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GRAVITATIONAL WAVES FROM COLLAPSING STARS

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The gravitational collapse of a massive star to a neutron star or black hole is one of the most promising sources of detectable gravitational radiation. A substantial fraction of the rest energy of the star could be radiated if the collapse were sufficiently nonspherical. In fact the pioneering gravitational wave bar detectors of the Weber type were resonant at kilohertz frequencies corresponding to the free fall collapse time scales of solar mass cores of massive main sequence stars, i.e.

$$t_{ff} \approx (R_C^3/GM_\odot)^{1/2} \sim (1\ \mathrm{KHz})^{-1}$$

The power radiated in gravitational waves, in general, for any physical process (such as an explosion, collapse, etc.) is given by

$$P_{GW} \sim \left(\frac{G}{c^5}\right) |\dddot{Q}|^2, \qquad (1)$$

where $\dddot{Q}$ is the triple-time derivative of the quadrupole moment of the system (the lowest order gravitational radiation being quadrupolar, the monopole and dipole orders vanishing due to conservation of energy and momentum respectively).

Eq.(1) can be written as:

$$P_{GW} \approx (G/C^5) \ M^2 \ V^4/t^2 \ ; \tag{2}$$

M,V and t being the characteristic masses, velocities and time scales associated with the physical process involved. We can write for the characteristic internal power, P_{int}, in quadrupole motion for the explosion as:

$$P_{int} \approx MV^2/t, \tag{3}$$

so that:

$$P_{GW} \simeq P_{int} \ \frac{P_{int}}{(C^5/G)} \simeq \frac{P_{int}^2}{(C^5/G)} \tag{4}$$

or

$$P_{GW} \simeq \frac{P_{int}^2}{3 \times 10^{59} \text{ ergs/s}} \ ; \tag{5}$$

where $(C^5/G) \approx 3 \times 10^{59}$ ergs/s, is the general relativistic upper limit, i.e. the maximal power or luminosity allowed in any physical process; also called the Gunn luminosity. The Gunn luminosity is reached in physical situations where the entire rest energy (Mc^2) of a star is radiated on time scales corresponding to its Schwarzschild radius, i.e. GM/C^3 ($\approx 10^{-5}$ secs for the sun).

In units of C = G = 1,

$P_{GW} \approx (E_{int}/t)^2$, or the energy radiated in gravitational waves E_{GW} for an explosion with total internal energy E_{int}, is:

$$E_{GW} \simeq P_{GW} \ t \approx (E_{int}^2/t) \tag{6}$$

The total number of gravitons (quanta of gravitational radiation, each of energy, $\hbar/t$ by Planck's law, $\hbar$ being Planck's constant) emitted is given by:

$$N_G \simeq \frac{E_{int}^2/t}{\hbar/t} \simeq \frac{E_{int}^2}{\hbar}$$

$$\simeq \left(\frac{E_{int}}{10^{16}\text{ergs}}\right)^2 \qquad (7)$$

Thus the total number of gravitons emitted depends only on the internal energy E_{int} of the explosion.

For a supernova explosion (or stellar collapse), $E_{int} \simeq 10^{51}$ ergs. $\therefore N_G \simeq 10^{70}$ and

$E_{GW} \simeq 10^{46}$ ergs, t being about a millisecond. For $t \simeq 10$ ms, the total energy released in gravitational waves by the supernova is $E_{GW} \simeq 10^{45}$ ergs. This is to be compared with the total integrated output in photon (i.e. electromagnetic) energy of the supernova, which is a $E_{Photon} \simeq$ few times 10^{49} ergs, and the total energy in neutrinos E_ν which is $E_\nu \simeq$ few times 10^{53} ergs, (for SN 1987A). For a hundred megaton hydrogen bomb ($E \simeq 4 \times 10^{24}$ ergs), E_{GW} is a mere 10^{-6} ergs! (although the destructive effects of such a bomb is about ten thousand times that of the Hiroshima bomb). Eq.(2) can also be expressed as:

$$P_{GW} \simeq G/C^5\,(M/R)^2\,V^6$$

$$\simeq (C^5/G)\,\left(\frac{R_{Schw}}{R}\right)^2\,(V/C)^6$$

$$\simeq (C^5/G)\,\left(\frac{R_{Schw}}{R}\right)^5 \qquad (8)$$

Thus P_{GW} tends to get maximised for compact objects with $R \approx R_{Schw}$, or for processes with $V \simeq C$.

