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Rotational motion of a satellite in an elliptical orbit under the influence of third body torque (I)

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Abstract. The rotational motion of a satellite in an elliptic orbit under the influence of the third body torque is being studied in Papers I and II. Paper I deals with the non-resonance case and Paper II deals with the resonance case and the chaotic nature of the given dynamical system. In Paper I, by using Melnikov's method we have shown that the equations of motion are non-integrable. Taking the third-body perturbation of the order of eccentricity e ($e << 1$) and using BKM method, it is observed that the amplitude of the oscillation remains constant up to second order of approximation. The main and parametric resonances have been shown to exist.

Key words : celestial mechanics—solar system

1. Introduction

Planar oscillations of a satellite in an elliptic orbit have been studied by Beletskii (1963), Cherousko, Zlatanstov *et al.* (1964), Singh (1973, 1983) and by Bhatnagar *et al.* (1993). None of them have taken the third body effect. Maciejewski (1992) has taken the effect of third body torque, but the satellite is assumed to move in a circular orbit. We have modified the problem of Maciejewski by taking the orbit of the satellite as elliptic. We have determined hyperbolic equilibrium solution and double asymptotic solutions corresponding to unperturbed Hamiltonian H_0. The non-integrability of the system has been shown through Melnikov's integral (1.2). Finally, we have found out the solution in the non-resonance case by BKM method taking the third body perturbation parameter ε of the order of the eccentricity e ($e << 1$).

The theorems on non-integrability and Melnikov's integral have been discussed by Bhatnagar *et al.* (1993). Because of the facts given in these theorems, the same method gives us more information about the chaotic nature of the dynamical system which we propose to study in paper II. An attempt is made to explain the tumbling of the satellite HYPERION with the help of the theory developed in these two papers.

2. Perturbed planar oscillations of a rigid satellite

2.1. *Equation of motion*

Let us consider a rigid satellite moving in an elliptic orbit (semi-major axis a, eccentricity e) under the influence of a central body of mass M and its moon of mass m whose orbit is assumed circular and coplanar with the orbit of the satellite (figure 1). The satellite is assumed to be a triaxial ellipsoid with principal moments of inertia $A < B < C$, and C is the moment of inertia about the spin axis which is perpendicular to the orbital plane. We approximate the influence of the moon by resolving potential of the torque with respect to the r/R ratio, where r is the radius of the satellite and R the radius of moon's orbit.

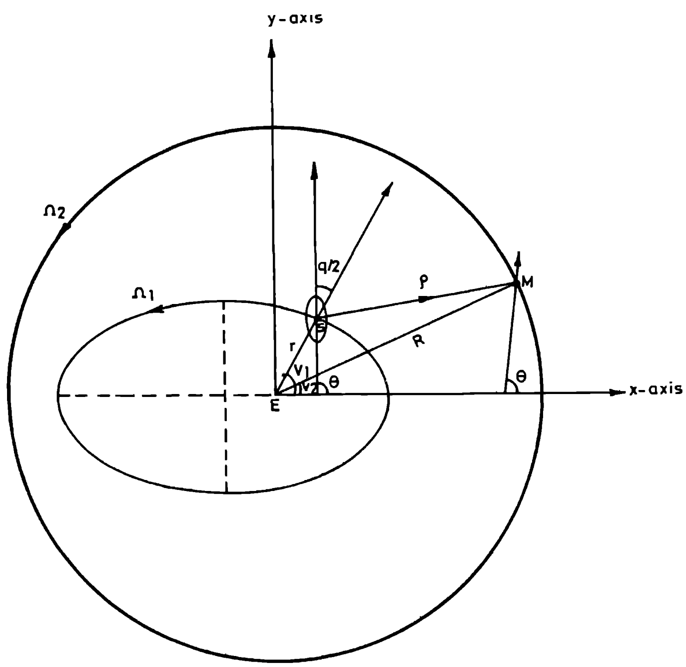

Figure 1. Satellite planar oscillation in elliptical orbit with third body perturbation

