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Reply to the paper "On naked singularities in spherically symmetric gravitational collapse" by C.S. Unnikrishnan

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Abstract. In a black-hole, the gravitational singularity at its centre is covered by an event horizon. This singularity is hidden from an observer who is far from the black-hole. On the other hand, a naked singularity is a gravitational singularity which is not covered by an event horizon, and is visible to a far-away observer. According to the cosmic censorship conjecture (CCC) of Penrose, the general theory of relativity does not admit the formation of naked singularities in gravitational collapse. Recently, Dwivedi and Joshi, and others, have found examples of naked singularities in collapse. One such example has been described in their paper, Dwivedi & Joshi (1992). The singularities described in this paper are strong curvature naked singularities.

Unnikrishnan (1993) has claimed that these examples are unphysical. He claims that although a naked singularity does result in these models, it must be a weak naked singularity rather than being strong. (Weak and strong are technical terms referring to the rate at which curvature blows up, as the singularity is approached).

The purpose of the present reply is to point out the mistake made by Unnikrishnan (U) in his analysis, leading to his erroneous conclusion. We point out here that the principal claim made by U in his paper, (namely that the singularity in this model is weak when naked, and it is hidden when it is strong) is baseless because it is not supported by any mathematical derivation. In particular, we show that the class of solutions considered by him is completely different from that of Dwivedi and Joshi and some other naked strong curvature examples. Thus, U's objections have no mathematical or physical validity.

Key words : gravitational collapse — cosmic censorship — naked singularities — Tolman-Bondi models

1. Introduction

The Cosmic Censorship Conjecture (CCC) was first proposed by Penrose (1969) and it states that a singularity forming in gravitational collapse must always be covered by an event horizon. This conjecture has not yet been proved. By now there are many versions of the conjecture. It is not known which of the versions, if any, will finally get proved. Thus the possibility remains open that naked singularities do result as the end states of gravitational collapse, in the general theory of relativity. If a naked singularity forms, then particle and photon trajectories starting at the singularity can travel all the way up to infinity, and hence "information" can be conveyed from the singularity to a far-away observer.

For the purpose of our discussion, it is important to note that there are several versions of CCC. Of these, the two that are relevant here are given below:

CCP-P : Naked singularities cannot form in gravitational collapse of matter satisfying reasonable equation of state and under the reasonable conditions such as the positivity of energy density.

CCC-N : Naked singularities, even if they form in gravitational collapse, must be gravitationally weak. They cannot be strong curvature singularities.

CCC-P is the version due to Penrose. The version CCC-N is due to Newman (1986). Note the very significant difference between the two versions. CCC-P does not allow for the formation of naked singularities, but CCC-N does. Dwivedi and Joshi (DJ) and U agree that there are examples in general relativity which show that CCC-P is violated and naked singularities do occur. The debate is over CCC-N. DJ have shown that CCC-N is violated: strong naked singularities can form. U claims that CCC-N holds, as there are no viable counter-examples. According to U, the only kind of naked singularities that can form are weak.

The terms strong and weak have a technical definition. In both the strong case and the weak case, the curvature at the singularity is infinite; the difference is in the rate at which the curvature blows up, as one approaches the singularity along a geodesic. If k is the affine parameter along the geodesic, K^a the tangent vector along the geodesic, and R_{ab} the Ricci tensor, then the singularity is said to be weak if the scalar $\psi \equiv k^2 R_{ab} K^a K^b$ goes to zero as $k \to 0$ ($k = 0$ at the singularity). The singularity is strong if in the limit $k \to 0$, ψ is non-zero.

DJ have demonstrated in a series of papers, the occurrence of strong curvature naked singularities in gravitational collapse, while analysing different features for the same. The papers of interest to us here are Dwivedi & Joshi (1992), together with those of Newman (1986) and Ori & Piran (1990). (Other papers on naked singularity referenced by U are not relevant to the present discussion on naked singularity). In their paper mentioned above, DJ demonstrate the occurrence of strong curvature naked singularities in a class of Tolman-Bondi inhomogeneous dust collapse models.

Let us dwell on some of these technical terms. Tolman-Bondi inhomogeneous models represent gravitationally collapsing matter for which the density is spherically symmetric, but inhomogeneous. Dust means that the matter is taken to be pressureless.

In his paper (Unnikrishnan 1993), U has contested the above results of DJ and the numerical work of Ori & Piran (1990). According to U, the naked singularity that results at the end of the gravitational collapse in these models is not strong, but weak. He says that these authors arrived at a strong singularity because they assumed an infinite density right at the start of the collapse!

