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Turbulent magnetic field averages for the Zeeman effect

H. Frisch[1], M. Sampoorna[1,2], and K. N. Nagendra[1,2]

[1] Laboratoire Cassiopée (CNRS, UMR 6202), Observatoire de la Côte d'Azur, BP 4229, 06304 Nice Cedex 4, France, e-mail: `frisch@oca.eu`
[2] Indian Institute of Astrophysics, Bangalore 560 034, India

Abstract. Stokes parameters measured in the Solar atmosphere are in general time or space averages over a magnetic field probability distribution function. Here we show how to write the Zeeman propagation matrix in a reference frame defined with respect to the direction of a mean magnetic field and how to average over a random magnetic field distribution. We concentrate on the case of a normal Zeeman triplet but indicate how to treat general Zeeman patterns. Numerical results are presented for Gaussian distributions having cylindrical symmetry about a mean field. Different models of probability distribution functions (PDF), are compared.

Key words. line: formation – polarization – magnetic fields – turbulence – radiative transfer – Sun: atmosphere

Observations and numerical simulations of magneto-convection show that the magnetic field in the solar atmosphere is highly intermittent down to scales which are much below the resolving power of modern telescopes. Thus observations always provide Stokes parameters averaged on some distribution $P(\boldsymbol{H})\, d\boldsymbol{H}$ of the vector magnetic field. We discuss here the calculation of mean Stokes parameters for the micro and macro-turbulent limits. In the micro-turbulent limit, i.e when the correlation scale of the magnetic field is much smaller than typical photon mean free path, the Zeeman propagation matrix can be averaged over the magnetic field distribution. This question was first addressed by Dolginov & Pavlov (1972) and Domke & Pavlov (1979). The polarized radiative transfer equation can then be solved with this mean propagation matrix. When the scale of variation of the magnetic field is large compared to typical photon mean free path – situation known as the macro-turbulent limit – averaging should be performed on the solution of the transfer equation. MISMA (Micro structured Magnetic Atmospheres; Sánchez Almeida et al. 1996) and multi-components models (Stenflo 1994) correspond to micro and macro-turbulent situations, respectively. For magnetic fields with scales of variation of the order of the line formation depth, macro-turbulent solutions can serve as building blocks for the calculation of mean Stokes parameters (Landi Degl'Innocenti 1994; Frisch et al. 2006).

Transfer equations for polarized radiation are usually written and solved in a reference frame defined with respect to the atmosphere

Send offprint requests to: H. Frisch

(orthogonal reference frame with the Z-axis along the normal to the atmosphere; see Fig. 1). The vector magnetic field distribution is usually defined with respect to a mean magnetic field $\boldsymbol{H}_0$ (or to a specified direction), i.e. in a reference frame where the Z-axis is aligned with the direction of a mean field, henceforth referred to as MRF (see Fig. 2). From a theoretical and numerical point of view, averaging is easier in this magnetic reference frame. Rotations between different reference frames are quite common in the analysis of polarimetric data (see e.g. Casini 2002). We show here how to obtain the Zeeman propagation matrix in the MRF. We work out in detail the case of a normal Zeeman triplet and indicate how to handle the general case. We then indicate how to perform averages over a vector magnetic field distribution.

1. Zeeman propagation matrix in the LOS reference frame

For simplicity we consider a normal Zeeman triplet and assume that the line of sight (LOS) is along the normal to the atmosphere. The Zeeman propagation matrix is of the form

$$\hat{\mathbf{\Phi}} = \begin{bmatrix} \varphi_{\rm I} & \varphi_{\rm Q} & \varphi_{\rm U} & \varphi_{\rm V} \\ \varphi_{\rm Q} & \varphi_{\rm I} & \chi_{\rm V} & -\chi_{\rm U} \\ \varphi_{\rm U} & -\chi_{\rm V} & \varphi_{\rm I} & \chi_{\rm Q} \\ \varphi_{\rm V} & \chi_{\rm U} & -\chi_{\rm Q} & \varphi_{\rm I} \end{bmatrix}. \quad (1)$$