Let the true anomaly be v_1, the orientation of the satellite's long axis θ. Then $\theta - v_1 = q/2$ measures the orientation of the satellite's long axis relative to the satellite's radius vector. The equation of motion of satellite's planar oscillation is

$$\left[1 + e\cos\left(\frac{\Omega_1}{\Omega}v\right)\right]\frac{d^2q}{dv^2} - 2e\frac{\Omega_1}{\Omega}\sin\left(\frac{\Omega_1}{\Omega}v\right)\frac{dq}{dv} - 4e\frac{\Omega_1^2}{\Omega^2}\sin\left(\frac{\Omega_1}{\Omega}v\right)$$

$$+ n^2\sin q - n^2\varepsilon\sin(v - q) = 0 \qquad \ldots (2.1.1)$$

where $v = \Omega t, \quad \Omega = 2(\Omega_1 - \Omega_2), \quad v_1 \equiv \Omega_1 t, \quad v_2 = \Omega_2 t,$

$$n^2 = 3\left(\frac{\Omega_1}{\Omega}\right)^2\left(\frac{B-A}{C}\right), \quad \varepsilon \equiv \left(\frac{\Omega_1}{\Omega_1}\right)^2\frac{m}{M},$$

Ω_1 and Ω_2 are angular orbital velocities of the satellite and moon respectively, ε is the parameter due to third body effect. Equation (2.1.1) is equivalent to the Hamilton's equations

$$\frac{dq}{dv} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dv} = -\frac{\partial H}{\partial q} \qquad \ldots (2.1.2)$$

where

$$H = H_0 + eH_1 + O(e^2),$$

$$H_0 = \frac{p^2}{2} - 2\frac{\Omega_1}{\Omega}p - n^2 \cos q,$$

$$H_1 = -\left[p^2 \cos\left(\frac{\Omega_1}{\Omega}v\right) + n^2 \cos q \cos\left(\frac{\Omega_1}{\Omega}v\right) + n^2\varepsilon_1 \cos(v - q)\right],$$

p = generalized momenta.

Here, ε has been taken of the order of e, i.e. $\varepsilon = \varepsilon_1 e$ $(0 < \varepsilon_1 << 1)$.

2.2. *Equilibrium and double asymptotic solution*

Equilibrium solution corresponding to H_0 is given by

$$\frac{dq}{dv} = 0, \quad \frac{dp}{dv} = 0.$$

So, by (2.1.2) the hyperbolic equilibrium solution is

$$q(v) = \pi, \quad p(v) = 2\frac{\Omega_1}{\Omega}.$$

Now, we determine the unperturbed double asymptotic solutions. Again from equation (2.1.2), we have

$$\frac{dp}{dq} = -\frac{n^2 \sin q}{p - 2(\Omega_1/\Omega)}.$$

On integration, we obtain

$$\frac{p^2}{2} - 2\frac{\Omega_1}{\Omega}p = n^2 \cos q + n^2 - 2\frac{\Omega_1^2}{\Omega^2}$$

But $$p = \frac{dq}{dv} = \pm 2n \cos\frac{q}{2}.$$

Hence, the unperturbed double asymptotic solutions are given by

$$p^{\pm}(v) = 2\frac{\Omega_1}{\Omega} \pm \frac{2n}{\cosh(nv)}, \quad \sin[q^{\pm}(v)] = \pm\frac{2\sinh(nv)}{\cosh^2(nv)},$$

$$\cos[q^{\pm}(v)] = \frac{2}{\cosh^2(nv)} - 1.$$

2.3. *(a) Evaluation of Melnikov's integral*

Melnikov's integral is

$$M^{\pm}(v_0) = \int_{-\infty}^{\infty} \{H_0, H_1\}\,[q^{\pm}(v - v_0), p^{\pm}(v - v_0), v]\,dv$$

$$= \pm 2\pi\frac{\Omega_1^2}{\Omega^2}\sin\left(\frac{\Omega_1}{\Omega}v_0\right)\left[3\,\text{cosech}\left(\frac{\pi}{2n}\frac{\Omega_1}{\Omega}\right) + 4\,\text{sech}\left(\frac{\pi}{2n}\frac{\Omega_1}{\Omega}\right)\right]$$

$$\pm 2\pi\varepsilon_1 \sin v_0\left[\text{sech}\left(\frac{\pi}{2n}\right) + \text{cosech}\left(\frac{\pi}{2n}\right)\right]. \qquad \ldots (2.3.1)$$

