l \cos\phi - 2G\sin\phi}{[a(1 - e\cos l)]^{1/2}} \tag{24}$$

$$\pm \pi_2 = \frac{2eL\sin l \sin\phi - 2G\sin\phi}{[a(1 - e\cos l)]^{1/2}}$$

where ϕ is given by equation (20).

The equations of motion for the complete Hamiltonian K are given by

$$\frac{dl}{d\tau}=\frac{\partial K}{\partial L}=\frac{\partial K_0}{\partial L}+\mu\frac{\partial K_1}{\partial L}$$

$$\frac{dg}{d\tau}=\frac{\partial K}{\partial G}=\frac{\partial K_0}{\partial G}+\mu\frac{\partial K_1}{\partial G} \tag{25}$$

$$\frac{dL}{d\tau}=-\frac{\partial K}{\partial l}=-\mu\frac{\partial K_1}{\partial l}$$

$$\frac{dG}{d\tau}=-\frac{\partial K}{\partial g}=-\mu\frac{\partial K_1}{\partial g}$$

where K_0 and K_1 are expressed in terms of the canonical elements.

Equation (25) forms the basis for the general theory of perturbation for the problem under consideration.

3. Existence of periodic orbits of first kind

Let

$$x_1 = L, x_2 = G, y_1 = l, y_2 = g.$$

Equation (19) may be written as

$$\frac{dx_i}{d\tau}=0, \quad \frac{dy_i}{d\tau}=\eta_i^{(0)}, \quad (i = 1, 2)$$

which gives

$$x_i^{(0)} = a_i, y_i^{(0)} = \eta_i^{(0)}\tau + w_i.$$

These are "generating solutions" of the problem of two bodies.

Let the general solution in the neighbourhood of the generating solutions be periodic with the period τ_0 $(1 + \alpha)$, where τ_0 is the period of the generating solutions and a is a negligible quantity of order μ.

The period of the general solutions will also be τ_0 if we change the independent variable τ to τ' with the help of the transformation

$$\tau' = \tau \, / \, (1 + \alpha).$$

Equation (25) then takes the form

$$\frac{dx_i}{d\tau'} = -(1+\alpha)\frac{\partial K}{\partial y_i},\ (i = 1, 2) \tag{26}$$

$$\frac{dy_i}{d\tau'} = (1+\alpha)\frac{\partial K}{\partial x_i},\ (i = 1, 2).$$

The general solutions in the neighbourhood of the generating solutions may be given by

$$x_i = a_i + \beta_i + \xi_i(\tau'),\ (i = 1, 2) \tag{27}$$

$$y_i = \eta_i^{(0)}\tau' + w_i + \gamma_i + \eta_i(\tau'),\ (i = 1,2).$$

Now following Bhatnagar (1969) we find that the orbits will be periodic for $\mu \neq 0$ if the following conditions (Duboshin 1964) are satisfied :

$$\frac{\partial[K_1]}{\partial w_i} = 0,\ (i = 1, 2) \tag{28}$$

$$\frac{\partial[K_1]}{\partial a_i} = 0,\ (i = 1, 2) \tag{29}$$

where

$$[K_1] = \frac{1}{\tau_0}\int_0^{\tau_0} K_1(\tau', a_i, \eta_i^{(0)}\tau' + w_i)\, d\tau' \tag{30}$$

$$\frac{\partial(\xi_2, \eta_1, \eta_2)}{\partial(\gamma_2, \beta_1, \beta_2)} = 0,\ \text{when } \mu = \beta_i = \gamma_i = 0 \tag{31}$$

The condition (31) can be written as (Bhatnagar 1971)

$$\frac{\partial_2[K_1]}{\partial w_2^2} . D \neq 0 \tag{32}$$

where

$$D = \begin{vmatrix} \frac{\partial^2 K_0}{\partial a_1^2} & \frac{\partial^2 K_0}{\partial a_1 \partial a_2} \\ \frac{\partial^2 K_0}{\partial a_2 \partial a_1} & \frac{\partial^2 K_0}{\partial a_2^2} \end{vmatrix} \quad (33)$$

We have

$$K_0 = a_1[-2(a_2 + C_0)]^{1/2} - 1 + I_1.$$

$$\therefore \quad D = -1/[2(G + C_0)] \neq 0. \quad (34)$$

Taking zero order terms, we have

$$r_1 = \mathrm{a}$$

$$r_2^2 = 1 + a^2 + 3aA \cos 2\phi - a \sin 2\phi$$

$$r_3^2 = 1 + a^2 + 3aA \cos 2\phi + a \sin 2\phi$$

$$\xi_1 \pi_1 - \xi_2 \pi_2 = -2G \sin 2\phi$$

$$\xi_1 \pi_2 + \xi_2 \pi_1 = 2G \cos 2\phi$$

$$2\phi = y_1 + y_2$$

$$x_i = a_i,\ y_i = \eta_i^{(0)} \tau' + w_i,\ (i = 1, 2)$$

$$[K_1] = \frac{1}{2}(\alpha + 1)\, G \sin 2\phi - \frac{3A}{2}(\alpha + 1)\, G \cos 2\phi + (1 - I_i)\,(\alpha + 1)$$
$$- (1 - I_2) a\, \alpha/r_2 - (1 - I_3) \mathrm{a}/r_3 + a(\alpha C_1' + C_2' \quad (36)$$

$$\therefore \quad \frac{\partial [K_1]}{\partial w_2} = \frac{1}{2}(\alpha - 1) G \cos 2\phi + \frac{3A}{2}(\alpha + 1) G \sin 2\phi$$
$$+ (1 - I_2)\,(-3aA \sin 2\phi - a \cos 2\phi) a\, \alpha/\mathrm{r}_2^3$$
$$+ (1 - I_3)\,(-3aA \sin 2\phi + a \cos 2\phi) a/r_3^3 \quad (37)$$

For $2\phi = 0$, π etc., we have

$$\frac{\partial[K_1]}{\partial w_2} = \frac{1}{2}(\alpha - 1)\, G - (1 - I_2)\, a^2\alpha/2r^3 + (1 - I_3)\, a^2/2r^3 \tag{38}$$

where

$$r^2 = 1 + a^2 + 3aA \tag{39}$$

$$\therefore \quad \frac{\partial[K_1]}{\partial w_2} = 0, \text{ if } G = \frac{a^2}{r^3}\left[1 - \frac{(\alpha I_2 - I_3)}{\alpha - 1}\right].$$

For $2\varnothing = 0, \pi$ etc and $G = \frac{a^2}{r^3}\left[1 - \frac{(\alpha I_2 - I_3)}{\alpha - 1}\right]$, we have

$$\frac{\partial^2[K_1]}{\partial w_2^2} = \frac{3a^2}{2r^5}[(\alpha I_2 + I_3) - (\alpha + 1)] + \frac{3a^2 A}{2r^3}(\alpha I_2 + I_3) \neq 0 \tag{40}$$

Using (34) and (40) we find that the condition (32) is satisfied.

It can be easily seen that the other conditions for the existence of the periodic orbits are also satisfied.

Therefore, there exists periodic orbits of first kind of the restricted four-body problem considered in this paper.

Acknowledgement

The author thanks the referee for giving constructive suggestions for the improvement of the paper.

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