The absorption coefficients, $\varphi_{\rm I,Q,U,V}$ can be written as (see e.g. Landi Degl'Innocenti 1976; Rees 1987; Jefferies et al. 1989; Stenflo 1994)

$$\begin{aligned} \varphi_{\rm I} &= \frac{1}{2}\varphi_0 \sin^2\theta + \frac{1}{4}(\varphi_{+1} + \varphi_{-1})(1+\cos^2\theta), \\ \varphi_{\rm Q} &= \frac{1}{2}[\varphi_0 - \frac{1}{2}(\varphi_{+1} + \varphi_{-1})] \sin^2\theta \cos 2\phi, \\ \varphi_{\rm U} &= \frac{1}{2}[\varphi_0 - \frac{1}{2}(\varphi_{+1} + \varphi_{-1})] \sin^2\theta \sin 2\phi, \\ \varphi_{\rm V} &= \frac{1}{2}(\varphi_{+1} - \varphi_{-1}) \cos\theta, \end{aligned} \quad (2)$$

where the φ_q ($q = 0, \pm 1$) are Voigt functions shifted by $q\Delta\nu_{\rm H}$ with $\Delta\nu_{\rm H}$ the Zeeman displacement by magnetic field present at the point where the absorption coefficients are calculated and θ and ϕ the polar angles of the magnetic field vector $\boldsymbol{H}$ (see Fig. 1). The $\chi_{\rm Q,U,V}$ are defined similar to $\varphi_{\rm Q,U,V}$ with the Voigt function replaced by the Faraday–Voigt function.

Following Domke & Pavlov (1979), one observes that the absorption coefficients can be rewritten as

$$\begin{aligned} \varphi_{\rm I} &= A_0 - \frac{1}{3}A_2(3\cos^2\theta - 1), \\ \varphi_{\rm V} &= A_1 \cos\theta, \\ \varphi_{\rm Q} &= A_2 \sin^2\theta \cos 2\phi, \\ \varphi_{\rm U} &= A_2 \sin^2\theta \sin 2\phi. \end{aligned} \quad (3)$$

The A_i contain the frequency variations of the coefficients and depend only on the magnitude of the magnetic field. They may be written as

$$\begin{aligned} A_0 &= \frac{1}{3}(\varphi_{-1} + \varphi_0 + \varphi_{+1}), \\ A_1 &= \frac{1}{2}(\varphi_{-1} - \varphi_{+1}), \\ A_2 &= \frac{1}{4}(-\varphi_{-1} + 2\varphi_0 - \varphi_{+1}). \end{aligned} \quad (4)$$

The angular dependence of the coefficients can be expressed in terms of Legendre polynomials $P_l(\cos\theta)$ and spherical harmonics $Y_{l,m}(\theta,\phi)$ (Domke & Pavlov 1979; Frisch et al. 2005, henceforth Paper I), namely

$$\begin{aligned} \varphi_{\rm I} &= A_0 - \frac{2}{3}A_2 P_2(\cos\theta), \\ \varphi_{\rm V} &= A_1 P_1(\cos\theta), \\ \varphi_{\rm Q} &= A_2\Big(\frac{32\pi}{15}\Big)^{1/2} \Re[Y_{2,2}(\theta,\phi)], \\ \varphi_{\rm U} &= A_2\Big(\frac{32\pi}{15}\Big)^{1/2} \Im[Y_{2,2}(\theta,\phi)]. \end{aligned} \quad (5)$$

$\Re$ stands for real part and $\Im$ for imaginary part. The Legendre polynomials are special cases of Y_{lm} corresponding to $m = 0$. The advantage of the expressions given in Eq. (5) over the expressions in Eq. (2) is that the $Y_{l,m}$ obey simple transformation laws under a rotation of the reference frame.

2. Zeeman propagation matrix in the magnetic reference frame (MRF)

We now write the coefficients of the Zeeman absorption matrix in the MRF (see Fig. 2). The Euler angles (α, β, γ) of the rotation which

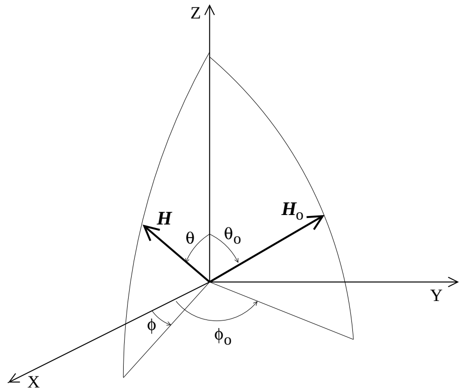

Fig. 1. Line of sight (LOS) reference frame (X, Y, Z). The LOS is taken as parallel to the Z-axis. The directions of the random vector magnetic field $\boldsymbol{H}$ and of the mean magnetic field $\boldsymbol{H}_0$ are defined by polar angles θ, ϕ and θ_0, ϕ_0.

