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Noise free response of the interferometric antenna to gravitational radiation from pulsars

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Abstract. We present here a full calculation of the noise free response of a laser interferometric gravitational wave detector which is subjected to gravitational radiation from a continuous source. The observation time is taken to be of the order of a few months. The long observation time implies that the motion of the detector is important and must be included in the response as a modulation effect. For simplicity we consider only two motions of the Earth, namely, the rotation of the Earth about its axis and the orbital motion about the Sun. The orbit is assumed to be circular. We consider the detector to be situated and oriented arbitrarily on the Earth, except that we assume the arms of the detector must lie in the tangent plane to the Earth at the point where the detector is situated. The gravitational wave incident on the detector is assumed to be a plane wave having arbitrary direction and polarization. We also present here the computation of the quadrupole wave form of a typical continuous source—a pulsar—which is modelled as an almost spherical object of uniform density, spinning about an arbitrary axis with uniform angular velocity. We use techniques of spherical tensors and Gel'fand functions developed in the literature to compute the wave form.

Key words : gravitational radiation—pulsars

1. Introduction

Direct detection of gravitational radiation (GR) from astrophysical sources is currently one of the most challenging problems in science. It is important, therefore, that different radiative processes in astrophysics be explored and, at least, conservative estimates be obtained for the radiated power and the dimensionless amplitude of GR from such processes. This is more so for the purposes of the current highly sensitive, laser interferometric detectors like LIGO, VIRGO, AIGO etc. However, what is equally important for the analysis of data obtainable from these detectors is an analytical treatment of the problem in question. This is crucial because the GR signal will be buried deep within the noise of the detector system.

Therefore, even for the detection of a GR signal, there is a special need of problem-oriented algorithms which make maximum use of the analytical treatment of the problem under consideration.

We can broadly classify astrophysical sources of GR as continuous, burst type and stochastic. An archetypal example of continuous source of GR is a pulsar while an asymmetric supernova explosion is that of the burst type of source. Beside these, many other astrophysical sources of both kinds have been considered in the literature (Thorne 1987; Schutz 1989) and pulsars are one continuous kind. In a simple but plausible model, we imagine an asymmetric pulsar being deformed into a slightly non-axisymmetric shape as a result of some astrophysical process. Provided that the axis of rotation does not coincide with the direction of the angular momentum of the pulsar, the pulsar will execute torque-free precession and the resultant time-varying mass-quadrupole then becomes the sources of GR from such a pulsar (if the wobble angle θ between the angular momentum direction and the rotation axis is small then the radiation will be emitted primarily at ω and 2ω where $\omega = \Omega_{em} - \Omega_{pre}$, Ω_{em} being the electromagnetic frequency and Ω_{pre} the precession rate. The electromagnetic observations of pulsars have shown no evidence for precession) (Zimmermann & Szedenits Jr. 1979; Zimmermann 1980; Shapiro & Teukolsky 1985). Needless to say, it is of much current interest to see how non-axisymmetric structures may form in pulsars (However, it is worth noting here that there is to this date no observational evidence for substantially large asymmetric structures as borne out by the low spin-down rates of known milli-second pulsars. This suggest that the amplitude of GR from these pulsars are probably $\lesssim 10^{-26} - 10^{-28}$). Of particular interest are the glitches in the rotation periods of pulsars interpreted as the release of crustal distortion, a dynamical process responsible for as, an example, an earthquake. It has also been argued that observed gamma ray burst are also a result of neutron starquakes (Hurding 1991).

In constructing the prototypes of laser interferometric detectors, and studying the noise and system characteristic, GR signals of the above nature were looked for. Noteworthy here is the work by J. C. Livas who used the 1.5 meters MIT laser interferometer to conduct low sensitivity, all sky, all frequency search for periodic signals and also for narrowband single direction sky search towards the galactic centre. For this purpose he developed a formalism that takes into account frequency modulation of the monochromatic GR signal from the rotation and orbital motion of the Earth in the Solar System Barycentre (SSB) frame. This consideration of only frequency modulation restricts the analysis for observation times $\lesssim 30$ minutes (However, the same motion also amplitude modulates any GR signal since the detector has an anisotropic antenna pattern). The conclusion by Livas is that within the experimental uncertainties, there is no conclusive evidence for GR emitting pulsars towards the galactic centre (Livas 1987). Recently pulsar search was made using continuous 100 hours data from Garching prototype gravitational wave detector in the direction of 1987A supernova. This analysis puts a constraint on the dimensionless amplitude *h*, of gravitational wave $\sim 9 \times 10^{-21}$ possibly produced by a periodic source in the SN 1987A (Niebauer *et al.* 1993). Since the typical GR signal from a pulsar is expected to be weak, for getting an appreciable signal-to-noise ratio one needs long integration times (observation may last for a few days to a few months) (Schutz 1991). This implies that both frequency modulation (FM) as well as amplitude modulation (AM) of the signal are going to be the important effects in the detection of the pulsar signal. Therefore it is very important and necessary to compute the response of the detector. This work will be a precursor to later work where the question of pulsar search will be addressed.

In this paper, we first consider the case of an almost spherical object of uniform density spinning about some arbitrary axis—a model for a pulsar. We then specialise in the case of spheroid. Although the GR from spinning object was calculated by Zimmermann (Zimmermann & Szedenits Jr. 1979; Zimmermann 1980) using the principal moments of inertia of the object, we present here a calculation which use the elegant properties of the Gel'fand functions (Gel'fand, Minlos & Shapiro 1963). We use the formalism of symmetric trace free (STF) tensors to actually compute the moment of inertia of an object having small deviation from sphericity but with uniform density. The plan of the paper is as follows : In section 2 the quadrupole formula is applied to compute the GR from an almost spherically spinning object. We use the formalism developed by Dhurandhar & Tinto (1989) to calculate the response of the detector. In this formalism which is based on the Newmann-Penrose (NP) formalism (Newmann & Penrose 1962), the wave and the detector are represented by STF tensors of second rank and the response of the detector is the scalar product of these two tensors. In this situation the computation of the response involves several reference frames and transformations between them. Since the second rank STF tensors in a 3 dimensions span a five dimensional space as compared with the nine dimensional space spanned by general second rank tensors, considerable economy and elegance is achieved by using the Gel'fand functions which form an irreducible unitary representation of the rotation group. We make use of STF tensors and the Gel'fand functions in calculating the full noise free response of the detector. In section 3, we specialise to the case of a spheroid whose semi-major axis is inclined at an angle α_0 to the rotation axis. In section 4, we first describe the formalism developed by Dhurandhar and Tinto, and calculate the wave and detector tensors in the SSB frame. We also give the transformation to galactic coordinates where the source direction is given in terms of the galactic longitude and latitude. Finally, in section 5 the response of the detector is obtained by taking the scalar product between the wave and detector tensors in the SSB frame. The response function incorporates both frequency modulation (FM) and amplitude modulation (AM). Further, we discuss quantitatively the consequences of the FM and AM of the signal in the context of signal detection.

2. Gravitational radiation from a spinning almost spherical object

We consider an object which is almost a sphere and which is spinning about some axis with uniform angular velocity ω. We also assume that it has uniform density ρ_0. This object is supposed to be a simple model for a pulsar and the aim is to compute first, the GR in the transverse traceless (TT) gauge (Misner, Thorne & Wheeler 1973) and finally compute response of a detector situated on the Earth. The entire problem involves several parameters consisting of orientations, rotations etc. But in this part of the discussion we will only be concerned with the derivation of the perturbed metric tensor in the TT gauge, denoted by h_{ik}^{TT} in the wave frame which we shall define later in the text. The quadrupole formula will be used to compute the wave form.

In this discussion we will consider three frames :

(1) The body frame (x', y', z') in which the object is static.

(2) The pulsar frame (x_p, y_p, z_p) in which the object is spinning about the z_p axis.

(3) The wave frame (X, Y, Z) in which the wave travels in the positive Z direction, so that the transverse gravitational wave field has components only in the (X, Y) plane.

The quadrupole formula states that the gravitational wave amplitudes are given by (Thorne 1980)

$$h_{ik}(t, \bar{x}) = 2\frac{G}{c^4 r}\ddot{I}_{ik}(t - \frac{r}{c}, \bar{x}') \qquad \ldots (2.1)$$

where the source coordinates are primed and the field coordinates are unprimed, $r = |\bar{x} - \bar{x}'|$ and the dots represent derivatives with respect to time. The TT components of the metric are obtained by taking the TT components of the right-hand side. We carry out the computation in the following steps :

(A) Compute the inertia tensor in the body frame,
(B) Transform the inertia tensor to the pulsar frame,
(C) Further transform only the time dependent part of the inertia tensor to wave frame,
(D) Finally project out the TT components.

This procedure will obtain for us the required metric perturbation h_{ik}^{TT}.

