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Location of equilibrium points in the perturbed photogravitational circular restricted problem of three bodies

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Abstract. In this paper we have found the location of triangular equilibrium points in the perturbed photogravitational restrict three body problem. We find that the position of equilibrium points are affected with the introduction of perturbation in coriolis and centrifugal forces and taking the more massive primary as radiating one. We have found the characteristic equation of motion and the critical value of the mass parameter. We get the result which is the generalisation of Bhatnagar and Hallan (1978).

Key words : Celestial mechanics—solar system

1. Introduction

It is well known that the restrical problem possesses five equilibrium points. Two of them make equilateral triangles with the primaries. They are stable for the mass ratio μ of the finite bodies of $\mu < \mu_0 = 0.03852$ (Szebehely 1967a).

In one of his papers Szebehely (1967b) established that the coriolis force is the stabilizing force when the centrifugal force is kept constant. Bhatnagar and Hallan (1978) considered the effect of perturbations in coriolis and centrifugal forces in the restricted problem of three bodies. They proved that the equilibrium points L_4 and L_5 form nearly equilateral triangles with the primaries.

Hence we thought to examine the restricted problem of three bodies in which the effect of perturbations in coriolis and centrifugal forces are taken into consideration along with the more massive primary as radiating one. As such our problem is called perturbed photogravitational restricted problem of three bodies.

In the second section we have found the co-ordinates of triangular equilibrium points. We find that the co-ordinates are affected with the introduction of perturbations and radiating factor. We get the generalised result problem of Bhatnagar and Hallan.

In the third section we have found out the characteristic equation of motion. From this we have calculated the characteristic roots ignoring the second and higher order terms in perturbations. Finally we find the critical value of the mass parameter which contains both perturbations and radiating factor.

2. The location of equilibrium points

Using non-disensional variables and synodic co-ordinate system (x, y) the equations of motion of the restricted problem are

$$\ddot{x} - 2\dot{y} - x = \frac{\delta F}{\delta x} \quad \text{and} \quad \ddot{y} + 2\dot{x} - y = \frac{\delta F}{\delta y} \qquad \ldots (1)$$

where $$F = \frac{1 - \mu}{r_1} + \frac{\mu}{r_2}$$

$$r_1^2 = (x - \mu)^2 + y^2 \quad \text{and} \quad r_2^2 = (x + 1 - \mu)^2 + y^2 \qquad \ldots (2)$$

and μ is the ratio of the mass of the smaller primary to the total mass of the primaries and $0 \leq \mu \leq {}^1/_2$.

Now we introduce the perturbations in the coriolis and the centrifugal forces with the help of the parameters α and β. The unperturbed value of each is unity. Also we consider the more massive primary as the radiating one. The radiation repulsive force Fp exerted on a particle can be represented in terms of gravitational attraction, Fg (Radzievskii 1950) as

$$Fp = Fg(1 - q). \qquad \ldots (3)$$

Here $q = 1 - (Fp/Fg)$ is a constant.
The assumption q = constant is equivalent to neglecting fluctuations in the beam of solar radiation and the effect of planet shadow.
Hence the equations of motion of our problem can be written as

$$\ddot{x} - 2\,\alpha\dot{y} = \frac{\delta\Omega}{\delta x} \quad \text{and} \quad \ddot{y} + 2\,\alpha\dot{x} = \frac{\delta\Omega}{\delta y} \qquad \ldots (4)$$

where $$\Omega = \frac{\beta}{2}\{(1 - \mu)\, r_1^2 + \mu r_2^2\} + \frac{q(1 - \mu)}{r_1} + \frac{\mu}{r_2}. \qquad \ldots (5)$$

Hence α, β and q may be taken as

$$\alpha = 1 + \varepsilon \quad ; \quad |\,\varepsilon\,| << 1,$$
$$\beta = 1 + \varepsilon' \quad ; \quad |\,\varepsilon'\,| << 1,$$
$$q = 1 - \varepsilon'' \quad ; \quad |\,\varepsilon''\,| << 1,$$

where ε, ε' represent the perturbations in the coriolis and centrifugal forces and ε'' radiation parameter.

At equilibrium points

$$\frac{\delta\Omega}{\delta x} = 0 \quad \text{and} \quad \frac{\delta\Omega}{\delta y} = 0.$$

The triangular points are the solution of the equations

$$\left.\begin{aligned} &\beta x - \frac{q(1-\mu)(x-\mu)}{r_1^3} - \frac{\mu(x+1-\mu)}{r_2^3} = 0 \\ &\beta - \frac{q(1-\mu)}{r_1^3} - \frac{\mu}{r_2^3} = 0. \end{aligned}\right\} \quad \ldots (6)$$

From these equations

$$r_1 = \left(\frac{q}{\beta}\right)^{1/3} \quad \text{and} \quad r_2 = \frac{1}{(\beta)^{1/3}}. \quad \ldots (7)$$

From equations (2) and (7), we get

$$\left.\begin{aligned} &x = \frac{1-q^{2/3}}{2\beta^{2/3}} + \mu - \frac{1}{2} \\ &y = \pm\left\{\frac{q^{2/3}}{\beta^{2/3}} + \frac{1-q^{2/3}}{2\beta^{2/3}} - \left(\frac{1-q^{2/3}}{2\beta^{2/3}}\right)^2 - \frac{1}{4}\right\}^{1/2} \end{aligned}\right\} \quad \ldots (8)$$

These are the co-ordinates of the triangular equilibrium points $L_{4,5}$.