It is easy to observe that for any values of mass parameter $n > 0$ and third body effect parameter ε_1 $(0 < \varepsilon_1 < 1)$, the above function has a simple zero. Thus, both pairs of asymptotic surfaces cross transversely and equations (2.1.2) are non-integrable.

2.3. *(b) Graphical representation of Melnikov's function*

For fixed value of $\Omega_1/\Omega = 0.4$.

(i) Figure 2 illustrates the graph of $M^{\pm}(v_0, \varepsilon_1, n)$ for a fixed value of $n = 0.1$, $\varepsilon_1 = 0.1$. In both cases the Melnikov function behaves almost like Sine functions; one corresponding to $M^{+}(v_0)$ and the other $M^{-}(v_0)$. They have simple zeros. We may observe that as $0 \leq v_0 \leq 6\pi$, the loops for $M^{\pm}(v_0)$ shrink in abscissa and expand in ordinate.

(ii) Figure 3 illustrates the graph of $M^{\pm}(v_0, \varepsilon_1, n)$ for a fixed value of $n = 0.1$, $v_0 = 0.1$. It has been observed that as the parameter due to third body effect changes from 0 to 1 $(0 \leq \varepsilon_1 \leq 1)$ then the value of Melnikov's function $M^{+}(v_0, \varepsilon_1, n)$ increases monotonically from 2.0787156 to 6.6752906 and $M^{-}(v_0, \varepsilon_1, n)$ decreases monotonically from –2.0787156 to –6.6752906.

(iii) Figure 4 illustrates the graph of $M^{\pm}(v_0, \varepsilon_1, n)$ for a fixed value of $v_0 = 0.1$, $\varepsilon_1 = 0.1$. It has been observed that as mass distribution parameter of satellite changes from 0.01 to 0.99 the value of Melnikov's function $M^{+}(v_0, \varepsilon_1, n)$ decreases rapidly for $0.01 \leq n \leq 0.05$ from 23.4114475 to 4.8594532 after that it decreases very slowly and $M^{-}(v_0, \varepsilon_1, n)$ increases rapidly for $0.01 \leq n \leq 0.05$ from –23.4114475 to –4.8594532 after that it increases very slowly. We may also observe that for axis symmetrical satellites, i.e. $A = B$, the value of $n = 0$ and the value of Melnikov's function $M^{+}(v_0, \varepsilon_1, n)$ and $M^{-}(v_0, \varepsilon_1, n)$ tending to $+\infty$ and $-\infty$ respectively.

