## Polarized Spectral Line Formation in Turbulent Magnetic Fields: The Zeeman and Hanle Effects

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**Abstract.** We present a short summary of work carried out on the effects of random magnetic fields with finite correlation length on spectral line polarization. The magnetic field is modeled by a step-wise Markovian random process defined by a probability distribution and a correlation length. Micro- and macro-turbulent limits are recovered when this length goes to zero and infinity, respectively. For the Zeeman effect, explicit expressions have been obtained for the mean emergent Stokes parameters and for their r.m.s. fluctuations. Examples illustrate the dependence of the mean Zeeman propagation matrix on the magnetic field distribution, and the dependence of mean Stokes parameters and their r.m.s. fluctuations on the correlation length of the magnetic field. For the Hanle effect, explicit expressions have also been obtained for the mean Stokes parameters. We outline the approach and give an explicit expression for the mean value of Stokes $Q$.

## 1. Introduction

For a long time, the Zeeman effect, and more recently the Hanle effect, have been used in Astrophysics to measure magnetic fields (e.g., Stenflo 1994; Landi Degl'Innocenti & Landolfi 2004). Observations of the solar magnetic field, and numerical simulations of solar magneto-hydrodynamical processes, all point to a magnetic field which is highly variable on all scales. This fact has motivated us to consider the Zeeman and Hanle effects in a medium where the magnetic field and the velocity field are random, with correlation lengths comparable to a typical photon mean free-path. The importance of this problem is well recognized (e.g., Landi Degl'Innocenti 1994, 2003; Landi Degl'Innocenti & Landolfi 2004).

In a random medium, the mean Stokes parameters can be calculated by averaging numerically the solutions of the radiative transfer equation for polarized radiation over all realizations of the random magnetic and velocity fields. This can be a very lengthy procedure, even with modern computers. The strategy that we adopted is to employ models of magnetic and velocity fields, and atmospheric models that are simple enough to allow us the construction of explicit expressions for mean observable quantities, but have enough flexibility to model various kinds of physical situations that are encountered in a magnetized medium. To satisfy these constraints, we describe the magnetic field through
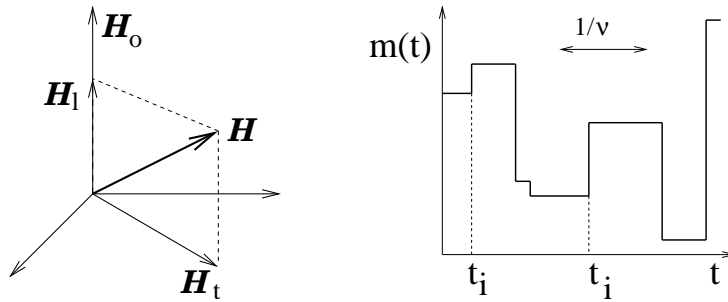
Figure 1.     *Left*: random magnetic field $\boldsymbol{H}$ with mean value $\boldsymbol{H}_0$. $\boldsymbol{H}_1$ and $\boldsymbol{H}_t$ are its longitudinal and transverse components, respectively. *Right*: a sample realization of a Kubo-Anderson process $m(t)$ with density $\nu$ (mean correlation length, $1/\nu$).
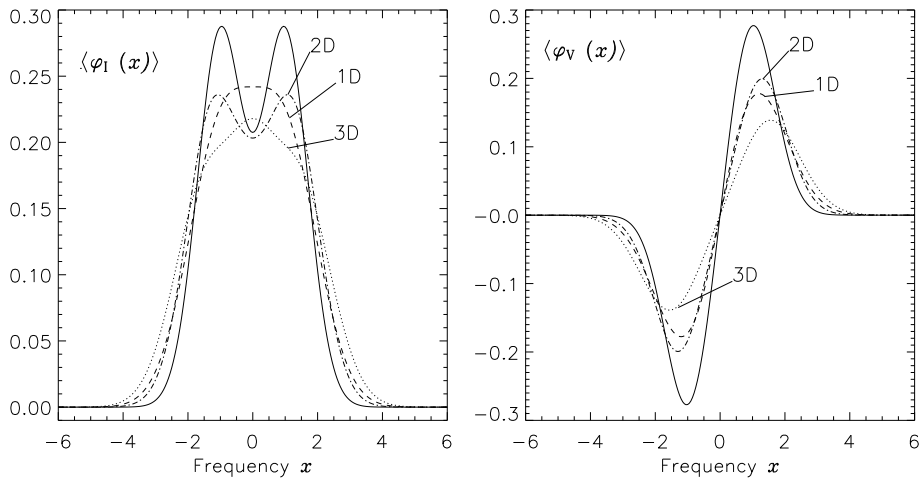