A. Inertia tensor in the body frame

Consider an almost spherical body denoted by the equation

$$r = a[1 + \varepsilon(\theta, \phi)], \qquad \ldots (2.2)$$

where (r, θ, ϕ) are the polar coordinates in the body frame, a is the approximate radius and $\varepsilon(\theta, \phi)$ is the deviation in the direction (θ, ϕ) from the spherical shape. We assume that $\varepsilon(\theta, \phi) << 1$ and in our computations we only retain the first order term in ε. Assuming an uniform density ρ_0, we have

$$I_{ik} = \rho_0 \int r^2 n_i n_k dV, \qquad \ldots (2.3)$$

where $n_i = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ is the unit vector in the direction (θ, ϕ) and $dV = r^2 dr d\Omega$ is the volume element. Therefore,

$$I_{ik} = \rho_0 \int n_i n_k d\Omega \int_0^{a[1+\varepsilon(\theta,\phi)]} r^4 dr \qquad \ldots (2.4a)$$

$$= \frac{\rho_0 a^5}{5} \int [1 + \varepsilon(\theta, \phi)]^5 n_i n_k d\Omega. \qquad \ldots (2.4b)$$

Retaining only up to the first order terms in $\varepsilon(\theta, \phi)$, we have,

$$I_{ik} \simeq \frac{\rho_0 a^5}{5} \int n_i n_k d\Omega + \rho_0 a^5 \int \varepsilon(\theta, \phi) n_i n_k d\Omega. \qquad \ldots (2.5)$$

Using the identity $\int n_i n_k d\Omega = \frac{4\pi}{3}\delta_{ik}$, the first term yields the inertia tensor for a sphere is equal to $\frac{1}{5}Ma^2\delta_{ik}$, where $M = \frac{4\pi}{3}\rho_0 a^3$ is the unperturbed mass of the sphere.

Hence we may write,

$$I_{ik} = \frac{1}{5}Ma^2\delta_{ik} + \rho a^5 J_{ik}, \qquad \text{... (2.6a)}$$

where

$$J_{ik} = \int \varepsilon(\theta, \phi) n_i n_k d\Omega. \qquad \text{... (2.6b)}$$

We will express J_{ik} in terms of spherical tensors. In what follows we will make use of the formalism found in Thorne's work on multiple expansions of GR (Thorne 1980). The spherical tensors or STF tensors and Gel'fand functions have elegant properties which can be used to our advantage, since several rotations are involved in the computation of the response of the detector. This formalism was found to be very useful in studying the response of detectors as functions of orientation *i.e.* in obtaining the antenna pattern and solving the inverse problem in its simplest form for five, four and three detectors situated in the same place (Dhurandhar & Tinto 1989; Tinto & Dhurandhar 1989).

We begin by expanding $\varepsilon(\theta, \phi)$ in terms of spherical harmonics :

$$\varepsilon(\theta, \phi) = \sum_{l,m} \varepsilon_{lm} Y^{lm}(\theta, \phi), \qquad \text{... (2.7a)}$$

where ε_{lm} are constants which are related to the $\varepsilon(\theta, \phi)$ by the inverse formulae :

$$\varepsilon_{lm} = \int \varepsilon(\theta, \phi) Y^{*lm}(\theta, \phi) d\Omega, \qquad \text{... (2.7b)}$$

where the integration is over a unit sphere. We note that

$$\varepsilon_{l,-m} = (-1)^m \varepsilon^*_{lm}. \qquad \text{... (2.7c)}$$

Here we will only need to invoke second rank STF tensors since the inertia tensor is of the second rank. The STF tensors of rank two span a five dimensional vector space and a convenient basis with useful orthogonality properties is given by the following definition,

$$\mathcal{Y}^{2m}_{ik} n_i n_k = Y^{2m}(\theta, \phi). \qquad \text{... (2.8a)}$$

Here repeated indices imply summation, $i,k = 1, 2, 3$ and $m = -2, -1, 0, 1, 2$. The inverse formulae are given by

$$\int Y^{2m} n_i n_k d\Omega = \frac{8\pi}{15} \mathcal{Y}^{2m}_{ik}. \qquad \text{... (2.8b)}$$

$\mathcal{Y}^{2m}_{ik}$ satisfy the orthogonality condition,

$$\mathcal{Y}^{2m}_{ik} \mathcal{Y}^{2n*}_{ik} = \frac{15}{8\pi}\delta_{mn}. \qquad \ldots (2.9)$$

The five tensors have been listed in appendix A.

For computing J_{ik} we need to expand $n_i n_k$ on the basis of Y^{lm}. We have the relation

$$n_i n_k = \frac{\sqrt{4\pi}}{3}\delta_{ik} Y^{*\,00} + \frac{8\pi}{15}\sum_{m=-2}^{2} Y^{*\,2m}\mathcal{Y}^{2m}_{ik}. \qquad \ldots (2.10)$$

Using equations (2.7a), (2.10) and the orthogonality properties of Y^{lm}'s, we have from equation (2.6b),

$$J_{ik} = \frac{\sqrt{4\pi}}{3}\delta_{ik}\varepsilon_{00} + \frac{8\pi}{15}\sum_{m=-2}^{2}\varepsilon_{2m}\mathcal{Y}^{2m}_{ik}. \qquad \ldots (2.11)$$

The full inertia tensor expressed in therms of the ε_{lm} is,

$$I_{ik} = \frac{1}{5}Ma^2\left[\left(1 + \frac{5}{\sqrt{4\pi}}\varepsilon_{00}\right)\delta_{ik} + 2\sum_{m=-2}^{2}\varepsilon_{2m}\mathcal{Y}^{2m}_{ik}\right]. \qquad \ldots (2.12)$$

We note that the ε_{00} term corresponds to the extra mass which has entered through the perturbation. However, this is of little consequence to further computations as this term remains invariant under rotations and so is time independent and will not contribute to the gravitational radiation. With this in mind one needs to focus only on the second term of (2.12) namely,

$$\delta I_{ik} = \frac{2}{5}Ma^2\sum_{m=-2}^{2}\varepsilon_{2m}\mathcal{Y}^{2m}_{ik}, \qquad \ldots (2.13)$$

which will contribute to the GR.

B. Calculation of the GR

Let the object rotate about the z_p axis with uniform angular velocity ω in the positive direction, *i.e.* the vector $\vec{\omega}$ is along the positive z_p axis. Then the inertia tensor I_p in the pulsar frame is related to the inertia tensor I_b in the body frame by the matrix formula

$$I_p = R^T I_b R, \qquad \ldots (2.14)$$

where the matrix R is given by

$$R(\omega t) = \begin{bmatrix} \cos\omega t & \sin\omega t & 0 \\ -\sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{bmatrix}. \qquad \ldots (2.15)$$

We need to only consider δI and hence in the pulsar frame,

$$\delta I_p = \frac{2}{5} M a^2 \sum_{m=-2}^{2} \varepsilon_{2m} [R^T(\omega t) \mathcal{Y}^{2m} R(\omega t)]. \quad \ldots (2.16)$$

For going over to the wave frame we need another rotation matrix. But if we choose the x axis in the pulsar frame to be the X axis in the wave frame, then only a single rotation through an angle i about the common x axis is needed to transform the pulsar frame to the wave frame (see figure 1). Actually the z_p axis and Z axis will be fixed, since they correspond to the rotation axis of the pulsar and the direction to the pulsar respectively. We choose the x_p axis and X axis perpendicular to the (z_p, Z) plane. The y_p and Y axes are chosen so that (x_p, y_p, z_p) and (X, Y, Z) form right handed triads. The matrix for rotation through an angle i about the x axis is given by

$$R_x(i) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos i & \sin i \\ 0 & -\sin i & \cos i \end{bmatrix}, \quad \ldots (2.17)$$

and hence the inertia tensor in the wave frame is

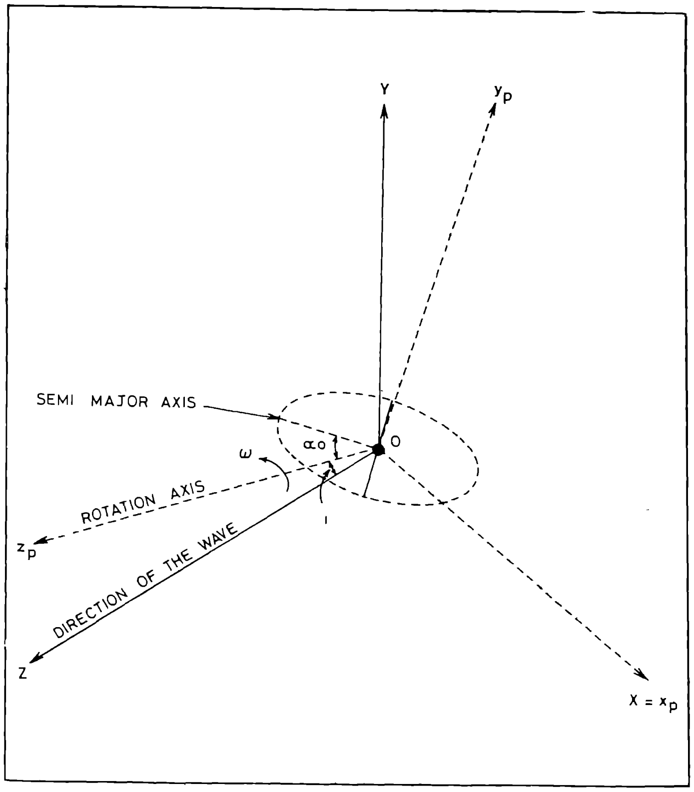

Figure 1. The figure shows the orientation of the pulsar frame (x_p, y_p, z_p) with respect to wave frame (X, Y, Z) The angle ι denotes the inclination of pulsar's spin axis with respect to the sky plane. The axis of symmetry of the pulsar makes an angle α_0 with the spin axis.