The purpose of this report is to show that the results of DJ are indeed correct, and that the claims of U are incorrect. In Section 2, we recall the work of DJ. In Section 3, we summarize the work of U. In Section 4, we compare DJ and U and show where U goes wrong. Concluding remarks are in Section 5.

We can state here, in summary, the mistake in the paper of U. There is a very large class of Tolman-Bondi solutions; these solutions are distinguished by a free function $F(r)$. U selects a special class of $F(r)$ functions which, he claims, lead to a weak naked singularity. He does not realize that his choice of $F(r)$ is special. DJ actually select a different class of $F(r)$, which leads to a strong naked singularity. U thinks that his choice of class of $F(r)$ is the most general possible, while it certainly is not. This is elaborated upon in the following sections. Apart from this, his claim on the weakness of the singularity is not tenable, because it is not supported by any mathematical derivation.

2. The work of DJ

DJ use the comoving coordinates (t, r, θ, ϕ) to describe the spherically symmetric collapse of an inhomogeneous dust cloud. The coordinate r has non-negative values and labels the spherical shells of dust and t is the proper time along the world lines of particles given by r = constant. The collapse of spherically symmetric inhomogeneous dust is described by the Tolman metric in comoving coordinates (i.e. $u^i = \delta^i_t$)

$$ds^2 = -dt^2 + \frac{R'^2}{1+f}dr^2 + R^2\,(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{1}$$

also

$$T^{ij} = \varepsilon\delta^i_t\,\delta^j_t\,, \quad \varepsilon = \varepsilon\,(r,\, t) = \frac{F'}{R^2 R'} \tag{2}$$

where T^{ij} is the stress-energy tensor, ε is the energy density and R is a function of both t and r, given by

$$\dot{R}^2 = \frac{F}{R} + f\,. \tag{3}$$

Here, the dot and prime denote partial derivatives with respect to t and r respectively. The quantities F and f are arbitrary functions of r. (What we denote as $F(r)$, U denotes as $M(r)$.) In further discussion here, we restrict to the class of solutions $f(r) \equiv 0$ as done by U. Also, DJ are concerned with the *dynamical gravitational* collapse of dust, in which case $\dot{R}(r, t) < 0$.

The Tolman-Bondi models admit a freedom of scaling in the following sense. One could arbitrarily relabel the dust shells given by $r =$ const. as $r \to g(r)$. Thus, at any constant time surface, say at $t = t_0$, $R(r, t_0)$ can be chosen to be an arbitrary function of r. This arbitrariness reflects essentially the freedom in the choice of units.

DJ make the following choice of scaling, at $t = 0$:

$$R(r,0) = r. \tag{4}$$

Obviously, one could have scaled on any surface $t = t_0$. Of course, once you have done the scaling on one $t = t_0$ surface, you are not allowed to scale on any other surface.

With the scaling as in (4), equation. (3) (set $f = 0$), can be easily integrated to get

$$R^{3/2}(r,t) = r^{3/2} - \frac{3}{2}\sqrt{F(r)/t}. \tag{5}$$

(Recall that we require $\dot{R} < 0$). Also, it is easily shown from (2) that

$$\varepsilon(r, t) = \frac{4/3}{\left(t - \frac{2}{3}\frac{G(r)}{H(r)}\right)\left(t - \frac{2}{3}\frac{G'(r)}{H'(r)}\right)}, \tag{6}$$

where $G(r) = r^{3/2}$, $G'(\tau) = (3/2)r^{1/2}$, and $H(r) = \sqrt{F(r)}$. DJ now make the following assumption: they write $F(r) \equiv r\lambda(r)$ and assume that $\lambda(0) \neq 0$. This means that near the origin, $F(r)$ goes as r, and it goes to zero at $r = 0$. From equation (6) it is now seen that the density at the centre, $r = 0$, behaves with time as

$$\varepsilon(0,t) = \frac{4}{3t^2} \tag{7}$$

Thus the central density becomes singular at $t = 0$. It is shown in the Appendix A that this is a naked singularity, and that it is a strong curvature naked singularity.

It is also apparent from equation (7) that the central density is finite at any $t < 0$. Thus the singularity is interpreted as having arisen from the evolution of dust collapse which had a finite density distribution in the past on an initial epoch.