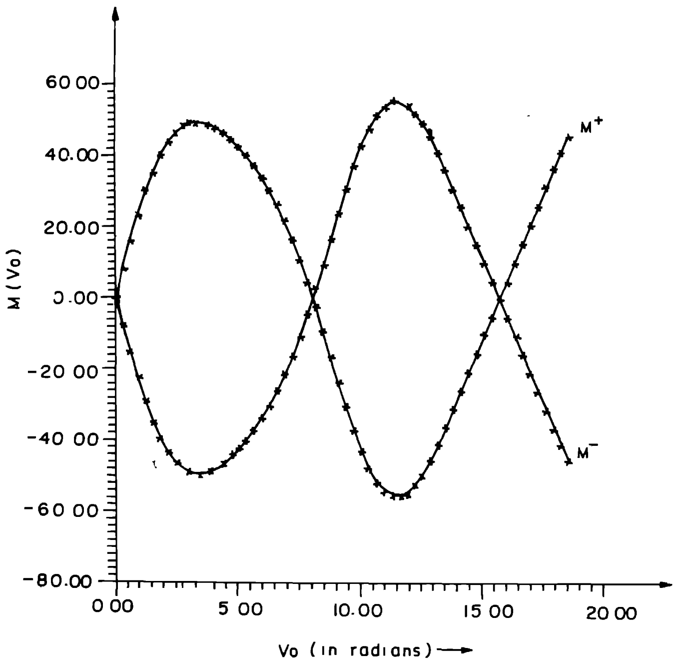

Figure 2. Melnikov's function for $\Omega_1/\Omega = 0.4$, $n = 0.1$, $\varepsilon_1 = 0\ 1$, $0 \le \nu_0 \le 6\pi$.

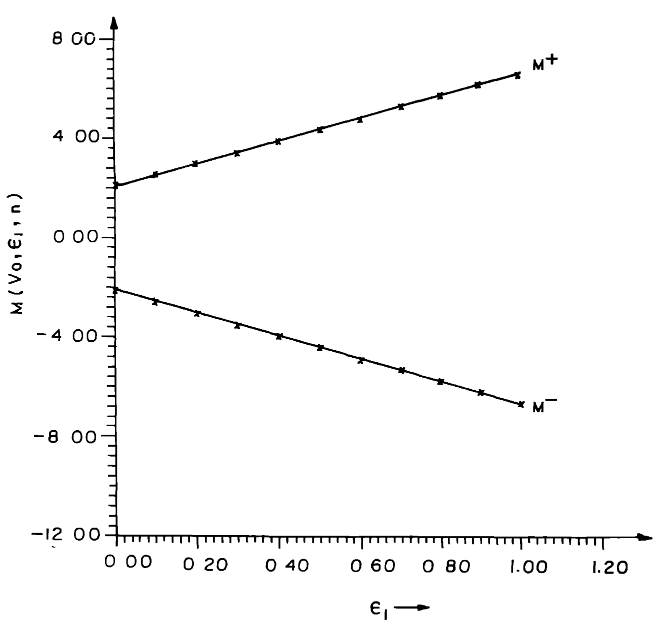

Figure 3. Melnikov's function for $\Omega_1/\Omega = 0.4$, $\nu_0 = 0\ 1$, $n = 0\ 1$.

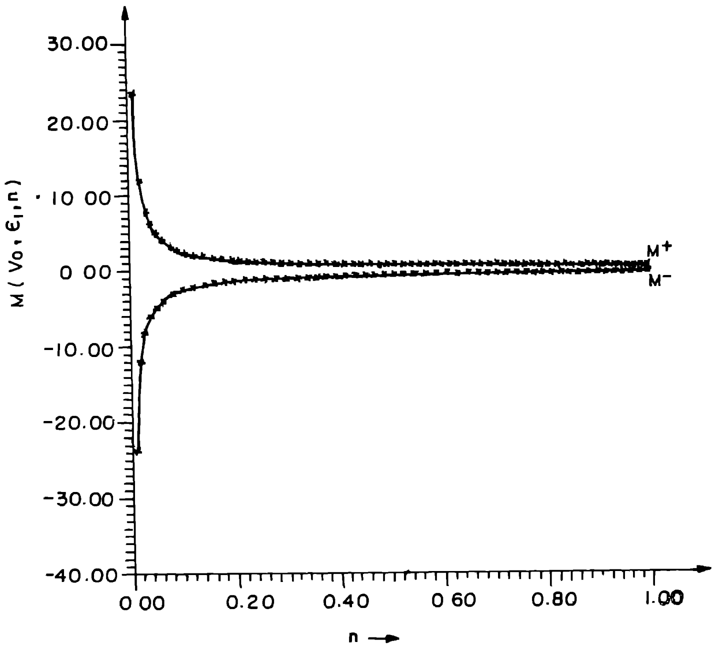

Figure 4. Melnikov's function for $\Omega_1/\Omega = 0.4$, $v_0 = 0.1$, $\varepsilon_1 = 0.1$.