Figure 2.     Dependence of the opacities $\langle \varphi_I \rangle$ (left) and $\langle \varphi_V \rangle$ (right) on various magnetic field distributions (see text). The full line corresponds to the mean-field solution. The frequency $x$ is in units of the Doppler width, $\Delta_D$. For this calculation, we used a longitudinal mean field, $\boldsymbol{H}_0$, with Zeeman shift $\Delta H_0 = 1.0\,\Delta_D$, and $H_0/\sqrt{2}\sigma = 1$. The line damping parameter is zero.

a Kubo-Anderson process (KAP). KAPs have been employed in many different branches of physics (Brissaud & Frisch 1974) and also by Landi Degl'Innocenti (1994) to represent random magnetic fields.

The results presented here are discussed in detail in Frisch, Sampoorna, & Nagendra (2005a,b) for the Zeeman effect, and in Frisch (2006) for the Hanle effect.

## 2.    The Zeeman propagation matrix in a random magnetic field

To avoid dealing with transfer equations with stochastic coefficients, it is sometimes assumed that the magnetic field has a scale of variation much smaller than the typical photon mean free path. In this micro-turbulent limit, the Zeeman

propagation matrix can be locally averaged over the distribution of the field vector. This problem was first considered in some detail by Dolginov & Pavlov (1972) and Domke & Pavlov (1979). Fluctuations of the magnetic field intensity and direction produce random Zeeman shifts of the $\sigma$ components, and random variations of the angular dependence of both $\pi$ and $\sigma$ components. When the random field $\boldsymbol{H}$ is invariant under rotation about the direction of $\boldsymbol{H}_0$ (see Fig. 1, left), it is possible to establish general and compact expressions for the mean elements $\langle\varphi_{\mathrm{I}}\rangle$, $\langle\varphi_{\mathrm{Q}}\rangle$, $\langle\varphi_{\mathrm{V}}\rangle$, etc., of the line propagation matrix $\boldsymbol{\Phi}$ (see Frisch et al. 2005a).

In Fig. 2, we show $\langle\varphi_{\mathrm{I}}\rangle$ and $\langle\varphi_{\mathrm{V}}\rangle$ for three different types of Gaussian distributions. The curves labeled "1D", "2D", and "3D", correspond to fluctuations that are, respectively, along the mean field, perpendicular to the mean field, and isotropically distributed. In the 1D case, the distribution function is $P(\boldsymbol{H}) = \exp[-(H - H_0)^2/2\sigma^2]/\sqrt{2\pi}\sigma$. Similar definitions hold in the 2D and 3D cases. In the 2D case, only $\boldsymbol{H}_{\mathrm{t}}$ is random, while $\boldsymbol{H}_{\mathrm{l}} \equiv \boldsymbol{H}_0$. The strength of the fluctuations is measured by the ratio $f = \sqrt{2}\sigma/H_0$. Figure 2 shows a case of moderate fluctuations ($f = 1$). We observe that $\langle\varphi_{\mathrm{I}}\rangle$ is very sensitive to the angular dependence of the distribution. For $\langle\varphi_{\mathrm{V}}\rangle$, the averaging produces a decrease in magnitude and a broadening of the peaks associated with a shift of their positions away from line center, which are particularly strong for the isotropic distribution.