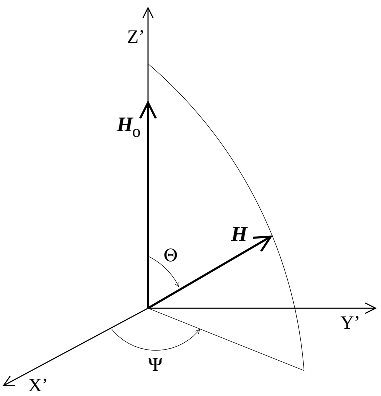

Fig. 2. Orthogonal right-handed magnetic reference frame X', Y', Z' (MRF). The vector magnetic field $\boldsymbol{H}$ is defined by the polar angles Θ and Ψ. The MRF is obtained by a rotation of the LOS reference frame defined by the Euler angles $\alpha = \phi_0, \beta = \theta_0, \gamma = 0$ (see Fig. 1).

brings the LOS reference frame to the MRF are $\alpha = \phi_0$, $\beta = \theta_0$, $\gamma = 0$, where θ_0 and ϕ_0 are defined in Fig. 1. In this rotation, the spherical harmonics transform according to (Varshalovich et al. 1988, p. 141) :

$$Y_{lm}(\theta,\phi) = \sum_{m'} Y_{lm'}(\Theta,\Psi) D^{(l)}_{m'm}(0,-\theta_0,-\phi_0), \quad (6)$$

where Θ and Ψ are the polar angles of the field $\boldsymbol{H}$ in the MRF (see Fig. 2). Combining Eq. (6) with Eq. (5), and using the explicit forms of the Y_{lm} and $D^{(l)}_{m'm}$ (see e.g. Brink & Satchler 1968; Varshalovich et al. 1988; Landi Degl'Innocenti & Landolfi 2004, henceforth LL04), we find

$$\begin{aligned}\varphi_{\rm I} = A_0 - \frac{2}{3}A_2\Big[&P_2(\cos\theta_0)P_2(\cos\Theta) \\ &- \frac{3}{4}\sin 2\theta_0 \sin 2\Theta \cos\Psi \\ &+ \frac{3}{4}\sin^2\theta_0 \sin^2\Theta \cos 2\Psi\Big], \quad (7)\end{aligned}$$

$$\begin{aligned}\varphi_{\rm V} = A_1\Big[&P_1(\cos\theta_0)P_1(\cos\Theta) \\ &- \sin\theta_0 \sin\Theta \cos\Psi\Big], \quad (8)\end{aligned}$$

$$\varphi_{\rm Q} = A_2\Big[\cos 2\phi_0\,[1] - \sin 2\phi_0\,[2]\Big], \quad (9)$$

$$\varphi_{\rm U} = A_2\Big[\sin 2\phi_0\,[1] + \cos 2\phi_0\,[2]\Big], \quad (10)$$

with

$$\begin{aligned}[1] = {}&\sin^2\theta_0 P_2(\cos\Theta) + \frac{1}{2}\sin 2\theta_0 \sin 2\Theta\cos\Psi \\ &+ \frac{1}{2}(1+\cos^2\theta_0)\sin^2\Theta\cos 2\Psi, \quad (11)\end{aligned}$$

$$\begin{aligned}[2] = {}&\sin\theta_0 \sin 2\Theta \sin\Psi \\ &+ \cos\theta_0 \sin^2\Theta \sin 2\Psi. \quad (12)\end{aligned}$$

The anomalous dispersion coefficients are given by Eqs. (8) to (10) where the Voigt functions in A_1 and A_2 are replaced by Faraday–Voigt functions.