3. Non-resonant planar oscillations of a satellite

Taking $q = \eta$, $n = \omega$ and ε = order of e, i.e. $\varepsilon = \varepsilon_1 e$ $(0 < \varepsilon_1 << 1)$ in equation (2.1.1), we get

$$\frac{d^2\eta}{dv^2} + \omega^2\eta = 4e\frac{\Omega_1^2}{\Omega^2}\sin\left(\frac{\Omega_1}{\Omega}v\right) + 2e\frac{\Omega_1}{\Omega}\sin\left(\frac{\Omega_1}{\Omega}v\right)\frac{d\eta}{dv}$$

$$- e\cos\left(\frac{\Omega_1}{\Omega}v\right)\frac{d^2\eta}{dv^2} + \omega^2(\eta - \sin\eta) + \omega^2\varepsilon_1 e\sin(v - \eta). \quad \ldots (3.1)$$

In equation (3.1), the non-linearity $(\eta - \sin\eta)$ is taken sufficiently weak and therefore it can also be taken of the order of e. So, by taking $\omega^2 = \alpha e$, equation (3.1) becomes

$$\frac{d^2\eta}{dv^2} + \omega^2\eta = e f\left(v, \eta, \frac{d\eta}{dv}, \frac{d^2\eta}{dv^2}\right) + e^2\bar{f}(v, \eta) \quad \ldots (3.2)$$

where $$f\left(v, \eta, \frac{d\eta}{dv}, \frac{d^2\eta}{dv^2}\right) = 4\frac{\Omega_1^2}{\Omega^2}\sin\left(\frac{\Omega_1}{\Omega}v\right) + 2\frac{\Omega_1}{\Omega}\sin\left(\frac{\Omega_1}{\Omega}v\right)\frac{d\eta}{dv}$$

$$- \cos\left(\frac{\Omega_1}{\Omega}v\right)\frac{d^2\eta}{dv^2} + \alpha(\eta - \sin\eta),$$

$$\bar{f}(v, \eta) = \alpha\varepsilon_1\sin(v - \eta).$$

The dynamical system described by equation (3.2) moves under forced vibrations due to the presence of the periodic sine forces on the right-hand side of the equation. We are benefited by the smallness of the eccentricity e, in equation (3.2) and hence the solution can be obtained by exploiting the BKM method. For $e = 0$, the generating solution of the zeroeth order is :

$$\eta = a \cos \psi, \qquad \psi = \omega v + \psi^*$$

where, amplitude a and phase ψ^* are constants which can be determined by the initial conditions. Proceeding exactly as in Bhatnagar *et al.* (1993) the solution of equation (3.2) is obtained in the form

$$\eta = a \cos \psi + e u_1(a, \psi, v) + e^2 u_2(a, \psi, v) + \ldots \qquad \ldots (3.3)$$

where the amplitude a and phase ψ are determined by the differential equations

$$\frac{da}{dv} = e A_1(a) + e^2 A_2(a) + \ldots \qquad \ldots (3.4)$$

$$\frac{d\psi}{dv} = \omega + e B_1(a) + e^2 B_2(a) + \ldots . \qquad \ldots (3.5)$$

Here

$$A_1(a) = 0, \qquad B_1(a) = \frac{\alpha[2J_1(a) - a]}{2a\omega},$$

$$u_1(a, \psi, v) = 4 \frac{\Omega_1^2}{\Omega^2} \frac{\sin [(\Omega_1/\Omega)v]}{[\omega^2 - (\Omega_1^2/\Omega^2)}$$

$$+ \frac{a\omega}{2} \frac{\Omega}{\Omega_1} \frac{[\omega - 2(\Omega_1/\Omega)]}{[2\omega - (\Omega_1/\Omega)]} \cos\left(\frac{\Omega_1}{\Omega} v - \psi\right)$$