Now the gravitational wave energy radiated by a non-spherical self gravitating system is given by:

$$\Delta E_{GW} \simeq Mc^2 \left(\frac{R_{Schw}}{R} \right)^{7/2} \qquad (9)$$

This corresponds to a relative strain 'h' (don't confuse with Planck's constant!) of: ($\delta \eta/\eta \sim h$, i.e. change in relative displacement between two particles of the detector. Unlike electromagnetic waves, where a single particle with charge can be used to detect the wave, one requires the relative separation changes with time of two particles to detect presence of GW)

$$h \simeq 6 \times 10^{-19} \left(\frac{\varepsilon}{0.1} \right)^{2/7} \left(\frac{M/M_\odot}{(R_d/50kpc)} \right), \qquad (10)$$

ε usually taken as 0.1, $\varepsilon \approx (R_{schw}/R)^{7/2}$
$\therefore$ for a 10 $M_\odot$ star collapsing in LMC,
$h \approx 6 \times 10^{-18}$
or $\Delta\eta \simeq 10^{-15}$ cm $\approx 10^{-2}$ fermi displacement in a 1.5 metre length Weber bar. The sensitivity of a Weber bar about $h \approx 10^{-16}$. Such bars cooled to millidegree Kelvin can have sensitivity $h \approx 10^{-20}$, (compare $h \approx 6 \times 10^{-18}$ for LMC SN 1987A). So such detectors if they had been in operation could have detected the gravitational waves from SN 1987A. (No coincidence detection for SN 1987A from two or more detectors was not possible as such detectors were simply not operating!).

The Weber bar was made from some high Q material like Al, sapphire etc. The impinging gravitational wave of the right frequency sets the fundamental mode of the bar into oscillation and a suitable sensor detects the displacement. Weber used piezoelectric crystals bonded to the bar. The signal was amplified and electronically scanned. The signals from two or more widely separated bars are compared and coincidence signal denotes burst of gravitational waves from astrophysical sources. With the latest laser interferometer type of detectors (where the relative change in length between two arms of the interferometer by passing gravitational waves can be measured) a sensitivity of $h \approx 10^{-22}$ is possible! If a gravitational wave hits an object of mass 'm' (with internal elastic forces) the object is set into oscilatory motion described by:

$$F_j = \frac{1}{2} m \ddot{h}^{TT}_{jk} \eta_K \qquad (11)$$

η_K is the displacement or separation between particles. $\delta\eta_j = \frac{1}{2} h^{TT}_{jk} \cdot \eta_K$, η_K is the change in separation due to passage of a GW. The relative acceleration between the particles caused by passage of the GW is:

$$\ddot{\eta}_j = \frac{1}{2} \ddot{h}^{TT}_{jk} \cdot \eta_K \qquad (12)$$

It is the Riemann tensor (analogous to the Maxwell tensor for electromagnetism) that drives the oscillatory motion according to:

$$\frac{d^2 x^j}{dt^2} = -R_{joko}\, x^K , \qquad (13)$$

If $\eta(x,t)$ is the displacement of the rod element then its motion is determined by the equation:

$$\frac{\partial^2 \eta}{\partial t^2} + \frac{1}{\tau} \frac{\partial \eta}{\partial t} - a^2 \frac{\partial^2 \eta}{\partial x^2} = 0 \qquad (14)$$

where a is the speed of sound in the rod, and τ the damping time. Solution is:

$$\eta = e^{-iwt} e^{-t/2\tau} u(x), \text{ the B.C.}$$

being $\partial\eta / \partial x|_{x = \pm L/2} = 0$ (ends of rod are free).