3. From the LOS to the MRF – an alternative method

We present here an other method for obtaining the elements of $\hat{\boldsymbol{\Phi}}$ in the MRF associated to the mean magnetic field. The components of the magnetic field in the LOS frame are

$$\begin{aligned}H_X &= H\sin\theta\cos\phi, \\ H_Y &= H\sin\theta\sin\phi, \\ H_Z &= H\cos\theta, \quad (13)\end{aligned}$$

and the components in the MRF are

$$
\begin{aligned}
H_{X'} &= H \sin\Theta \cos\Psi, \\
H_{Y'} &= H \sin\Theta \sin\Psi, \\
H_{Z'} &= H \cos\Theta. \qquad (14)
\end{aligned}
$$

The magnitude H of the magnetic field is invariant under rotation. The vectors $\boldsymbol{H}_{\rm LOS} = (H_X, H_Y, H_Z)^T$ and $\boldsymbol{H}_{\rm MRF} = (H_{X'}, H_{Y'}, H_{Z'})^T$ are related by

$$\boldsymbol{H}_{\rm LOS} = [\hat{R}(0, -\theta_0, -\phi_0)]^{-1}\, \boldsymbol{H}_{\rm MRF}, \qquad (15)$$

with

$$\hat{R}(0, -\theta_0, -\phi_0) = \begin{bmatrix} \cos\theta_0\ \cos\phi_0 & \cos\theta_0\ \sin\phi_0 & -\sin\theta_0 \\ -\sin\phi_0 & \cos\phi_0 & 0 \\ \sin\theta_0\ \cos\phi_0 & \sin\theta_0\ \sin\phi_0 & \cos\theta_0 \end{bmatrix}, \qquad (16)$$

(see e.g. Jefferies et al. 1989; LL04). Since $\hat{R}$ is unitary, $[\hat{R}]^{-1} = [\hat{R}]^T$. From the equations given above, we can obtain the following simple relations connecting (θ, ϕ) with (Θ, Ψ) :

$$
\begin{aligned}
\sin\theta\cos\phi &= \cos\theta_0 \cos\phi_0 \sin\Theta\cos\Psi \\
&- \sin\phi_0 \sin\Theta \sin\Psi \\
&+ \sin\theta_0 \cos\phi_0 \cos\Theta, \qquad (17)
\end{aligned}
$$

$$
\begin{aligned}
\sin\theta\sin\phi &= \cos\theta_0 \sin\phi_0 \sin\Theta\ \cos\Psi \\
&+ \cos\phi_0 \sin\Theta \sin\Psi \\
&+ \sin\theta_0 \sin\phi_0 \cos\Theta, \qquad (18)
\end{aligned}
$$

$$\cos\theta = \cos\theta_0 \cos\Theta - \sin\theta_0 \sin\Theta \cos\Psi. \qquad (19)$$

These basic relations yield all the angle factors in Eqs. (2) or (3). Easy algebra leads to Eqs. (7) to (12).

4. General Zeeman pattern

The coefficients of the matrix $\hat{\boldsymbol{\Phi}}$ written in Eq. (2) can be deduced from the classical harmonic oscillator representation of the Zeeman effect. They hold for a two-level atom with unpolarized ground level when the angular momentum of the lower and upper level are $J_{\rm l} = 0$ and $J_{\rm u} = 1$. When these conditions are not satisfied, the elements of $\hat{\boldsymbol{\Phi}}$ have fairly complicated expressions (see e.g. LL04 pp. 289, 377). Their structure is however similar to the structure of Eq. (5) in the sense that the dependence on frequency and strength of the magnetic field can be separated from the dependence on the direction of the magnetic field. The angular dependence can be expressed in terms of functions (or tensors) that obey simple transformation laws under a rotation of the reference frame. A convenient way of doing this is by means of the spherical tensors for polarimetry $\mathcal{T}^K_Q(i, \boldsymbol{\Omega})$ (Landi Degl'Innocenti 1984; Bommier 1997; LL04, p. 208). The frequency variations can then be expressed in terms of generalized profiles $\Phi^{KK'}_Q(\nu)$ (Landi Degl'Innocenti et al. 1991; LL04, Appendix A.13). The generalized profiles are weighted linear combinations of Zeeman component profiles φ_q ($q = 0, \pm 1$). The spherical tensors are combinations of Wigner rotation matrices. For example, for a normal Zeeman triplet, the absorption coefficients can be written as

$$\varphi(i, \boldsymbol{\Omega}) = \sum_{K=0}^{K=2} \mathcal{T}^K_0(i, \boldsymbol{\Omega})\, \Phi^{0K}_0(\nu). \qquad (20)$$