$$\delta I_{\rm w} = \frac{2}{5} M a^2 \sum_{m=-2}^{2} \varepsilon_{2m} R_{\rm x}^{\rm T}(i)\, R_{\rm z}^{\rm T}(\omega t) \mathscr{Y}^{2m} R_{\rm z}(\omega t) R_{\rm x}(i). \qquad \ldots (2.18)$$

We now follow Goldstein's convention for Euler angles θ, ϕ, ψ where ϕ is the rotation about z axis, then θ is the rotation about the new x axis, and finally ψ is the rotation about the still newer z axis. All the rotations are in the positive sense. The above equations show that final rotation $R_{\rm z}(\omega t) R_{\rm x}(i)$ has $\phi = \omega t$ and $\theta = i$. We now need to know how $\mathscr{Y}^{2m}$ transform under a rotation $R(\theta, \phi, \psi)$. We merely state the formula,

$$R^{\rm T}(\theta, \phi, \psi) \mathscr{Y}^{2m} R(\theta, \phi, \psi) = \sum_{n=-2}^{2} T_{mn}^{2}(\theta, \phi, \psi) \mathscr{Y}^{2n}. \qquad \ldots (2.19)$$

The functions $T_{mn}^{2}(\theta, \phi, \psi)$ appearing in equation (2.19) are just the Gel'fand functions and are related to spin weighted spherical harmonics, Jacobi polynomials etc. (Gel'fand, Minlos & Shapiro 1963; Goldberg *et al.* 1967). These functions again provide a representation of the rotation group which is unitary and irreducible. They are given by

$$T_{mn}^{l}(\theta, \phi, \psi) = e^{-im\phi} P_{mn}^{l}(\cos\theta) e^{-in\psi}, \qquad \ldots (2.20)$$

where

$$P_{mn}^{l}(\mu = \cos\theta) = \frac{(-10)^{l-m} i^{n-m}}{2^{l}(l-m)!} \sqrt{\frac{(l-m)!\,(l+n)!}{(l+m)!\,(l-n)!}} (1-\mu)^{(m-n)/2} (1+\mu)^{-(m+n)/2}$$

$$\times \frac{d^{l-n}}{d\mu^{l-n}} [(1-\mu)^{l-m} (1+\mu)^{l+m}], \; -l \le m, n \le l. \qquad \ldots (2.21)$$

We will only need the functions for $l = 2$. The relevant 5×5 matrix $P_{mn}^{2}(\cos\theta)$ has been listed in appendix B.

In terms of these functions we may rewrite equation (2.18). Thus,

$$\delta I_{\rm w} = \frac{2}{5} M a^2 \sum_{m,n} \varepsilon_{2m} T_{mn}(i, \omega t, 0) \mathscr{Y}^{2n} \qquad \ldots (2.22a)$$

$$= \frac{2}{5} M a^2 \sum_{m,n} \varepsilon_{2m} e^{-im\omega t} P_{mn}(\cos i) \mathscr{Y}^{2n}, \qquad -2 \le m, n \le 2. \qquad \ldots (2.22b)$$

Here, we have dropped the superscript 2 of T_{mn}^{2} since it never changes. To get the wave tensor, we have to carry out further operations of taking the second derivative with respect to time and projecting out the TT part. The TT part of a tensor for a direction n_i is given by

$$I_{ik}^{\rm TT} = \mathscr{P}_{im} \mathscr{P}_{kn} I_{mn} - \frac{1}{2} \mathscr{P}_{ik} (\mathscr{P}_{mn} I_{mn}), \qquad \ldots (2.23)$$

where $\mathscr{P}_{ik} = (\delta_{ik} - n_i n_k)$, n_i being the direction of the propagation of the wave.

In our case $n_I = (0, 0, 1)$, (we use capital indices for the wave frame) and we need to project out $\mathcal{Y}^{2n}$ in (2.22). It is easy to see that

$$(\mathcal{Y}^{2,\pm2})^{TT} = \mathcal{Y}^{2,\pm2}, \qquad \ldots (2.24a)$$

and

$$(\mathcal{Y}^{2m})^{TT} = 0, \qquad m = -1, 0, 1. \qquad \ldots (2.24b)$$

These relations simplify our formulae to

$$\ddot{I}^{TT} = \delta\ddot{I}^{TT}$$

$$= -\frac{2}{5}Ma^2\omega^2 \sum_{m=-2}^{2} m^2\varepsilon_{2m}\left[T_{m2}(i, \omega t, 0)\mathcal{Y}^{22} + T_{m,-2}(i, \omega t, 0)\mathcal{Y}^{2,-2}\right] \ldots (2.25)$$

We have used here the relations,

$$T_{m2}(i, \omega t, 0) = -im\omega T_{m2}(-i, \omega t, 0).$$

At a distance r along the Z axis from the origin the perturbed metric amplitude in the TT gauge can be written as

$$h_{IK}^{TT}\left(t - \frac{r}{c}\right) = -\frac{4}{5}\frac{G}{c^4}\frac{Ma^2\omega^2}{r}\left(\beta + \mathcal{Y}_{IK}^{22} + \beta - \mathcal{Y}_{IK}^{2-2}\right), \qquad \ldots (2.26)$$

where

$$\beta_\pm = \sum_m m^2\varepsilon_{2m}T_{m,\pm2}(i, \omega t, 0). \qquad \ldots (2.27)$$

We observe that $\beta_\pm \times (-\frac{4}{5}\frac{G}{c^4}\frac{Ma^2\omega^2}{r})$ are just the amplitudes of the positive and negative handed circularly polarizations. Therefore, the above equation expresses the wave as a linear combination of left and right handed circularly polarized waves. The linear polarizations are obtained by taking the XX and XY components of equation (2.26)

$$h_{XX}^{TT} = -\frac{1}{5}\frac{G}{c^4}\frac{Ma^2\omega^2}{r}(\beta_+ + \beta_-)\sqrt{\frac{15}{2\pi}}, \qquad \ldots (2.28a)$$

$$h_{XY}^{TT} = -\frac{1}{5}\frac{G}{c^4}\frac{Ma^2\omega^2}{r}(\beta_+ - \beta_-)\sqrt{\frac{15}{2\pi}}. \qquad \ldots (2.28b)$$

We need only to simplify the expressions for $\beta_\pm$. Using the fact that $\varepsilon_{2,-m} = (-1)^m\varepsilon_{2m}^*$,

$$\beta_+ \pm \beta_- = \sum_{m=1}^{2} m^2[\varepsilon_{2m}(T_{m2} \pm T_{m,-2}) + (-1)^m \varepsilon_{2m}^*(T_{-m,2} \pm T_{-m,-2})]. \quad \ldots (2.29)$$

We observe that $m = 2$ term gives the $2\omega t$ contribution, while the $m = 1$ term gives the ωt dependence. $m = 0$ term is absent corresponding to ε_{20} since this is just the 'dipole' term which is independent of time. Explicitly,

$$\beta_+ + \beta_- = \sin 2i \, \mathrm{Im}\,(\varepsilon_{21}e^{-i\omega t}) + 4(1 + \cos^2 i)\, \mathrm{Re}\,(\varepsilon_{22}e^{-i2\omega t}), \quad \ldots (2.30s)$$

$$i(\beta_+ - \beta_-) = -8 \cos i \, \mathrm{Im}\,(\varepsilon_{22}e^{-i2\omega t}) + 2 \sin i \, \mathrm{Re}\,(\varepsilon_{21}e^{-i\omega t}). \quad \ldots (2.30b)$$

In the next section, we apply these results to a spheroid rotating about an axis inclined at an angle α_0 with respect to the rotation axis.

3. Gravitational radiation from a spinning spheroid

We consider a spheroid whose two semi-minor axes are equal of length a and a semi-major axis of length $a(1 + \delta)$. The quantity δ is called the ellipticity of the spheroid. We assume that it's semi-major axis is inclined to the spin axis by an angle α_0.

From the results of the previous section it is sufficient to compute the two quantities ε_{21} and ε_{22} for this object in the body frame in order to obtain the h_{IK}^{TT}. We do the calculation in two steps :

(i) We first compute the ε_{2m} for the principal axes of the spheroid.

(ii) Then we transform ε_{2m} to the body frame in which the z axis is the spin axis.

The equation of the spheroid in the principal axes is

$$x^2 + y^2 + \frac{z^2}{(1 + \delta)^2} = a^2. \quad \ldots (3.1)$$

Transforming to polar coordinates and retaining up to the first order terms in δ, we have

$$r = a(1 + \delta \cos^2 \theta). \quad \ldots (3.2)$$

The quantity $\varepsilon(\theta, \phi)$ is then just equal to $\delta \cos^2 \theta$.