3. The work of U

We summarize in this section the work reported by U in his paper. Our response to U's results is not given in this section, but in Section 4. U is concerned with dust models with $f = 0$ with the same basic equations as given in Section 2. In his abstract he states that strong curvature naked singularity classes are unphysical. He also claims in his paper that in dust collapse the resulting naked singularity is always *weak*, and not *strong.* According to U, DJ found a strong naked singularity because they started with an *infinite density* and that the collapse in DJ's case is not *dynamical*.

The paper then proceeds to demonstrate the occurrence of a singularity. According to U, DJ do not start from a finite density distribution which is Taylor expandable. He says that they find a singularity at $t = 0$, as in equation (7), and claim that this singularity arises from a finite density distribution in the past. U says that DJ will not get a strong naked singularity if they actually start their collapse from a finite, Taylor expandable density in the past. U would explicitly like to start from a finite, Taylor expandable, density distribution at $t = 0$, evolve the collapse forward, and check what kind of singularity results.

So he starts by writing down the density distribution at $t = 0$, and by assuming that it can be expanded in a Taylor series. U's principal claim is the following: The end state of the collapse is a *weak* naked singularity, and not a strong one. (Recall that according to DJ a *strong* naked singularity results. So the results of DJ and U are in direct contradiction. Only one of them can be right!).

U then proceeds as follows. He has the same set of basic equations as DJ. That is, U also has the equations (1-6), as in Section 2 above. However, now note the following crucial difference between DJ's and U's meaning of equation (4). Equation (4) is the scaling equation

$$R(r,0) = r.$$

Both DJ and U do this scaling at $t = 0$ (although U does not state this explicitly, it is implicit in the derivation of equation (4) of his paper). However, at $t = 0$, DJ have a singularity, for whom the initial surface lies in the past. But for U, $t = 0$ is the starting epoch of collapse, and the singularity forms at $t > 0$. Thus, DJ and U do the scaling $R = r$ at *different* epochs: for the former, this scaling is done at the singularity epoch, for the latter, at a non-singular epoch.

After choosing the scaling as above, U substitutes the Taylor expansion of the density at $t = 0$ epoch in the density evolution equation (6). He gets

$$\varepsilon(r,t) = \frac{4/3}{\left[t-\frac{2}{3}\sqrt{\frac{3}{\rho_0}}\left(1-\frac{3}{20}\frac{\rho_0''r^2}{\rho_0}\right)\right]\left[t-\frac{2}{3}\sqrt{\frac{3}{\rho_0}}\left(1-\frac{7}{20}\frac{\rho_0''r^2}{\rho_0}\right)\right]}$$
(8)

(Trivial numerical errors in U's paper have been corrected). From equation (8) it is clear that the central density is

$$\varepsilon(0,t) = \frac{4/3}{\left(t - \frac{2}{3}\sqrt{\frac{3}{\rho_0}}\right)^2}. \qquad (9)$$

Thus the singularity forms at the centre after a time $t_s = \frac{2}{3}\sqrt{\frac{3}{\rho_0}}$, which is determined by the initial central density ρ_0. It also follows from Equation (8) that at $t = t_s$, the radial dependence of the density is

$$\varepsilon(r,t_s) \; \alpha \; \frac{1}{r^4}. \qquad (10)$$

From here, and by using the equation (5) for the evolution of $R(t)$, it can be shown that

$$\varepsilon(r,t_s) \; \alpha \; \frac{1}{R^{12/7}}. \qquad (11)$$

U then makes the following points :

(1) The central density, as in equation (9), does not vary as $1/t^2$.

(2) The singularity is naked and weak. He claims that this follows automatically from results of Newman (1986).

He then asks the question : What is the physical interpretation of the work of DJ and Ori & Piran (1990)? How come DJ found a strong naked singularity whereas U claims only a weak naked singularity after starting with a finite central density and a Taylor expandable density distribution? U concludes that the singularities of DJ do not arise from *dynamical collapse* of physical density functions (since, U claims, he started with the most general density function), but arise because the starting density functions were intrinsically singular. In other words, DJ found a singularity, because they started with an infinite density. U adds that in the model of DJ, the function dR/dr is infinite at $r = 0$, for all epochs prior to the formation of singularity.

In the next section we show the mistake made by U, which led him to such objections. We show that these objections are completely wrong.

4. The refutation of the claims of U

The comparison of DJ and U's work becomes easier if we rewrite the DJ results of Section 2 in the following manner. We want to demonstrate that DJ also start from a finite, Taylor expandable density distribution. Recall that DJ have a singularity at $t = 0$. Let us now define a new time axis, giving a constant shift : $t \rightarrow t' = t + \alpha$, where α is a positive real number. In this new coordinate system, the singularity forms at a time α, and the collapse can be assumed to start at any time $t' < \alpha$. Let us assume that collapse starts at the time $t' = 0$.