3. Critical mass

Putting $x = a + \xi$; $y = b + \eta$ ($\xi, \eta \ll 1$) in equation (4), where (a, b) are the co-ordinates of points of equilibrium under study, we get the variational equations as

$$\left.\begin{aligned} &\xi - 2\alpha\eta = \Omega^0_{xx}\xi + \Omega^0_{xy}\eta \\ &\text{and} \\ &\eta + 2\alpha\xi = \Omega^0_{xy}\xi + \Omega^0_{yy}\eta \end{aligned}\right\} \quad \ldots (9)$$

Here only linear terms in ξ and η have been taken. The second partial derivatives of Ω are denoted subscripts and the superscript 0 indicates that the derivatives are to be evaluated at the point under study. The characteristic equation corresponding to equation (9) is

$$\lambda^4 - (\Omega^0_{xx} + \Omega^0_{yy} - 4\alpha^2)\lambda^2 + \Omega^0_{xx}\Omega^0_{yy} - (\Omega^0_{xy})^2 = 0. \quad \ldots (10)$$

We have, for L_4

$$\begin{aligned} \Omega^0_{xx} = \frac{3}{4}\,\beta^{1/3}\{&3\mu(1-q)^{2/3} + 2\mu\beta^{2/3}(q^{-2/3} - q^{2/3}) + \mu\beta^{4/3}(1 - q^{-2/3}) \\ &+ \mu(q^{4/3} - q^{-2/3}) + (q^{-2/3} + q^{2/3}) - 2 \\ &+ 2\beta^{2/3}(1 - q^{-2/3}) + \beta^{4/3}\,q^{-2/3}, \end{aligned} \quad \ldots (11)$$

$$\Omega^0_{yy} = \frac{3}{4}\,\beta^{1/3}\{4q^{2/3}\beta^{2/3} + (1 - q^{2/3}) + 2\beta^{2/3} - (1 - 2q^{2/3} + q^{4/3}) - \beta^{4/3}\}$$
$$\times\ \{q^{-2/3}(1 - \mu) + \mu\} \qquad \ldots (12)$$

and

$$\Omega^0_{xy} = \Big[\frac{9}{16}\,\beta^{2/3}\{-(q^{2/3} - \beta^{2/3})^2 + 2\beta^{2/3} + 2q^{2/3} - 1\}$$
$$\times\ \{(1 - \mu)q^{-2/3}(1 - q^{2/3} - \beta^{2/3}) + \mu(1 - q^{2/3} - \beta^{2/3})^2\}\Big]^{1/2} \qquad \ldots (13)$$

Therefore, the characteristic equation becomes

$$\lambda^4 - \lambda^2(3\beta - 4\alpha^2) + \frac{9}{4}\,\beta^{8/3}(4 - \beta^{2/3}q^{2/3})\,\mu(1 - \mu) = 0.$$

Its roots are

$$\lambda^4 = \frac{(3\beta - 4\alpha^2) \pm \sqrt{(3\beta - 4\alpha^2 - 9\,\beta^{8/3}(4 - \beta^{8/3}q^{2/3})\,\mu(1 - \mu)}}{2}.$$

Putting $\alpha = 1 + \varepsilon$, $\beta = 1 + \varepsilon'$, $q = 1 - \varepsilon''$, and ignoring the second and higher order terms in ε, ε', ε'' we have

$$\lambda^2 = \frac{(-1 + 3\varepsilon' - 8\varepsilon) \pm \sqrt{(1 - 6\varepsilon' + 16\varepsilon) - 9(3 + \frac{22}{3}\varepsilon' + \frac{2}{3}\varepsilon'')\,\mu(1 - \mu)}}{2}$$

The discriminant is zero if μ satisfies the equation

$$3(9 + 22\varepsilon' + 2\varepsilon'')\mu^2 - 3(9 + 22\varepsilon' + 2\varepsilon'')\mu + (1 - 6\varepsilon' + 16\varepsilon) = 0. \qquad \ldots (14)$$

Thus the critical value of the mass parameter.

$$\mu_c = \mu_0 + \frac{4\left(36\varepsilon - 19\varepsilon' + \frac{1}{2}\varepsilon''\right)}{27(69)^{1/2}} \qquad \ldots (15)$$

where

$$\mu_0 = \frac{1}{2}\left(1 - \frac{(69)^{1/2}}{9}\right).$$

Hence the equilibrium point L_4 is stable for all mass ratios $\mu < \mu_c$. If the term within the bracket is +ve, range of stability increases ($\mu_c > \mu_0$) and if it is –ve it decreases ($\mu_c < \mu_0$). This will depend upon the perturbations ε, ε', ε''. This means coriolis force is not always a stabilizing force as claimed by Szebehely (1967b). It happens so because Szebehely has taken perturbations only in the coriolis forces keeping the centrifugal force constant. We may also note that coriolis and centrifugal forces, in general, enter the equations of motion when written in a synodic system of co-ordinates and if we write the equations of motion, in a problem where coriolis force is perturbed (for example if we take one of the primaries as an oblate body) then the centrifugal force is also perturbed. Szebehely's model in that sense is not consistent. If we ignore the radiation effect then the range of stability increases or decreases depending upon whether the perturbation point (ε, ε') lies on one or the other side of the line $36\varepsilon - 19\varepsilon' = 0$. Which confirms the result of Bhatnagar and Hallan (1978).

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