From equation (2.7b), we compute ε_{2m}. The result is as follows :

$$\varepsilon_{20} = \sqrt{\frac{2}{3}} \sqrt{\frac{8\pi}{15}}\, \delta, \quad \varepsilon_{2m} = 0, \quad m \neq 0. \quad \ldots (3.3)$$

At $t = 0$, we take the semi-major axis of the spheroid to lie in the (x_b, z_b) plane inclined at an angle α_0 with the z_b axis and $(\pi/2) - \alpha_0$ with the x_b axis (see figure 2). That is to obtain the inertia tensor in the body axes from the principal axes, we apply the rotation

$$I_b = R_y^T(\alpha_0) I^{\text{principal axes}} R_y(\alpha_0), \quad \ldots (3.4)$$

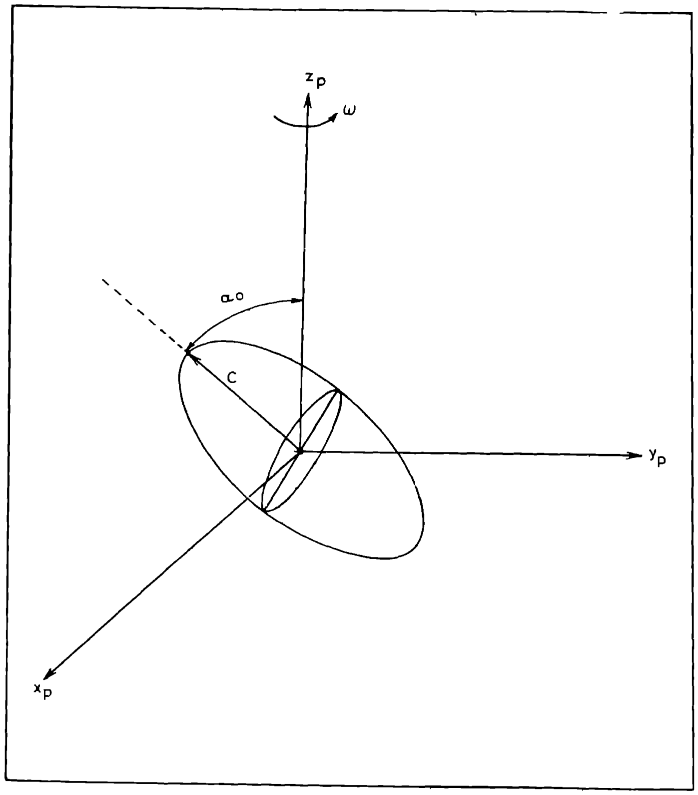

Figure 2. The figure shows orientation of an spheroid in pulsar coordinate system (x_p, y_p, z_p) The angle α_0 is the angle between the axis of symmetry of the spheroid and the z_p axis.

where

$$R_y(\alpha_0) = \begin{bmatrix} \cos\alpha_0 & 0 & -\sin\alpha_0 \\ 0 & 1 & 0 \\ \sin\alpha_0 & 0 & \cos\alpha_0 \end{bmatrix}. \quad \ldots (3.5)$$

A rotation of α_0 about y axis is equivalent to a rotation through Euler angles $\phi = \pi/2$, $\theta = \alpha_0$, $\psi = -\pi/2$. Hence the tensor $\mathscr{Y}^{2m}$ transforms as,

$$\mathscr{Y}^{2m} \rightarrow T_{mn}\left(\frac{\pi}{2}, \alpha_0, -\frac{\pi}{2}\right)\mathscr{Y}^{2n},$$

under this rotation. This provides the transformation law for ε_{2m}

$$\varepsilon_{2n}^{\text{body}} = \sum_m T_{mn}\left(\frac{\pi}{2}, \alpha_0, -\frac{\pi}{2}\right)\varepsilon_{2m}^{\text{principal axes}}, \quad \ldots (3.6a)$$

$$= \sum_m i^{n-m} P_{mn}(\cos\alpha_0)\varepsilon_{2m}^{\text{principal axes}}. \quad \ldots (3.6b)$$

Using the expression for P_{mn} from appendix B and equation (3.3) we therefore have in the body frame,

$$\varepsilon_{21} = \sqrt{\frac{2\pi}{15}}\,\delta \sin 2\alpha_0, \qquad \ldots (3.7a)$$

$$\varepsilon_{22} = \sqrt{\frac{2\pi}{15}}\,\delta \sin^2 \alpha_0, \qquad \ldots (3.7b)$$

where we have dropped the superscript 'body', from the ε_{mn}'s.

Using (2.28), (2.29) and (3.7a, b) we obtain the wave amplitudes as given below :

$$h^{TT}_{XX} = -\frac{8}{5}\frac{G}{c^4}\frac{Ma^2\omega^2}{r}\,\delta \sin^2 \alpha_0 \left(\frac{1+\cos^2 i}{2}\right) \cos 2\omega t + \frac{1}{5}\frac{G}{c^4}\frac{Ma^2\omega^2}{r}\,\delta \sin 2\alpha_0 \sin 2i \sin \omega t, \qquad \ldots (3.8a)$$

$$h^{TT}_{XY} = -\frac{8}{5}\frac{G}{c^4}\frac{Ma^2\omega^2}{r}\,\delta \sin^2 \alpha_0 \cos i \sin 2\omega t - \frac{2}{5}\frac{G}{c^4}\frac{Ma^2\omega^2}{r}\,\delta \sin 2\alpha_0 \sin i \cos \omega t. \qquad \ldots (3.8b)$$

We note that the $2\omega t$ term is dominant in the expression. Further point to note is that the mass entering into these formulae is the mass of the sphere of the radius a. The actual mass of the spheroid is greater by the factor δ. However, this correction will only introduce a term of $O(\delta^2)$ in the formulae. Therefore, the mass M appearing in the formulae can be taken as the mass of the object, since our aim is to obtain results to the first order in δ.

4. Wave and detector tensor in the solar system barycentric frame

A. The Dhurandhar-Tinto formalism

In this section, we first briefly review the formalism set up by Dhurandhar and Tinto and then in the later subsections apply it to the problem at hand.

Consider a plane gravitational wave travelling along the Z axis. In the TT gauge the metric perturbations h^{TT}_{IK} has components lying only in the (X, Y) plane. In the wave axes we have two amplitudes which characterise the wave as follows :

$$h_+ = h^{TT}_{XX} = -h^{TT}_{YY}, \quad \text{and} \quad h_\times = h^{TT}_{XY} = h^{TT}_{YX}.$$

In this formalism the wave tensor denoted by W is defined by

$$W = \frac{1}{2}h_+(e_X \otimes e_X - e_Y \otimes e_Y) + \frac{1}{2}h_\times(e_X \otimes e_Y + e_Y \otimes e_X), \qquad \ldots (4.1)$$

where e_X and e_Y are unit vectors in the X and Y direction respectively. We observe that W is a STF tensor.

The detector too can be represented by a STF tensor D, the form of which differs for the types of detectors; the interferometer and the bar. For an interferometer with its arms in the directions of the unit vector n_1 and n_2 the detector tensor D^{INT} is defined as

$$D^{\mathrm{INT}} = n_1 \otimes n_1 - n_2 \otimes n_2. \qquad \ldots (4.2)$$

The detector axes (x, y, z) are chosen so that the arms of the interferometer lie in the (x, y) plane and the x axis bisects the arms. For an interferometer with its arms at right angles, we have,

$$n_1 = \frac{1}{\sqrt{2}}(e_x + e_y), \qquad \ldots (4.3a)$$

$$n_2 = \frac{1}{\sqrt{2}}(e_x - e_y). \qquad \ldots (4.3b)$$

where e_x and e_y are unit vectors in the x and y directions.
Therefore, the detector tensor for the interferometer can be written as

$$D^{\mathrm{INT}} = e_x \otimes e_y + e_y \otimes e_x. \qquad \ldots (4.4)$$

For a bar detector whose longitudinal axis is the direction n, the detector tensor D^{BAR} is given by

$$D^{\mathrm{BAR}} = n \otimes n.$$

The response of the interferometric detector $\delta l/l$ where δl is the change in the arm length, l the length of the arm, is then simply given by the scalar product between the wave and detector tensors. For the bar the response is proportional to the scalar product. Here we limit ourselves to the case of the interferometer. The response $R(t)$ is given by

$$R(t) = W^{ij} D_{ij}, \qquad \ldots (4.5)$$

where the superscript 'INT' has been dropped from D.