The rod is driven on resonance

$$R_{xoxo} \sim e^{-iw_n t} ; \quad B_n = e^{-iw_n t}$$

and the displacement can be written as a superposition of normal modes:

$$\eta = \sum_n B_n(t)\, u_n(x),$$ the u_n being orthogonal. The energy flux in gravitational waves scales as $|R_{xoxo}/\omega\eta|^2$ $\alpha\ 1/n^2$ and the measurable quantity is $\delta\eta/\eta = h$, the relative strain produced by passage of the wave. The superscript TT in the above equations: (11) and (12) stands for transverse traceless; as analogous to the equations for electromagnetic waves:

$$A_o = o,\ A_{i,i} = o \text{ and } \Box A_i = o, \tag{15}$$

we have for the gravitational case:

$$h^{TT}_{o\mu} = o,\ h^{TT}_{jk,k} = o\ \Box\ h^{TT}_{jk} = o, \tag{16}$$

i.e Eq. (16) show that the number of degrees of freedom as a result of the 'transverse traceless gauge' for the gravitational wave is reduced to two (i.e. the two states of polarisation) analogous to the radiation gauge in electromagnetism which gives the photon two degrees of polarisation (right or left).

Now apart from the gravitational waves emitted by the collapse, any freshly formed neutron star or pulsar can also emit an intense flux of gravitational waves. We have for spheroidal collapse:

$$P_{GW} \approx \frac{2}{375}\ \frac{GM^2}{c^5} \langle (\dot{\dot{a}}^2 - \dot{\dot{\dot{c}}}^2) \rangle\ ,\ e^2 = 1 - c^2/a^2 \tag{17}$$

e being the oblateness of the object
or P_{GW} can be written as:

$$P_{GW} \simeq \frac{32G}{5c^5}\ I_m^2\ \ e^2\ \Omega^6\ , \tag{18}$$

where $\Omega = 2\pi / P$, P is the period of the neutron star formed. For an initial $e \simeq 0.1$ to 0.01, and $P \simeq 10$ milliseconds

$$P_{GW} \simeq 10^{44} - 10^{45} \text{ ergs/sec} \tag{19}$$

Eq.(19) implies that a substantial fraction of the rotational energy of the newly formed compact object can be radiated in about a day, the gravitational radiation damping time in this case being about 10^4 to 10^5 seconds. The corresponding h for such an object formed at LMC distance is $h \approx 10^{-20}$. Now the angular momentum is

$$J = \left(\frac{2M}{5}\right)^{3/2} (Ga)^{1/2} e,$$ for a Maclaurin spheroid in limit $e \to o$ during axisymmetric collapse with conserved J and M. e should grow during collapse. The angular momentum provides eventual centrifugal support for equatorial a-axis while polar c-axis collapses unimpeded. The collapse of a slowly rotating stellar core to a black hole releases energy in gravitational waves of:

$$E_{GW} \sim (J/GM^2/C)^4 Mc^2, \qquad (20)$$

This is maximised for some $J = J_{max}$ when rotational energy comparable to gravitational energy and e becomes large just above nuclear densities: Note $dE_{GW}/dt \propto J^4$.

One could also have an efficient source of gravitational waves if magnetic fields are strong enough to produce nonspherical collapse. Maximum radiation efficiencies $\Delta E_{GW}/Mc^2$ at most one per cent.

As a final curiosity, the coalescence of two neutron stars (one of the proposed scenarios for millisecond pulsar formation), the binary pulsar may eventually reach that stage, gives rise to a two millisecond pulse of gravitational radiation of energy $E_{GW} = 3 \times 10^{52}$ ergs. This would be detectable in Weber type bars over a distance of 100 Mpc, i.e. one such event in ten million galaxies should be detectable!

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