The index i corresponds to the Stokes parameters ($i = 0, 1, 2, 3$ for I, Q, U, V) and $\boldsymbol{\Omega}$ is the direction of the magnetic field. For a given value of i, the $\mathcal{T}^K_Q(i, \boldsymbol{\Omega})$ transform according to Eq. (6), the indices K and Q playing the role of l and m, respectively. The A_i are simply related to the Φ^{0K}_0. Using Tables for $\mathcal{T}^K_Q$ given in Bommier (1977) or LL04 (Table 5.6, p.211), Eqs. (3) or (5) lead to :

$$A_0 = \Phi^{00}_0;\ A_1 = \sqrt{\frac{3}{2}}\Phi^{01}_0;\ A_2 = -\frac{3}{2\sqrt{2}}\Phi^{02}_0. \qquad (21)$$

5. Averaging over the vector magnetic field distribution

The calculation of mean values of $\hat{\boldsymbol{\Phi}}$ (or of the solution of the transfer equation) requires an integration over the probability distribution function (PDF) $P(\boldsymbol{H})\, d\boldsymbol{H}$ of the vector magnetic field. Figures 3 and 4 show examples of mean absorption coefficients. They have been obtained with Gaussian PDF, cylindrically symmetrical about a mean field $\boldsymbol{H}_0$.

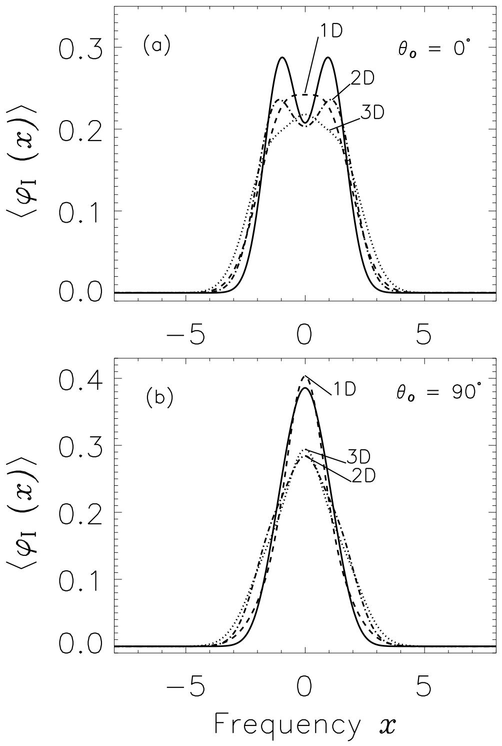

Fig. 3. Dependence of $\langle\varphi_{\rm I}\rangle$ on the magnetic field distribution. Frequency x in Doppler width units. Full line corresponds to mean field $\boldsymbol{H}_0$. The other curves are labeled by the angular distribution (see text). Panels (a) and (b) correspond to longitudinal and tranverse mean field.

Because of this assumption of cylindrical symmetry, all the terms depending on Ψ in Eqs. (7) to (12) average to zero. The mean absorption coefficients are thus given by

$$\begin{aligned}\langle\varphi_{\rm I}\rangle &= \bar{A}_0 - \frac{1}{3}\bar{A}_2(3\cos^2\theta_0 - 1),\\ \langle\varphi_{\rm V}\rangle &= \bar{A}_1\cos\theta_0,\\ \langle\varphi_{\rm Q}\rangle &= \bar{A}_2\sin^2\theta_0\cos 2\phi_0,\\ \langle\varphi_{\rm U}\rangle &= \langle\varphi_{\rm Q}\rangle\tan 2\phi_0, \end{aligned}\tag{22}$$

where

$$\begin{aligned}\bar{A}_0 &= \Big\langle A_0(\nu, H)\Big\rangle,\\ \bar{A}_1 &= \Big\langle A_1(\nu, H)\cos\Theta\Big\rangle,\\ \bar{A}_2 &= \Big\langle A_2(\nu, H)\frac{1}{2}(3\cos^2\Theta - 1)\Big\rangle. \end{aligned}\tag{23}$$

The notation $\langle\ \rangle$ represents an integration over Θ and the field strength H weighted by the vector magnetic field PDF. Similar expressions can be found in Paper I (Eqs. (16) and (17)) (see also Domke & Pavlov 1979).