## 3. The Kubo-Anderson magnetic field model

We modeled the magnetic field with a KAP, which is a step-wise, stationary constant, Markov process, characterized by a correlation length and a probability distribution function. Figure 1 (right) shows a typical realization of a scalar KAP, $m(t)$, where the jumping points, $t_i$, are distributed following a Poisson law with density $\nu$. For our problem, the random magnetic field is characterized by a density $\nu$, and a distribution function $P(\boldsymbol{H})$. Micro- and macro-turbulence are recovered when the correlation length, $1/\nu$, goes to zero and infinity, respectively.

## 4. Mean Stokes parameters and their r.m.s. fluctuations

Frisch et al. (2005b) showed that a KAP magnetic field model, associated with a Milne-Eddington atmosphere, yields explicit expressions for the mean Stokes vector $\boldsymbol{I}$, and for the r.m.s. fluctuations of Stokes parameters around their mean values. An explicit expression for mean Stokes parameters was also given by Landi Degl'Innocenti (1994), but their dispersion was not considered.

We recall that the ratio $\beta$ of line opacity to continuum opacity is a constant in a Milne-Eddington model, and that line and continuum have the same linear, unpolarized source function $S = (B_0 + B_1\tau_{\mathrm{c}})$, where $\tau_{\mathrm{c}}$ is the continuum optical depth along the line-of-sight (LOS). The KAP model is described above. The transfer equation for the Stokes vector $\boldsymbol{I}$ along the LOS can be written as

$$\frac{\mathrm{d}\boldsymbol{I}}{\mathrm{d}\tau_{\mathrm{c}}} = (\boldsymbol{E} + \beta\boldsymbol{\Phi})(\boldsymbol{I} - \boldsymbol{S}) \,, \tag{1}$$
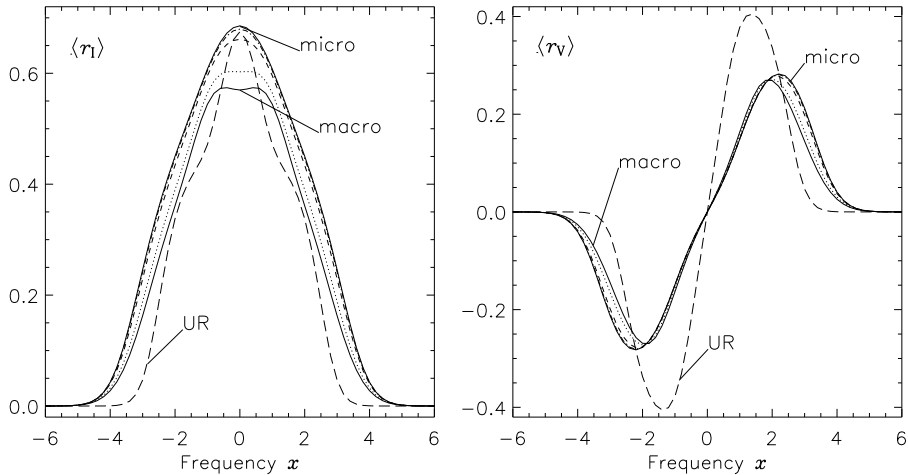
Figure 3.    Dependence of $\langle r_{\mathrm{I}} \rangle$ and $\langle r_{\mathrm{V}} \rangle$ on the magnetic field correlation length, $1/\nu$. Isotropic magnetic field distribution with fluctuations of moderate strength. Longitudinal mean field. Same magnetic field parameters as in Fig. 2 ($\Delta H_0 = 1.0\,\Delta_{\mathrm{D}}$, $H_0/\sqrt{2}\sigma = 1$). Line strength $\beta = 10$. Gaussian line absorption profile. Long-dashed line: Unno-Rachkovsky solution with $\boldsymbol{H}_0$.