In the t' coordinate, equation (5) becomes

$$R^{3/2}(r,t') = r^{3/2} - \frac{3}{2}\sqrt{F(r)}\,(t' - \alpha). \tag{12}$$

and equation (6) becomes

$$\varepsilon(r,t') = \frac{4/3}{\left(t' - \alpha - \frac{2}{3}\frac{G(r)}{H(r)}\right)\left(t' - \alpha - \frac{2}{3}\frac{G'(r)}{H'(r)}\right)}. \tag{13}$$

Henceforth we drop the prime on t. It is very obvious that all we have done is a time-shift, so that now in DJ's model also, the collapse begins at $t = 0$, and singularity forms at $t = \alpha$. Thus we can trivially rewrite (12) as

$$R^{3/2}(r,t) = r^{3/2} - \frac{3}{2}\sqrt{F(r)}\,(t - \alpha), \tag{14}$$

and (13) as

$$\varepsilon(r,t) = \frac{4/3}{\left(t - \alpha - \frac{2}{3}\frac{G(r)}{H(r)}\right)\left(t - \alpha - \frac{2}{3}\frac{G'(r)}{H'(r)}\right)}. \tag{15}$$

Now, let us note the following. At the time at which collapse begins, i.e. at $t = 0$, equation (14) implies

$$R^{3/2}(r,0) = r^{3/2} + \frac{3\alpha}{2}\sqrt{F(r)} \tag{16}$$

and at the singular epoch, $t = \alpha$, equation (14) gives $R = r$. $R = r$ is of course the scaling that DJ make at the singular epoch. On the other hand, at the starting epoch, DJ choose the scaling given by equation (16). In other words, at the epoch when $R = r$, the central density is finite in U's case, but infinite in DJ's case. We recall that in DJ, the class considered is $F(r) = r\lambda(r)$, $\mu\nu(0) \neq 0$.

At the starting epoch t = 0, the density profile in DJ's case is given, from (15), by

$$\varepsilon(r,0)=\frac{4/3}{\left(\alpha+\frac{2}{3}\frac{G(r)}{H(r)}\right)\left(\alpha+\frac{2}{3}\frac{G'(r)}{H'(r)}\right)}\,. \tag{17}$$

Since $H(r)$ goes as $r^{1/2}$ near $r = 0$ in DJ's model, it follows that the central density $\varepsilon(0,0) \equiv \rho_0$ is finite at the starting epoch. Also, $\rho_0 = 4/3\alpha^2$, i.e. $\alpha=2/\sqrt{3\rho_0}$. Also, since H(r) goes as $r^{1/2}$ near $r = 0$, we can write $\varepsilon(r,0)$ near $r = 0$, as

$$\varepsilon(r,0) = \frac{1}{(\alpha+Ar)(\alpha+Br)}\,. \tag{18}$$

It follows that the density function is Taylor expandable to all orders, with all the derivatives finite at the centre.

Thus by writing DJ's model in an explicit manner, we have shown that they start with a finite density which is Taylor expandable to all orders. And of course, they obtain a strong naked singularity, as we saw before.

And yet, U claims that if you start with a finite, Taylor expandable density, you can only get a weak naked singularity! What is going on? Here is the answer. Recall that in DJ's model, the free function $F(r)$ is assumed to go as r near the origin $r = 0$. However, in the model of U, $F(r)$ does not go as r, it goes as r^3 near $r = 0$. It is the behaviour of $F(r)$ near $r = 0$ which determines the strength of the singularity. If $F(r)$ goes as r, the singularity is strong. If $F(r)$ goes as r^3, the singularity is weak. Note that in both U and DJ, the density at the centre $\varepsilon \sim (t-\alpha)^{-2}$.

To see that in U's model $F(r)$ goes as r^3 near the origin, we go back to his Taylor expansion for the density distribution (equation (6) in his paper). Use this expansion in the energy equation, equation (2), at $t = 0$:

$$\varepsilon(r,0) = \frac{F'}{R^2R'}\,. \tag{19}$$

To be able to evaluate this integral, we need to know the scaling. If we use U's scaling, $R = r$, we immediately find that to leading order, $F(r)$ goes as r^3 near the centre. But if a different scaling had been chosen by U, he would find $F(r)$ going, not as r^3, but as some other power. DJ choose $F(r)$ going as r, and they also choose the scaling in equation (16), different from $R = r$. Note that the scaling $R = r$ will always give $F(r)$ going as r^3 near $r = 0$, if done at a finite density epoch. You have to scale differently from $R = r$, at the regular epoch if you want $F(r)$ to go as other than r^3.