The results shown in Figs. 3 and 4 correspond to three different choices for the angular distribution of the magnetic field fluctuations :

(i) Fluctuations along the direction of the mean field $\boldsymbol{H}_0$, referred to as *1D distribution*. In this case $\boldsymbol{H}$ is aligned with $\boldsymbol{H}_0$. Hence $\Theta = \Psi = 0$. The averaging is done only over the field strength H, with H varying between $-\infty$ and $+\infty$.

(ii) Fluctuations perpendicular to the direction of the mean field, referred to as *2D distribution*. In this case $\cos\Theta = H_0/H$. The integration is thus only over H, with H varying between H_0 and ∞.

(iii) Fluctuations with an isotropic distribution, referred to as *3D distribution*. Now the integration has to be carried out over Θ and H, with H varying between 0 and ∞.

Explicit expressions of the PDF corresponding to these three cases can be found in Paper I. The results shown in Figs. 3 and 4 have been calculated with $\Delta\nu_{H_0} = 1$ (in Doppler width units), random magnetic field fluctuations with strength $f = \sigma/H_0 = 1$, where σ is the width of the Gaussian distribution. The damping parameter of the absorption profile φ_0 is zero.

In a random magnetic field, in general both the strength and the direction of the field are fluctuating. Fluctuations of the former affect only the σ-components (producing a broadening and decrease in amplitude), while fluctuations of the latter affect all the components (producing essentially a decrease in amplitude). Thus 1D fluctuations do not affect the π-component of $\langle\varphi_{\rm Q}\rangle$ nor the π-component of $\langle\varphi_{\rm I}\rangle$ for $\theta_0 = 90°$. With 2D and 3D distributions, all the components decrease in amplitude. As can be seen in Fig. 3, the line center of $\langle\varphi_{\rm I}\rangle$ for $\theta_0 = 0°$ is very sensitive to the angular distribution of the magnetic field PDF. For 2D distribution, the Zeeman components are still separated because the projection of the magnetic field on the vertical stays equal to H_0

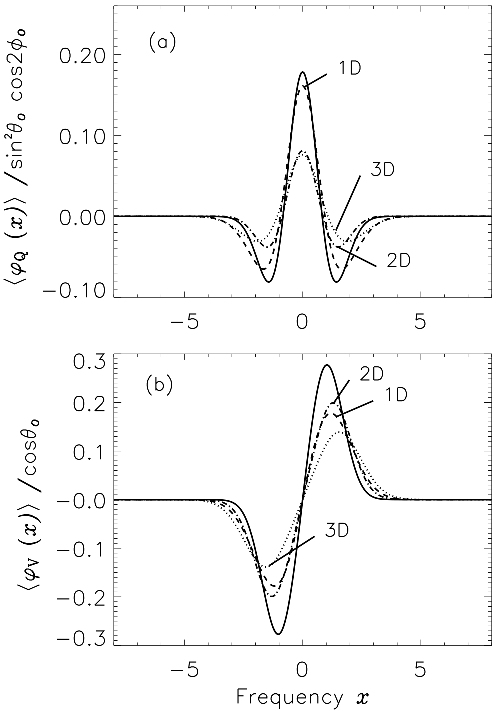

Fig. 4. Dependence of $\langle \varphi_{\rm Q} \rangle$ and $\langle \varphi_{\rm V} \rangle$ on the magnetic field distribution. Same model parameters as in Fig. 3. Panels (a) and (b) correspond to transverse and longitudinal cases.

while the transverse component is zero on the average with a Gaussian distribution.

When the magnetic field PDF does not possess cylindrical symmetry about the mean field, averaging the matrix $\hat{\mathbf{\Phi}}$ requires an integration over Θ, Ψ and H, weighted by the vector magnetic field PDF (see Eqs. (7) to (12)).

To calculate the mean Stokes parameters, one must still solve a polarized radiative transfer equation. In the micro-turbulent limit the elements of $\hat{\mathbf{\Phi}}$ should be replaced by their mean values. In the the macro-turbulent limit one should make use of the elements of $\hat{\mathbf{\Phi}}$ given in Eqs. (7) to (12) and then perform the averaging of emergent intensity over the vector magnetic field PDF.

In the case of a Gaussian PDF, the numerical averaging over H, Θ and Ψ can be performed with Gauss–Legendre quadratures. For Θ and Ψ, 7 to 9 quadrature points are sufficient. The H integration is more demanding and the number of grid points depends on the mean field and fluctuations strength (see Paper I and Frisch et al. 2006).

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