where $\boldsymbol{E}$ is the $4 \times 4$ unit matrix, $\boldsymbol{\Phi}$ the line propagation matrix, and $\boldsymbol{S} = S\boldsymbol{U}$ with $\boldsymbol{U} = (1,0,0,0)$. It is convenient to express the results in terms of the reduced intensity at the surface, $\boldsymbol{r}(0) = [\boldsymbol{I}_c(0) - \boldsymbol{I}(0)]/B_1$. Its average over all realizations of the magnetic field can be written as

$$\langle \boldsymbol{r}(0) \rangle = (1+\nu)\left\langle \lambda\boldsymbol{\Phi}(\boldsymbol{E} + \lambda\boldsymbol{\Phi})^{-1} \right\rangle \left[ \boldsymbol{E} + \nu \left\langle \lambda\boldsymbol{\Phi}(\boldsymbol{E} + \lambda\boldsymbol{\Phi})^{-1} \right\rangle \right]^{-1} \boldsymbol{U} \,, \quad (2)$$

where $\lambda = \beta/(1+\nu)$. In the r.h.s. of Eq. (2), averages are over the distribution $P(\boldsymbol{H})$. In the micro-turbulent limit, one recovers the Unno-Rachkovsky (UR) solution calculated with the mean propagation matrix, $\langle \boldsymbol{\Phi} \rangle$, and in the macro-turbulent limit, the UR solution averaged over the distribution of magnetic fields. Equation (2) shows that fairly strong lines ($\beta \sim 10$) are needed to observe differences between the micro- and macro-turbulent limits, and also that the micro-turbulent limit is reached when $\nu \geq \beta$ (correlation length smaller than unity in line optical depth units).

    An essential ingredient to prove Eq. (2) is that the magnetic field be piecewise constant with uncorrelated random values in each interval. The mean propagation operator then satisfies a convolution equation that yields an explicit expression for the Laplace transform of the mean propagation operator. When the source function is linear, this Laplace transform immediately provides the surface value of the mean Stokes vector. Equation (2) also holds if the velocity field is a KAP with the same correlation length as the magnetic field. Then the averaging is over the joint magnetic and velocity field distribution.

    In Fig. 3 we show the dependence on $\nu$ of the mean residual Stokes parameters, $\langle r_{\mathrm{I}}(0) \rangle$ and $\langle r_{\mathrm{V}}(0) \rangle$, for moderate fluctuations. We remark that $\langle r_{\mathrm{V}}(0) \rangle$ is not very sensitive to the correlation length of the magnetic field, a property which holds even for stronger fluctuations, and that the departure from the UR
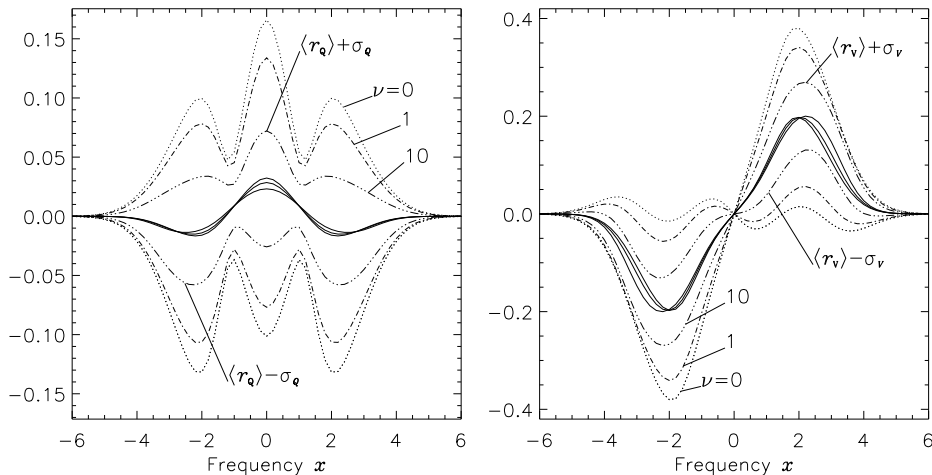
Figure 4.     Dispersion around the mean Stokes profiles. Full lines show the mean profiles and discontinuous lines the mean values plus or minus the square root of the dispersion. Same model parameters as in Fig. 3, but the mean field has an inclination of 45° and azimuth of 30° about the ray direction.

solution is very sensitive to the strength $f$ of the magnetic field fluctuations. It remains small when $f$ is small, but becomes quite large already for $f$ around unity, as shown in Fig. 3. For Stokes $I$, one observes a somewhat larger sensitivity to the value of $\nu$, roughly independent of the strength of fluctuations. Other examples can be found in Frisch et al. (2005b).