It is important to realize that different forms of the function $F(r)$ characterize different Tolman-Bondi spacetimes. *After the function F(r) has been selected to have a particular form, the requirement that the energy density be finite and non-vanishing on the scaling surface constrains the allowed choices of the scaling.* This is clear from

(19). If we take $\varepsilon(r,0)$ to be ρ_0 as the leading order in a Taylor expansion, it follows that $F(r)\ \alpha\ R^3$ (r,0). So, if $F\ \alpha\ r$, then the scaling has to be as $r^{1/3}$. On the other hand, if $F\ \alpha\ r^3$, the scaling has to be as r. One has to first choose $F(r)$, and then the scaling; not the other way round.

Thus, the mistake in U's paper is now clear. U, in his paper, has a special class of Tolman-Bondi models, those with $F(r) \sim r^3$ near $r^3 = 0$. This class is completely different from the class selected by DJ, for whom $F(r) \sim r$ near $r = 0$, and which class leads to a strong naked singularity. The objections of U are thus simply not valid, as he is not addressing the same class of solutions as DJ. By studying his class of Tolman-Bondi solutions he cannot draw any conclusions about a different class of Tolman-Bondi solutions.

To make things clearer, let us look at the following example which is a special case of DJ's work. Take the case

$$F(r) = r. \tag{20}$$

It then follows from equation (16) that the scaling is

$$R^{3/2} = r^{3/2} + \frac{3\alpha}{2}\, r^{1/2}, \tag{21}$$

and from equation (17) that

$$\varepsilon(r,0) \equiv \rho(r) = \frac{4/3}{\left(\alpha + \frac{2}{3}\, r\right)\ (\alpha + 2r)}. \tag{22}$$

Thus $\rho_0 = 4/3\alpha^2$, $\rho_0' = -32/9\alpha^3$, etc. Note that all the derivatives are finite at the centre. In this model the density at the epoch $t = 0$ is Taylor expandable (see also Appendix B), so U's primary requirement is satisfied. Yet this is an example of a *strong* naked singularity. It is interesting to note the following point: equation (2) implies that near $r = R = 0$, $F\ \alpha\ R^3$, and it again follows that the dependence of $F(r)$ on r will be determined by the scaling at the initial epoch.

We now return to U's next objection. From equation (16) it follows that in DJ's model, for all $t < \alpha$, the derivative dR/dr is infinite at $r = 0$. Thus, from equation (1), it follows that one of the metric components is singular at $r = 0$, at all epochs prior to the formation of the singularity. However, this is not a problem at all, because although the metric component blows up, the density and curvature invariants are all finite. One is reminded of the Schwarzschild metric, where at $r = 2M$, one of the metric components is infinite, but all the same, this is in no way unphysical, since the curvature is finite. The coordinate singularity at $r = 2M$ in the Schwarzschild metric can be removed by going to another coordinate system, say the Kruskal coordinates. Similarly the coordinate singularity at $r = 0$ in DJ's models can be removed by going to another coordinate system. This change of coordinates will of course not affect the invariant results that the singularity is naked and strong; it will only make the analysis more difficult.

As a matter of fact, it must be noted that if the gradient R' is calculated, not at a constant time epoch at the centre, but along a null geodesic (light ray) coming out from the regular centre at $r = 0$ in the limit of approaching the origin, it is actually finite at $r = 0$. This is easily seen from equation (A9) and in any case a more physical thing to do.

These arguments invalidate all of U's objections. We would like to state at this stage that U does not have a single valid mathematical/technical/physical objection to the work of DJ, and Ori and Piran.

5. Critical comments and conclusion

(1) *The principal claim made by U, that the singularity in his model is weak, is completely baseless and leads nowhere.* This is because, as explained above, in order to verify the strength of the naked singularity, one must calculate the growth of curvatures along the families of non-spacelike trajectories coming out of the naked singularity. U has not carried out any such analysis for his model and thus his claim is not supported by any mathematical derivation whatsoever! To illustrate this point, the correct method for calculating the strength of a naked singularity is given in Appendix A. Without such an analysis for any given model, no claims on the strength of the naked singularity can be made.