B. The wave tensor in the SSB frame

We choose the Solar System Barycentre frame (x_b, y_b, z_b) such that orbital plane of the Earth lies in the (x_b, y_b) plane. Therefore the orbital angular velocity vector $\vec{\omega}_{\mathrm{orb}}$ points towards the positive z_b direction. We assume a circular orbit for the Earth around the Sun with the Sun at its centre. The SSB frame is obtained by rotating the wave frame by the Euler angles θ, ϕ, ψ (see figure 3). The wave tensor in the SSB frame is obtained as follows :

$$W_{\mathrm{SSB}} = R^{\mathrm{T}}_{\mathrm{SSB}}(\theta, \phi, \psi) W R_{\mathrm{SSB}}(\theta, \phi, \psi), \qquad \ldots (4.6)$$

where $R_{\mathrm{SSB}}(\theta, \phi, \psi)$ is the orthogonal transformation matrix connecting (X, Y, Z) to

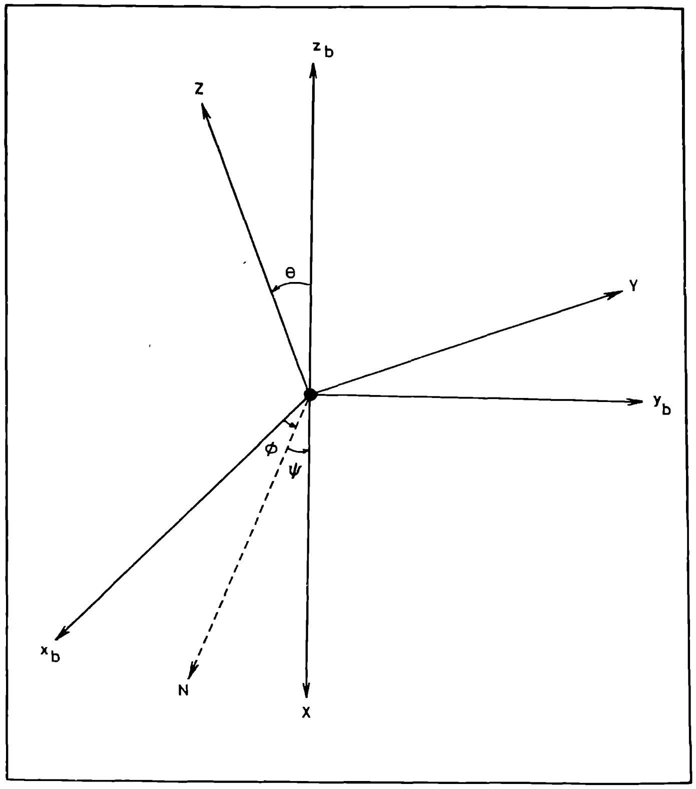

Figure 3. The figure shows the orientation of the wave frame (X, Y, Z) with respect to the Solar System Barycentre (SSB) frame (x_b, y_b, z_b). The orientation of the wave frame is specified by the three Euler angles θ, φ, ψ, which are needed to rotate the wave frame to SSB frame. The Angles (θ, φ) give the direction of incoming wave in the SSB frame, while ψ represents the polarization angle.

(x_b, y_b, z_b) axes. We give below the matrix R_{SSB} and also list the components of the wave tensor in the SSB frame.

$$R_{SSB} = \begin{bmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi & \sin\theta\sin\phi \\ \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi & -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi & -\sin\theta\cos\phi \\ \sin\theta\sin\psi & \sin\theta\cos\psi & \cos\theta \end{bmatrix}, \quad \ldots (4.7)$$

$$\begin{aligned} W_{x_b x_b} = \tfrac{1}{2}\, & [(\cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi)^2 \\ & - (\cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi)^2]\, h_+ \\ & + [(\cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi) \\ & \times (\cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi)]\, h_\times\,, \end{aligned} \quad \ldots (4.8)$$

$$\begin{aligned} W_{y_b y_b} = \tfrac{1}{2}\, & [(-\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi)^2 \\ & - (-\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi)^2]\, h_+ \\ & + [(-\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi) \\ & \times (-\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi)]\, h_\times\,, \end{aligned} \quad \ldots (4.9)$$

$$W_{z_b z_b} = \tfrac{1}{2}\,[-\sin^2\theta\cos 2\phi]h_+ + \tfrac{1}{2}\,[-\sin^2\theta\sin 2\phi]h_\times, \qquad \ldots (4.10)$$

$$\begin{aligned} W_{x_b y_b} = \tfrac{1}{2}\ &[(\cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi) \\ &\times(-\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi) \\ &-(\cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi) \\ &\times(-\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi)]\ h_+ \\ &+\tfrac{1}{2}\ [(\cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi) \\ &\times(-\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi) \\ &+(-\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi) \\ &\times(\cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi)]\ h_\times, \end{aligned} \qquad \ldots (4.11)$$

$$\begin{aligned} W_{x_b z_b} = \tfrac{1}{2}\ &[(\cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi)(\sin\theta\sin\phi) \\ &+(\cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi)(\sin\theta\cos\phi)]\ h_+ \\ &+\tfrac{1}{2}\ [(\cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi)(\sin\theta\sin\phi) \\ &-(\cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi)(\sin\theta\cos\phi)]\ h_\times, \end{aligned} \qquad \ldots (4.12)$$

$$\begin{aligned} W_{y_b z_b} = \tfrac{1}{2}\ &[(-\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi)(\sin\theta\sin\phi) \\ &+(-\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi)(\sin\theta\cos\phi)]\ h_+ \\ &+\tfrac{1}{2}\ [(-\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi)(\sin\theta\sin\phi) \\ &-(-\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi)(\sin\theta\cos\phi)]\ h_\times. \end{aligned} \qquad \ldots (4.13)$$

The angles θ, ϕ give the direction of incoming wave in the SSB frame. This SSB frame is nothing but astronomer's ecliptic coordinate system. The angle θ is the ecliptic colatitude ($\beta = \pi/2 - \theta$) and ϕ is the ecliptic longitude (λ). We give below the relations between galactic longitude l, galactic latitude b, the equatorial coordinates, right ascension α, declination δ and ecliptic coordinates λ and β (Green 1985) :

$$\delta = \sin^{-1}\{\sin\delta_G\sin b + \cos\delta_G\cos b\cos(\theta_p - l)\}, \qquad \ldots (4.14)$$

$$\alpha = \tan^{-1}\left\{\frac{\cos b\cos(\theta_p - l)}{\sin\delta_G\sin b - \sin\delta_G\cos b\cos(\theta_p - l)}\right\}. \qquad \ldots (4.15)$$

The (α_G, δ_G) are the galactic North pole coordinates and θ_p is the position of the galactic centre in the equatorial coordinates given at epoch 1950.0. Their values are given below as,

$$\alpha_G = 192^\circ.25, \qquad \delta_G = 27^\circ.4, \qquad \theta_p = 123^\circ.$$

The ecliptic coordinates of the pulsar are related to the α and δ by

$$\lambda = \tan^{-1}\left\{\frac{\sin\alpha\cos\kappa + \tan\delta\sin\kappa}{\cos\alpha}\right\}, \qquad \ldots (4.16)$$

$$\beta = \sin^{-1}\{\cos\delta\cos\kappa - \cos\delta\sin\kappa\sin\alpha\}. \qquad \ldots (4.17)$$

The obliquity of ecliptic is $\kappa = 23^\circ.441$. The source direction if given in terms of l and b can be connected to the SSB coordinates θ and ϕ by using the above relations.

C. The detector tensor in the SSB frame

We assume that the arms of the detector lie in the tangent plane to the Earth at the site of the detector. Otherwise the orientation as well as the position of the detector is allowed to be arbitrary. Two angles are needed to specify the position of the detector. This fixes the tangent plane in which the detector's arm must lie, leaving only one degree of the freedom of rotation in the plane. The orientation is then fully specified if we specify one angle with respect to some direction fixed in the tangent plane. In order to do this systematically we define two set of axes, one connected with detector and the other with the Earth.

1. DETECTOR AXES (x, y, z)

With respect to the detector, the axes are chosen so that the detector arms lie in the (x, y) plane with the x axis bisecting the two arms of the interferometer (Schutz & Tinto 1987).

In relation to the Earth the z axis points to the zenith with the (x, y) plane tangent to the Earth at the site of the detector. The x, y axes are chosen so that the x axis makes an angle γ with the local meridian.

2. EARTH AXES (x_E, y_E, z_E)

The z_E axis is chosen as the axis of the rotation of the Earth. The x_E axis coincides with that of the x axis of the SSB frame, *i.e.* the x_b axis. The SSB and the Earth axes are connected by a single rotation angle κ about their common x axis (see figure 4).

In order to describe the detector orientation completely we need two more angles related to its position on the Earth. Let α be the angle the line joining the centre of the Earth to the detector makes with the spin-axis of the Earth, measured from the North pole. Hence α is just the co-latitude. Let β be the angle between the plane containing the detector position, the centre of the Earth and the z_E axis, and the (x_E, z_E) plane. The angle β is just the azimuthal angle which keeps changing as the Earth rotates. Thus $\beta = \beta_0 + \omega_{rot} t$, where ω_{rot} is the angular velocity of the rotation of the Earth, and β_0 the value of β at $t = 0$. We now write the detector tensor in the Earth frame. It is given by the equation

$$D_{\text{Earth}} = C^{\text{T}} D_{\text{detector}} C, \qquad \ldots (4.18)$$

where C is the orthogonal matrix of transformation given by

$$C = \begin{bmatrix} \cos\alpha\cos\beta\cos\gamma - \sin\beta\sin\gamma & -\cos\alpha\cos\beta\sin\gamma - \sin\beta\cos\gamma & \sin\alpha\cos\beta \\ \cos\alpha\sin\beta\cos\gamma + \cos\beta\sin\gamma & -\cos\alpha\sin\beta\sin\gamma + \cos\beta\cos\gamma & \sin\alpha\sin\beta \\ -\sin\alpha\cos\gamma & \sin\alpha\sin\gamma & \cos\alpha \end{bmatrix} \qquad \ldots (4.19)$$

The components of the detector tensor in the Earth frame are as given below :