$$- \frac{a\omega}{2} \frac{\Omega}{\Omega_1} \frac{[\omega + 2(\Omega_1/\Omega)]}{[2\omega + (\Omega_1/\Omega)]} \cos\left(\frac{\Omega_1}{\Omega} v + \psi\right)$$

$$+ \frac{\alpha}{2\omega^2} \sum_{k=1}^{\infty} \frac{(-1)^k J_{2k+1}(a) \cos (2k + 1)\psi}{k(k + 1)},$$

$$A_2(a) = 0,$$

$$B_2(a) = \frac{\alpha^2}{2a\omega^3} \sum_{k=1}^{\infty} \frac{J_{2k+1}(a) J'_{2k+1}(a)}{k(k + 1)} - \frac{\alpha^2}{8a^2\omega^3} (2J_1(a) - a)^2$$

$$+ \frac{3\omega[\omega^2 - (\Omega_1^2/\Omega^2)]}{4[4\omega^2 - (\Omega_1^2/\Omega^2)]},$$

$J'_{2k+1}(a) = dJ_{2k+1}(a)/da$, and $J_k(a)$ is the Bessel's function of the kth order.

Thus, in the first approximation, the solution is given by

$$\eta = a \cos \psi \qquad \ldots (3.6)$$

where the amplitude a and phase ψ are given by

$$\frac{da}{dv} = 0, \quad \frac{d\psi}{dv} = \omega + \frac{\omega}{2a}[2J_1(a) - a] \qquad \ldots (3.7)$$

and, in the second approximation, the solution is obtained as

$$\eta = a \cos \psi + \frac{1}{2}\sum_{k=1}^{\infty} \frac{(-1)^k J_{2k+1}(a)}{k(k+1)} \cos (2k+1)\psi$$

$$+ 4e\frac{\Omega_1^2}{\Omega^2}\frac{1}{[\omega^2 - (\Omega_1^2/\Omega^2)]}\sin\left(\frac{\Omega_1}{\Omega}v\right)$$

$$+ e\frac{\omega a}{2}\frac{\Omega}{\Omega_1}\frac{[\omega - 2(\Omega_1/\Omega)]}{[2\omega - (\Omega_1/\Omega)]}\cos\left(\frac{\Omega_1}{\Omega}v - \psi\right)$$

$$- e\frac{\omega a}{2}\frac{\Omega}{\Omega_1}\frac{[\omega + 2(\Omega_1/\Omega)]}{[2\omega + (\Omega_1/\Omega)]}\cos\left(\frac{\Omega_1}{\Omega}v + \psi\right) \qquad \ldots (3.8)$$

where amplitude a and phase ψ are given by

$$\frac{da}{dv} = 0, \quad \frac{d\psi}{dv} = \omega + \frac{\omega}{2a}[2J_1(a) - a] + \frac{\omega}{2a}\sum_{k=1}^{\infty}\frac{J_{2k+1}(a)J'_{2k+1}(a)}{k(k+1)}$$

$$-\frac{\omega}{8a^2}[2J_1(a) - a]^2 + e^2\frac{3\omega}{4}\frac{[\omega^2 - (\Omega_1^2/\Omega^2)]}{[4\omega^2 - (\Omega_1^2/\Omega^2)]}. \qquad \ldots (3.9)$$

It may be observed from equations (3.7) and (3.9) that the amplitude of the oscillation remains constant even up to the second approximation. Moreover, it is observed that the main resonance occurs at $\omega = \pm(\Omega_1/\Omega)$ and parametric resonance at $\omega = \pm{}^1/_2(\Omega_1/\Omega)$.

Conclusion

We conclude that the non-linear rotational equations of motion of the planar oscillation of a satellite in an elliptic orbit under the influence of third body torques are non-integrable. By assuming the third body perturbation parameter ε of the order of the eccentricity $e(e \ll 1)$, we have found out the solutions by BKM method and have observed that amplitude remains constant even up to the second order of approximation. The main resonance occurs at $\omega = \pm(\Omega_1/\Omega)$ and parametric resonance at $\omega = \pm{}^1/_2(\Omega_1/\Omega)$.

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