(2) He also appears to claim that his model is the same as that of Newman (1986), and that is why the naked singularity is weak just as in Newman's model. Even this claim is simply not correct. This is because, Newman's model imposes the condition of evenness on the functions involved, such as $\rho(r)$. U's model requires no such constraints and in fact, he allows the first derivatives as well as other higher order odd derivative terms to be non-zero in his model. Thus, this is simply not Newman's model. So any conclusions from Newman (1986), including the weakness of the naked singularity, cannot be blindly applied in U's model. Further, it should also be clearly understood that Newman has shown the weakness of the naked singularity only along the *radial null geodesics* coming out of the singularity. Whether the singularity is weak also along the non-radial and timelike geodesics, is still an open question, as stated very clearly in his paper by Newman himself. Infact, Newman (1986) has made no general claims on the weakness of the naked singularity as attributed to him by U! Thus, the argument by U, that the singularity in his model is weak, as shown by the analysis of Newman (1986), is completely false.

(3) Apart from claiming on the strength of the naked singularity (dealt with above), U also offers a 'physical interpretation' of the existing examples of strong curvature naked singularities. According to this interpretation of U, the existing examples are unphysical because they have intrinsically singular and infinite density at the origin right at the starting epoch! We point out below the basic mathematical error

committed by U in arriving at such an absurd interpretation. (Ofcourse, we have already shown above explicitly, that the starting central density is finite and well behaved in DJ's model.) In his starting equation (6) of Taylor expandable density function, it is more than obvious and a basic fact of elementary calculus that the quantity ρ_0 is finite. Subsequent derivations leading up to his equation (8) are based on this assumption. And yet, towards the end of the paper, in order to arrive at a 'correct interpretation' he goes ahead and sets ρ_0 equal to infinity in his equation (8)! What can be more ridiculous than this?

(4) In his equations (6) to (9), U demonstrates the radial dependence of energy density at the time of formation of the singularity. This calculation has no connection or any linkage to his main claim on the strength of the naked singularity. So one would ask what is the purpose of these equations and what do they show, if any thing?

(5) Throughout his paper, U does not clarify which version of CCC he has in mind. One has to read between the lines and conclude that U adheres to the version CCC-N, i.e. he accepts the occurrence of weak naked singularities, but not strong ones.

(6) Now that it is clear that strong curvature naked singularities do form in Tolman-Bondi inhomogeneous models with dust, we wish to comment on broader issues. The philosophy behind the counter examples to cosmic censorship is not that one should outright accept the occurrence of naked singularities in nature. Rather, these counter examples will help in formulating and arriving at a suitably provable version of the cosmic censorship conjecture, if there is one. This point of view has been emphasized by one of the present authors again and again in several papers.

Also considering the structure of the equations here, it may well be that for some region of the the parameter space, the end-state of gravitational collapse is a black hole, and for another region of the parameter space, a naked singularity. This needs to be investigated further.

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Appendix A

1. The existence of naked singularity

In this Appendix, we illustrate the development of a naked singularity in gravitational collapse of dust from regular initial data. This is done by means of a specific example, which is a special case of the general class of models treated in DJ.

The inhomogeneous, spherically symmetric collapse of dust is described by the Tolman—Bondi models, which are characterised by two free functions, $F(r)$ and $f(r)$ of the radial coordinate r. These functions describe the inhomogeneous distribution of matter in space, which would then evolve in time (through the Einstein field equations).

For the sake of clarity, we confine ourselves presently to the case $f(r) = 0$. The models treated in DJ work with a general form of $f(r)$ and $F(r)$ has the form

$$F(r) = r\,\lambda(r) \tag{A1}$$

where $\lambda(r)$ is a smooth function with $\lambda(0) \neq 0$. This amounts to the condition that the mass function $F(r)$ has a linear behaviour initially for small values of r. It is clear that the example considered here forms a special case of the general class of models considered in DJ.

The coordinates t and r are comoving coordinates, where r with nonnegative real values labels the spherical dust shells; and t is the proper time along the lines r = const., i.e. the world lines of dust. The quantity $R = R(r, t)$ is defined by setting $A = 4\pi R^2$ where A is the area of the two-sphere t = const., r = const. It is allowed to make an arbitrary relabeling of the spherical dust shells by $r \to g(r)$. Using this scaling freedom, we fix without loss of generality a labeling by requiring that on the surface $t = 0$, the coordinate r coincides with the radius R:

$$R(r, 0) = r\,. \tag{A2}$$

(As shown in the text, a different choice of scaling or a different surface does not alter the qualitative nature of the conclusions.)