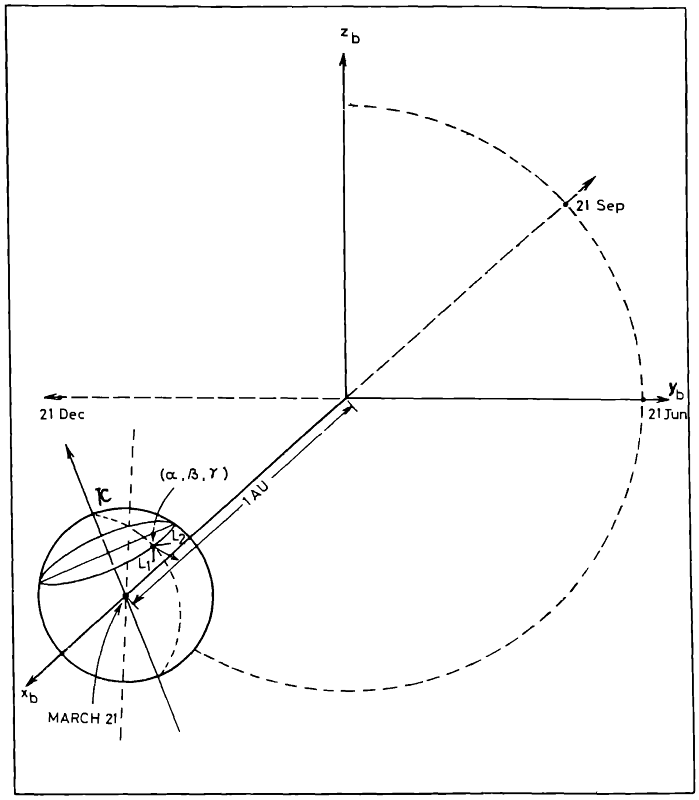

Figure 4. The figure shows orientation of the detector on the Earth. The detector orientation is specified by the angles (α, β, γ) with respect to the Earth frame (x_E, y_E, z_E) with z_E as spin axis of the Earth. Here α is the colatitude, β is the current longitude $(\beta = \beta_0 + \omega_{rot}t)$ and γ is the angle the x axis of the detector makes with the local meridian. The Earth's spin axis has eclipticity κ (the equatorial plane of the Earth is inclined by an angle κ with respect to orbital plane of the Earth). These angles specify the orientation of detector in SSB frame The SSB frame and the Earth frame have a common x axis. The x axis of the SSB frame is taken to lie along the line joining the Sun to the Earth on 21 March (Vernal equinox). The z axis of the SSB frame is normal to the orbital plane of the Earth.

$$D_{x_E x_E} = 2(\cos\alpha\cos\beta\cos\gamma - \sin\beta\sin\gamma)(-\cos\alpha\cos\beta\sin\gamma - \sin\beta\cos\gamma), \qquad \ldots (4.20)$$

$$D_{y_E y_E} = 2(\cos\alpha\sin\beta\cos\gamma + \cos\beta\sin\gamma)(-\cos\alpha\sin\beta\sin\gamma + \cos\beta\cos\gamma), \qquad \ldots (4.21)$$

$$D_{z_E z_E} = (-\sin^2\alpha\sin 2\gamma), \qquad \ldots (4.22)$$

$$D_{x_E y_E} = [(\cos\alpha\cos\beta\cos\gamma - \sin\beta\sin\gamma)(-\cos\alpha\sin\beta\sin\gamma + \cos\beta\cos\gamma) + (-\cos\alpha\cos\beta\sin\gamma - \sin\beta\cos\gamma)(\cos\alpha\sin\beta\cos\gamma + \cos\beta\sin\gamma)], \qquad \ldots (4.23)$$

$$D_{x_E z_E} = [(\cos\alpha\cos\beta\cos\gamma - \sin\beta\sin\gamma)(\sin\alpha\sin\gamma) + (-\cos\alpha\cos\beta\sin\gamma - \sin\beta\cos\gamma)(-\sin\alpha\cos\gamma)], \qquad \ldots (4.24)$$

$$D_{y_E z_E} = [(\cos\alpha\sin\beta\cos\gamma + \cos\beta\sin\gamma)(\sin\alpha\sin\gamma) + (-\cos\alpha\sin\beta\sin\gamma + \cos\beta\cos\gamma)(-\sin\alpha\cos\gamma)]. \qquad \ldots (4.25)$$

The spin axis of the Earth makes an angle κ with z_b axis of the SSB frame (*i.e.* equatorial plane of the Earth is inclined by an angle κ with respect to the orbital plane). So in order to get the detector tensor in the SSB frame one more rotation namely, κ rotation is needed. The rotation matrix is

$$R(\kappa) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\kappa & -\sin\kappa \\ 0 & \sin\kappa & \cos\kappa \end{bmatrix}. \qquad \ldots (4.26)$$

The detector tensor components in the SSB frame are listed below :

$$D_{x_b x_b} = 2(\cos\alpha\cos\beta\cos\gamma - \sin\beta\sin\gamma)(-\cos\alpha\cos\beta\sin\gamma - \sin\beta\cos\gamma),$$

$$\begin{aligned} D_{y_b y_b} = {} & \cos^2\kappa[2(\cos\alpha\sin\beta\cos\gamma + \cos\beta\sin\gamma) \\ & \times(-\cos\alpha\sin\beta\sin\gamma + \cos\beta\cos\gamma)] \\ & + \sin^2\kappa[-\sin^2\alpha\sin 2\gamma] \\ & + (\sin 2\kappa)\,[(\cos\alpha\sin\beta\cos\gamma + \cos\beta\sin\gamma)(\sin\alpha\sin\gamma) \\ & + (-\cos\alpha\sin\beta\sin\gamma + \cos\beta\cos\gamma)(-\sin\alpha\cos\gamma)], \end{aligned} \qquad \ldots (4.27)$$

$$\begin{aligned} D_{z_b z_b} = {} & \sin^2\kappa[2(\cos\alpha\sin\beta\cos\gamma + \cos\beta\sin\gamma) \\ & \times(-\cos\alpha\sin\beta\sin\gamma + \cos\beta\cos\gamma)] \\ & + \cos^2\kappa[-\sin^2\alpha\sin 2\gamma] - (\sin 2\kappa)\,[(\cos\alpha\sin\beta\cos\gamma + \cos\beta\sin\gamma) \\ & \times(\sin\alpha\sin\gamma) + (-\cos\alpha\sin\beta\sin\gamma + \cos\beta\cos\gamma) \\ & \times(-\sin\alpha\cos\gamma)], \end{aligned} \qquad \ldots (4.28)$$

$$\begin{aligned} D_{x_b y_b} = {} & \cos\kappa[(\cos\alpha\cos\beta\cos\gamma - \sin\beta\sin\gamma) \\ & \times(-\cos\alpha\sin\beta\sin\gamma + \cos\beta\cos\gamma) \\ & + (\cos\alpha\sin\beta\cos\gamma + \cos\beta\sin\gamma) \\ & \times(-\cos\alpha\cos\beta\sin\gamma - \sin\beta\cos\gamma)] \\ & + \sin\kappa[(\cos\alpha\cos\beta\cos\gamma - \sin\beta\sin\gamma)(\sin\alpha\sin\gamma) \\ & + (-\cos\alpha\cos\beta\sin\gamma - \sin\beta\cos\gamma)(-\sin\alpha\cos\gamma)], \end{aligned} \qquad \ldots (4.29)$$

$$\begin{aligned} D_{x_b z_b} = {} & -\sin\kappa[(\cos\alpha\cos\beta\cos\gamma - \sin\beta\sin\gamma) \\ & \times(-\cos\alpha\sin\beta\sin\gamma + \cos\beta\cos\gamma) \\ & + (\cos\alpha\sin\beta\cos\gamma + \cos\beta\sin\gamma) \\ & \times(-\cos\alpha\cos\beta\sin\gamma - \sin\beta\cos\gamma)] \\ & + \cos\kappa[(\cos\alpha\cos\beta\cos\gamma - \sin\beta\sin\gamma)(\sin\alpha\sin\gamma) \\ & + (-\cos\alpha\cos\beta\sin\gamma - \sin\beta\cos\gamma)(-\sin\alpha\cos\gamma)], \end{aligned} \qquad \ldots (4.30)$$

$$\begin{aligned} D_{y_b z_b} = {} & -(\cos\kappa\sin\kappa)\,[2(\cos\alpha\sin\beta\cos\gamma + \cos\beta\sin\gamma) \\ & \times(-\cos\alpha\sin\beta\sin\gamma + \cos\beta\cos\gamma)] + (\cos\kappa\sin\kappa)\,[-\sin^2\alpha\sin 2\gamma] \\ & + (\cos 2\kappa)\,[(\cos\alpha\sin\beta\cos\gamma + \cos\beta\sin\gamma)(\sin\alpha\sin\gamma) \\ & + (-\cos\alpha\sin\beta\sin\gamma + \cos\beta\cos\gamma)(-\sin\alpha\cos\gamma)]. \end{aligned} \qquad \ldots (4.31)$$

All these calculations were performed with the help of the symbolic manipulation package 'Mathematica'. This completes the discussion about the wave and detector tensors. In the next section we discuss the response of the detector.