Explicit expressions can also be obtained for the dispersion around the mean Stokes parameters, $\sigma_X^2 = \langle X^2 \rangle - \langle X \rangle^2$, where $X$ stands for $I$, $Q$, $U$ or $V$. They can be established by a summation method over all possible realizations of the random field (Frisch et al. 2005b). In Fig. 4 we show the dispersions around the mean values for the same model as in Fig. 3. It is clear that the fluctuations are very sensitive to the value of $\nu$. They reach their maximum values in the macro-turbulent limit and go to zero in the micro-turbulent limit.

## 5.    The Hanle effect in a random magnetic field

In the Hanle effect, polarization is created by a scattering process (resonance polarization in the presence of a magnetic field), so the photons can return several times to the same turbulent element. Thus the method developed for the Zeeman effect is not directly applicable. One can circumvent this difficulty with the assumption that the magnetic field is a KAP along a photon trajectory, as in Frisch & Frisch (1976), and study the stationary solution, as time goes to infinity, of a time-dependent transfer equation for the six-component vector introduced by Faurobert-Scholl (1991, also Nagendra, Frisch, & Faurobert-Scholl 1998), representing the polarized radiation in place of the usual three Stokes parameters. This approach leads to an integral equation for a mean source vector, conditioned by the random value of the magnetic field. This integral equation is not easy to solve, but it yields explicit expressions for the mean Stokes parameters when combined with some physically realistic approximations, such

as neglecting the influence of the magnetic field on Stokes $I$, keeping only the contributions from $I$ and $Q$ in the source terms for $Q$ and $U$, and solving the integral equation for $Q$ with a Neumann series expansion, limited to the two first terms. If $\tau$ is the frequency-averaged line optical depth, and $\boldsymbol{\Omega}$ the direction of the LOS defined by its polar angles $\theta$ and $\chi$ ($\mu = \cos\theta$), the mean value of Stokes $Q$ can be written as

$$\langle Q(\tau, x, \boldsymbol{\Omega})\rangle \simeq \frac{3}{2\sqrt{2}} W_2 (1 - \mu^2)\, I_2(\tau, x, \mu)\ , \tag{3}$$

where $W_2$ is the atomic depolarization factor (unity for a normal Zeeman triplet) and $I_2(\tau, x, \mu)$ one of the six components of the radiation field averaged over the random magnetic field. In the r.h.s. of Eq. (3) we ignored small terms depending on $\chi$. The field $I_2(\tau, x, \mu)$ satisfies a standard transfer equation with a source function, $S_2$, independent of the direction of the LOS. For complete redistribution it can be written as

$$S_2(\tau) \simeq (1 - \epsilon_{\mathrm{p}})\langle \mathcal{M}_{00}^2 \rangle\, C_{\mathrm{I}}(\tau) + (1 - \epsilon_{\mathrm{p}})^2\, W_2 \int_0^\infty K_{2\mathrm{s}}(|\tau - \tau'|)\, C_I(\tau')\, d\tau'\ . \tag{4}$$

The first term is a generalized single-scattering approximation where $\epsilon_{\mathrm{p}}$ is the depolarization rate by elastic collisions, $\langle \mathcal{M}_{00}^2 \rangle$ the average of the Hanle phase matrix element $\mathcal{M}_{00}^2(\boldsymbol{H})$ (with the notations of Landi Degl'Innocenti & Landolfi 2004) over the magnetic field distribution, and $C_I(\tau)$ a measure of the anisotropy of Stokes $I$ (it is the dominant term in the frequency-averaged spherical tensor, $J_0^2(\tau)$). The second term takes into account photons that have been scattered twice, and through the kernel $K_{2\mathrm{s}}$ contains the autocorrelation of $\mathcal{M}_{00}^2(\boldsymbol{H})$, hence the correlation length of the magnetic field. This term comes as a correction to the single-scattering approximation. More general expressions including magnetic and velocity field correlations can be found in Frisch (2006).

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