The relevant collapse equations from DJ are then simplified as below. Equation (3) integrates to,

$$t = \frac{2}{3}\,\frac{r^{3/2}}{3\sqrt{F}}\left[1 - \left(\frac{R}{r}\right)^{3/2}\right] \tag{A3}$$

The partial derivatives R' and $\dot{R}'$ are then written as below :

$$R' = \eta P - \left[\frac{1-\eta}{\sqrt{\lambda}} + \eta\frac{t}{r}\right]\dot{R} \tag{A4}$$

$$\dot{R}' = \frac{\lambda}{2rP^2}\left[\frac{1-\eta}{\sqrt{\lambda}} + \eta\frac{t}{r}\right] \tag{A5}$$

where we have put

$$R(r,t) = rP(r,t)\,,\ \eta=\eta(r) = \frac{rF'}{F} \tag{A6}$$

$$F(r) = r\lambda(r)\ .$$

As we are concerned with the dynamical gravitational collapse, we take $\dot{R}(r,t) < 0$. The space-time singularity occurs when the collapsing shells reach zero radius, and the corresponding time is given from (A3) as,

$$t_0(r) = \frac{r^{3/2}}{\sqrt{F}}\,. \tag{A7}$$

Our main purpose is to examine whether this singularity could be naked. For that we examine if the light rays and material particle trajectories (null and timelike geodesics) could come out of the singularity at $t = 0$, $r = 0$, which forms whenever the initial behaviour of $F(r)$ is smaller than r^3. The equations of outgoing radial null geodesics in the space-time, with k as affine parameter, are given by,

$$\frac{dK^t}{dk} + \dot{R}'\,K^r\,K^t = 0 \tag{A8}$$

$$\frac{d^t}{d^k} = \frac{K^t}{K^r} = R' \tag{A9}$$

where $K^t = d^t/d^k$ and $K^r = dr/dk$ are tangents to the outgoing null geodesics. If these null geodesics terminate in the past with a definite tangent at the singularity (in which case the singularity would be naked), then using (A9) and L' Hospital rule we get

$$X_0 = \lim_{t\to 0,\ r\to 0} \frac{t}{r} = \lim_{t\to 0,\ r\to 0} \frac{dt}{dr} = \lim_{t=0,\ r=0} R' \tag{A10}$$

where $X = t/r$ is a new variable. The function $P(r,t) = P(X,r)$ is then given using (A3) and (A7) by

$$X - \frac{2}{3\sqrt{\lambda}} = \frac{2P^{3/2}}{3\sqrt{\lambda}}\,. \tag{A11}$$

Writing $Q = Q(X) = P(X,0)$, which is given by the above equations (A10) can be simplified to the condition

$$V(X_0) = 0 \tag{A12}$$

where

$$V(X) \equiv Q + X\sqrt{\frac{\lambda_0}{Q}} - X. \tag{A13}$$

Here $\lambda_0 = \lambda(0) \neq 0$. The null geodesic equations could now be written in the form $r=r(X)$,

$$\frac{dX}{dr} = \frac{R'-X}{r}. \qquad \text{(A14)}$$

This has been analyzed in DJ to show that an infinite family of null geodesics will come out of the singularity $r = 0$, $t = 0$ (making it naked) if and only if (A12) admits at least one real positive root. Such a condition will be realized for a non-zero measure subspace of the possible values of physical parameters involved here. When this is not realized, the evolution will be a black hole. Such a singularity could be either locally or globally naked depending on the global features of the function $\lambda(r)$. For details we refer to DJ.

2. Curvature strength

We now calculate the curvature strength of this naked singularity. This determines the physical seriousness of the naked singularity. For example, if no curvature invariants diverged in the limit of approach to the singularity, it would not be considered to have any physical implications but would be regarded as a mere mathematical pathology. In order to be physically meaningful, the naked singularity must be a curvature singularity. Two main distinctions are as below :

Strong curvature singularity : The energy density $\psi = R_{ab}K^aK^b$ diverges as $1/k^2$ or faster in the limit of approach to the naked singularity along the trajectories coming out from the same; where $k = 0$ at the singularity.