5. Response of the detector in the SSB frame

Since the wave and detector tensors are known in the SSB frame, the response of the detector is just a scalar product of these two STF tensors. From equation (4.5),

$$R(t) = W^{ij}_{\rm SSB} D_{{\rm SSB}\,ij}, \qquad \text{... (5.1)}$$

where t is the time when the signal is detected at the detector. We define the barycentric time $t_{\rm b}$ as the time measured by a clock situated at the centre of SSB frame. For a given phase of the wave the SSB frame will register a time $t_{\rm b}$. Since the detector is not at the origin of the SSB frame the same phase of the wave will be registered at different time $t_{\rm d}$ at the detector. The time t appearing in the response below is the time measured at the detector site. We have,

$$t = t_{\rm b} - \frac{\Delta\vec{r}\cdot\vec{n}}{c} - \frac{\vec{r}_{\rm tot}(0)\cdot\vec{n}}{c}, \qquad \text{... (5.2)}$$

where $\Delta\vec{r}$ is the change in radius vector from initial position to the current position at time t, $\vec{r}_{\rm tot}(0)$ the radius vector from the centre of SSB frame to the initial detector position and $\vec{n}$ the unit vector in the direction of the source. The relation between barycentric time and detector time is given in appendix C. The response can be expressed as linear combination of the two polarizations,

$$R(t) = F_+ h_+(t) + F_\times h_\times(t), \qquad \text{... (5.3a)}$$

where F_+ and $F_\times$ are fairly complicated functions of θ, ϕ, ψ, α, β, γ, κ and are given as below:

$$\begin{aligned}
F_+ = \tfrac{1}{2}\, & [\{(\cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi)^2 \\
& - (\cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi)^2\} \\
& \times\{2(\cos\alpha\cos\beta\cos\gamma - \sin\beta\sin\gamma) \\
& \times(-\cos\alpha\cos\beta\sin\gamma - \sin\beta\cos\gamma)\}] \\
& + \tfrac{1}{2}\,[\{(-\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi)^2 \\
& - (-\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi)^2\} \\
& \times[\cos^2\kappa\{2(\cos\alpha\sin\beta\cos\gamma + \cos\beta\sin\gamma) \\
& \times(-\cos\alpha\sin\beta\sin\gamma + \cos\beta\cos\gamma)\} \\
& + \sin^2\kappa\{-\sin^2\alpha\sin 2\gamma\} \\
& + (\sin 2\kappa)\{(\cos\alpha\sin\beta\cos\gamma + \cos\beta\sin\gamma)(\sin\alpha\sin\gamma) \\
& + (-\cos\alpha\sin\beta\sin\gamma + \cos\beta\cos\gamma)(-\sin\alpha\cos\gamma)\}]] +
\end{aligned}$$

Continued

$$
\begin{aligned}
&+ \tfrac{1}{2} \Big[[-\sin^2 \theta \cos 2\phi] \, [\sin^2 \kappa[2(\cos \alpha \sin \beta \cos \gamma + \cos \beta \sin \gamma) \\
&\quad \times (-\cos \alpha \sin \beta \sin \gamma + \cos \beta \cos \gamma)] + \cos^2 \kappa[-\sin^2 \alpha \sin 2\gamma] \\
&\quad - (\sin 2\kappa) \, [(\cos \alpha \sin \beta \cos \gamma + \cos \beta \sin \gamma) \, (\sin \alpha \sin \gamma) \\
&\quad + (-\cos \alpha \sin \beta \sin \gamma + \cos \beta \cos \gamma) \, (-\sin \alpha \cos \gamma)]]\Big] \\
&\quad + \Big[[(\cos \psi \cos \phi - \cos \theta \sin \phi \sin \psi) \\
&\quad \times (-\sin \psi \cos \phi - \cos \theta \cos \phi \cos \psi)] \\
&\quad - (\cos \psi \sin \phi + \cos \theta \cos \phi \sin \psi) \\
&\quad \times (-\sin \psi \sin \phi + \cos \theta \cos \phi \cos \psi)] \\
&\quad \times [\cos \kappa[(\cos \alpha \cos \beta \cos \gamma - \sin \beta \sin \gamma) \\
&\quad \times (-\cos \alpha \sin \beta \sin \gamma + \cos \beta \cos \gamma) \\
&\quad + (\cos \alpha \sin \beta \cos \gamma + \cos \beta \sin \gamma) \\
&\quad \times (-\cos \alpha \cos \beta \sin \gamma - \sin \beta \cos \gamma)] \\
&\quad + \sin \kappa[(\cos \alpha \cos \beta \cos \gamma - \sin \beta \sin \gamma) \, (\sin \alpha \sin \gamma) \\
&\quad + (-\cos \alpha \cos \beta \sin \gamma - \sin \beta \cos \gamma) \, (-\sin \alpha \cos \gamma)]]\Big] \\
&\quad + \Big[[(\cos \psi \cos \phi - \cos \theta \sin \phi \sin \psi) \, (\sin \theta \sin \phi) \\
&\quad + (\cos \psi \sin \phi + \cos \theta \cos \phi \sin \psi) \, (\sin \theta \cos \phi)] \\
&\quad \times [-\sin \kappa[(\cos \alpha \cos \beta \cos \gamma - \sin \beta \sin \gamma) \\
&\quad \times (-\cos \beta \sin \beta \sin \gamma + \cos \beta \cos \gamma) \\
&\quad + (\cos \alpha \sin \beta \cos \gamma + \cos \beta \sin \gamma) \\
&\quad \times (-\cos \alpha \cos \beta \sin \gamma - \sin \beta \cos \gamma)] \\
&\quad + \cos \kappa[\cos \alpha \cos \beta \cos \gamma - \sin \beta \sin \gamma) \, (\sin \alpha \sin \gamma) \\
&\quad + (-\cos \alpha \cos \beta \sin \gamma - \sin \beta \cos \gamma) \, (-\sin \alpha \cos \gamma)]]\Big] \\
&\quad + \Big[[(-\sin \psi \cos \phi - \cos \theta \sin \phi \cos \psi) \, (\sin \theta \sin \phi) \\
&\quad + (-\sin \psi \sin \phi + \cos \theta \cos \phi \cos \psi) \, (\sin \theta \cos \phi)] \\
&\quad \times [-(\cos \kappa \sin \kappa) \, [2(\cos \alpha \sin \beta \cos \gamma + \cos \beta \sin \gamma) \\
&\quad \times (-\cos \alpha \sin \beta \sin \gamma + \cos \beta \cos \gamma)] \\
&\quad + (\cos \kappa \sin \kappa) \, [-\sin^2 \alpha \sin 2\gamma] \\
&\quad + (\cos 2\kappa) \, [(\cos \alpha \sin \beta \cos \gamma + \cos \beta \sin \gamma) \, (\sin \alpha \sin \gamma) \\
&\quad + (-\cos \alpha \sin \beta \sin \gamma + \cos \beta \cos \gamma) \, (-\sin \alpha \cos \gamma)]]\Big], \qquad \ldots (5.3b)
\end{aligned}
$$

and

$$
\begin{aligned}
F_\times = &\Big[[(\cos \psi \cos \phi - \cos \theta \sin \phi \sin \psi) \, (\cos \psi \sin \phi + \cos \theta \cos \phi \sin \psi)] \\
&\times [2(\cos \alpha \cos \beta \cos \gamma - \sin \beta \sin \gamma) \\
&\times (-\cos \alpha \cos \beta \sin \gamma - \sin \beta \cos \gamma)]\Big] \\
&+ \Big[[(-\sin \psi \cos \phi - \cos \theta \sin \phi \cos \psi) \\
&\times (-\sin \psi \sin \phi + \cos \theta \cos \phi \cos \psi)] \\
&\times [\cos^2 \kappa[2(\cos \alpha \sin \beta \cos \gamma + \cos \beta \sin \gamma) \\
&\times (-\cos \alpha \sin \beta \sin \gamma + \cos \beta \cos \gamma)] +
\end{aligned}
$$