Weak naked singularity : This is given by

$$R_{ab}K^aK^b \; \alpha \; 1/k$$

and in this case,

$$\lim_{k\to 0} k^2 R_{ab}K^aK^b = 0.$$

We now show that the naked singularity in the class of dust collapse considered by DJ is a strong curvature singularity. The quantity $k^2 \psi$ is given by,

$$k^2\psi = k^2 R_{ab}K^aK^b = \frac{k^2 F'(K^t)^2}{R^2 R'} = \frac{k^2 F'(K^t)^2}{r^2 P^2 R'}$$

Note that in the limit $k \to 0$ we have $t \to 0$ and $r \to 0$ and the quantities F' and R' tend to finite limits λ_0 and X_0 respectively. Thus, we need to examine the limit,

$$\lim_{k\to 0} k^2 \psi = \frac{\lambda_0}{X_0} \lim_{k\to 0} \left[\frac{kK^t}{R}\right]^2 \qquad \text{(A15)}$$

Note that the tangent vector component K^t could be blowing up in the limit of approach to the naked singularity. Then, the use of (A4, A5) and (A8, A9) and some algebra gives using the L' Hospital rule,

$$\lim_{k\to 0} \frac{k}{R.1/K^t} = \lim \frac{R'}{R\dot{R}'+\dot{R}R'+R'}$$

Again a simplification using (A4, A5) and noting that lim $P = P(X_0, 0) = Q_0$ is finite (which follows from (A11) gives,

$$\lim_{k\to 0} k^2\psi = \frac{\lambda_0 X_0}{(h_0+2)^2 Q_0^2} \neq 0$$

where

$$h_0+2 = \frac{\lambda_0 X_0}{2Q_0^2} + 1 .$$

It follows that this is a strong curvature naked singularity.

Appendix B

On the regularity of density function in DJ

Here we discuss the regularity and smoothness properties of the density $\varepsilon(r, t)$ in the models of DJ with their original scaling $R = r$ on the surface $t = 0$, and with $F(r) = r\lambda(r)$, $\lambda_0 = \lambda(0) \neq 0$. Calculating R and R' from (5) and substituting in (2) gives,

$$\varepsilon(t, r) = \frac{F'}{r^2 - \frac{t}{2}\left(3\sqrt{rF} + \frac{r^{3/2}F'}{\sqrt{F}}\right) + \frac{3}{4} F' t^2} . \tag{B1}$$

Since $F'(0) = \lambda_0$ is finite and non-zero, it follows that a singularity forms at $r = 0$ at a time $t = 0$ when $\lambda(r)$ goes slower than r^2 near the center $r = 0$. The central density in this case is given by,

$$\varepsilon(t, 0) = \frac{4/3}{t^2} \tag{B2}$$

which is seen to be finite at all earlier epochs, blowing up at $t = 0$. The gravitational collapse in this case starts from any of the earlier epochs $t_0 < 0$ with a finite density distribution and evolves into a naked singularity at $t = 0$.

To illustrate the behaviour of $\varepsilon(r)$ as a function of r on any such earlier epoch (say $t = -1$) from which the collapse starts, we choose a special case $F(r) = r$ (which is consistent with the general models considered by DJ). Then (B1) gives,

$$\varepsilon(-1, r) = \frac{1}{r^2 + 2r + 3/4} . \tag{B3}$$

The derivatives of $\varepsilon(r)$ with respect to r are given by,

$$\varepsilon'(-1,r) = \frac{-2(r+1)}{(r^2+2r+3/4)^2}\,;\quad \varepsilon''(-1,r) = \frac{8(r+1)^2}{(r^2+2r+3/4)^3} - \frac{2}{(r^2+2r+3/4)^2}$$

etc. It is seen that these are all finite and continuous at $r = 0$. Thus $\varepsilon(r)$ is a regular, Taylor expandable function on the initial surface from which the collapse starts.

A clarification needs to be made on a reference to non-analytic property of the density function, given in the paper of DJ. They make the following statement: 'It is seen that the density function is non-analytic, though non-singular, at $r = 0$ at these earlier epochs'. This is to be understood in the following sense. Even though the density is finite and Taylor expandable in r at any constant time epoch before the singularity, the derivatives ε', ε'' etc. blow up in the limit of approach to the singularity. To see this, consider again the special example discussed above, in which case we have,

$$\varepsilon(t,r) = \frac{1}{r^2 - 2rt + \frac{3}{4}t^2}$$

$$\varepsilon'(t,r) = \frac{-2(r-t)}{\left(r^2 - 2rt + \frac{3}{4}t^2\right)^2}$$

etc.

It is seen that at $r = 0$, ε' blows up as t^{-3} as $t \to 0$, and so will other higher derivatives. This is the onset of non-analytic property just before the occurrence of the naked singularity at $t = 0$.

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