Continued

$$+ \sin^2 \kappa(-\sin^2 \alpha \sin 2\gamma] + (\sin 2\kappa)$$
$$\times [(\cos \alpha \sin \beta \cos \gamma + \cos \beta \sin \gamma) (\sin \alpha \sin \gamma)$$
$$+ (-\cos \alpha \sin \beta \sin \gamma + \cos \beta \cos \gamma) (-\sin \alpha \cos \gamma)]]\Big]$$
$$+ \tfrac{1}{2} \Big[[-\sin^2 \theta \sin 2\phi]$$
$$\times \Big[\sin^2 \kappa[2(\cos \alpha \sin \beta \cos \gamma + \cos \beta \sin \gamma)$$
$$\times (-\cos \alpha \sin \beta \sin \gamma + \cos \beta \cos \gamma)]$$
$$+ \cos^2 \kappa[-\sin^2 \alpha \sin 2\gamma] - (\sin 2\kappa)$$
$$\times [(\cos \alpha \sin \beta \cos \gamma + \cos \beta \sin \gamma) (\sin \alpha \sin \gamma)$$
$$+ (-\cos \alpha \sin \beta \sin \gamma + \cos \beta \cos \gamma) (-\sin \alpha \cos \gamma)]]\Big]$$
$$+ \Big[[(\cos \psi \cos \phi - \cos \theta \sin \phi \sin \psi)$$
$$\times (-\sin \psi \sin \phi + \cos \theta \cos \phi \cos \psi)$$
$$+ (-\sin \psi \cos \phi - \cos \theta \sin \phi \cos \psi)$$
$$\times (\cos \psi \sin \phi + \cos \theta \cos \phi \sin \psi)]$$
$$\times \Big[\cos \kappa[\cos \alpha \cos \beta \cos \gamma - \sin \beta \sin \gamma)$$
$$\times (-\cos \alpha \sin \beta \sin \gamma + \cos \beta \cos \gamma)$$
$$+ (\cos \alpha \sin \beta \cos \gamma - \cos \beta \sin \gamma) (-\cos \alpha \cos \beta \sin \gamma - \sin \beta \cos \gamma)]$$
$$+ \sin \kappa[(\cos \alpha \cos \beta \cos \gamma - \sin \beta \sin \gamma) (\sin \alpha \sin \gamma)$$
$$+ (-\cos \alpha \cos \beta \sin \gamma - \sin \beta \cos \gamma) (-\sin \alpha \cos \gamma)]]\Big]$$
$$+ \Big[[-(\cos \psi \cos \phi - \cos \theta \sin \phi \sin \psi) (\sin \theta \cos \phi)$$
$$+ (\cos \psi \sin \phi + \cos \theta \cos \phi \sin \psi) (\sin \theta \sin \phi]$$
$$\times \Big[-\sin \kappa[(\cos \alpha \cos \beta \cos \gamma - \sin \beta \sin \gamma)$$
$$\times (-\cos \alpha \sin \beta \sin \gamma + \cos \beta \cos \gamma)$$
$$+ (\cos \alpha \sin \beta \cos \gamma + \cos \beta \sin \gamma) (-\cos \alpha \cos \beta \sin \gamma - \sin \beta \cos \gamma)]$$
$$+ \cos \kappa[(\cos \alpha \cos \beta \cos \gamma - \sin \beta \sin \gamma) (\sin \alpha \sin \gamma)$$
$$+ (-\cos \alpha \cos \beta \sin \gamma - \sin \beta \cos \gamma) (-\sin \alpha \cos \gamma)]]\Big]$$
$$+ \Big[[(-\sin \psi \sin \phi + \cos \theta \cos \phi \cos \psi) (\sin \theta \sin \phi)$$
$$- (-\sin \psi \cos \phi - \cos \theta \sin \phi \cos \psi) (\sin \theta \cos \phi)]$$
$$\times \Big[(-\cos \kappa \sin \kappa) [2(\cos \alpha \sin \beta \cos \gamma + \cos \beta \sin \gamma)$$
$$\times (-\cos \alpha \sin \beta \sin \gamma + \cos \beta \cos \gamma)]$$
$$+ (\cos \kappa \sin \kappa) [-\sin^2 \alpha \sin 2\gamma] + (\cos 2\kappa)$$
$$\times [(\cos \alpha \sin \beta \cos \gamma + \cos \beta \sin \gamma) (\sin \alpha \sin \gamma)$$
$$+ (-\cos \alpha \sin \beta \sin \gamma + \cos \beta \cos \gamma) (-\sin \alpha \cos \gamma)]]\Big]. \qquad \ldots (5.3c)$$

We note that the total response is a function of several angles arising from the various orientations comprising the motion of the detector relative to the source. It is a function of the position of the source, and the orientation of the detector on Earth, orientation of the spin axis of the Earth and the orientation of the orbital plane. If we assume that the source emits a wave of constant amplitude and frequency, the response gets modulated due to the motion of the detector. If we assume that the source is fixed in the sky, then the response is both frequency and amplitude modulated. The response will show a constant Doppler shift, if the source is in uniform motion, or when the motion can be considered to be approximately uniform over the period of observation, usually ~10^7 sec. We discuss now two modulations appearing in the response, namely, (a) Frequency modulation (FM), and (b) Amplitude modulation (AM).

(a) *Frequency modulation* : It arises due to translatory motion of the detector acquired from the motion of the Earth. We have only considered two motions of the Earth namely, its rotation about the spin axis and the orbital motion about the Sun. Hence the response is doubly frequency modulated with one period corresponding to 1 day and the other period corresponding to a year. The FM smears out a monochromatic signal into a small bandwidth around the signal frequency of the monochromatic waves. It also redistributes the power in a small bandwidth. The study of FM due to rotation of the Earth about its spin axis, for one day's observation shows that the Doppler spread in the bandwidth for 1 kHz signal will be 0.029 Hz. The Doppler spread in the bandwidth due to orbital motion for one day observation will be 1.74×10^{-3} Hz (Schutz 1991). The consequence of the Doppler spread in signal detection will be discussed elsewhere separately. Since any observation is likely to last longer than a day it will be very important to incorporate this effect in the data analysis algorithms.

(b) *Amplitude modulation* : The amplitude modulation arises due to the anisotropic response of the detector, i.e. the detector possesses a quadrupole antenna pattern. For a given incident wave, a detector in different orientation will record different amplitudes in the response. The functions F_+ and $F_\times$ appearing in the expression of the response completely characterise AM for the two polarizations. Since the expressions for F_+ and $F_\times$ are quite complicated, we will consider some special cases to get some idea of AM. For the ideal case when the wave is optimally incident on the detector F_+ and $F_\times$ can individually have maximum value of unity.

We consider the following special cases : For cases I to III the wave is taken to travel in the positive y_b direction of the SSB frame.

For case I, the detector is situated on the equator with one arm pointing along it and another along the North-South direction.

(a) $\alpha = \frac{\pi}{2}, \ \beta_0 = 0, \ \gamma = \frac{\pi}{4}, \ \text{and} \ \theta = \frac{\pi}{2}, \ \phi = 0, \ \psi = 0.$

This gives,

$$F_+ = \tfrac{1}{2}\left[(1 + \sin^2 \kappa) \sin^2 (\omega_{rot} t) + \cos 2\kappa\right], \qquad \text{... (5.4a)}$$

$$F_\times = -\tfrac{1}{2} \sin \kappa \sin (2\omega_{rot} t). \qquad \text{... (5.4b)}$$

(b) $\alpha = \frac{\pi}{2}, \ \beta_0 = 0, \ \gamma = \frac{\pi}{4}, \ \text{and} \ \theta = \frac{\pi}{2}, \ \phi = \frac{\pi}{4}, \ \psi = 0.$

The polarization of the wave is rotated by 45° from that of case (a). We get,

$$F_+ = \tfrac{1}{2} \sin \kappa \sin (2\omega_{rot} t), \qquad \text{... (5.5a)}$$

$$F_\times = \tfrac{1}{2}\,[(1 + \sin^2\kappa)\sin^2(\omega_{rot}t) + \cos 2\kappa]. \qquad \ldots (5.5b)$$

For case II, the detector is situated on the equator with the arms symmetrically placed about the North-South direction.

$$\alpha = \frac{\pi}{2},\ \beta_0 = 0,\ \gamma = 0,\ \text{and}\ \theta = \frac{\pi}{2},\ \phi = 0,\ \psi = 0.$$

For this case,

$$F_+ = -\tfrac{1}{2}\sin 2\kappa\cos(\omega_{rot}t), \qquad \ldots (5.6a)$$

$$F_\times = -\cos\kappa\,\sin(\omega_{rot}t). \qquad \ldots (5.6b)$$

For all the above cases the AM results in about 40% drop in the amplitude of the signal as compared to optimal incidence (i.e. F_+ or $F_\times = 1$).

For case III, the detector is situated at the North pole.

$$\alpha = 0,\ \beta_0 = 0,\ \gamma = 0,\ \text{and}\ \theta = \frac{\pi}{2},\ \phi = 0,\ \psi = 0.$$

And we have,

$$F_+ = -\tfrac{1}{2}(1 + \sin^2\kappa)\sin(2\omega_{rot}t), \qquad \ldots (5.7a)$$

$$F_\times = \sin\kappa\,\cos(2\omega_{rot}t). \qquad \ldots (5.7b)$$

There is approximately 57% drop in the amplitude of the signal as compared to optimal incidence.

For case IV, the wave is incident along the spin axis of the Earth, with the detector at the North pole. The wave is therefore incident normally on the detector.

$$\alpha = 0,\ \beta_0 = 0,\ \gamma = 0,\ \text{and}\ \theta = \kappa,\ \phi = 0,\ \psi = 0.$$

$$F_+ = -\sin(2\omega_{rot}t), \qquad \ldots (5.8a)$$

$$F_\times = \cos(2\omega_{rot}t). \qquad \ldots (5.8b)$$

This case corresponds to the maximum possible response of the detector.

In the context of signal detection the FM and AM play a very important role. This can be seen as follows. Suppose we take a monochromatic signal recorded in the data stream without any FM or AM, a straight forward Fourier transform of the data would result in a peak in the Fourier domain of height $T/2$, where T is the total observation time. The effect of both modulations is to 'spread out' the peak thus reducing the height of the peak. The effect of FM is however much more severe. As our computations show [this will be published elsewhere] just taking account of the rotational motion of the Earth, the peak gets dispersed over a frequency band Δf given by $(\Delta f/f_r) = 2\pi f_0 R/c$, where f_0 is the frequency of the signal, R the radius of the Earth and c the velocity of light.

6. Concluding remarks

We have considered an almost spherical object, spinning about an arbitrary axis emitting gravitational waves. We have then calculated the quadrupole wave form, and specialised to the case of a spheroid whose axis of symmetry makes a non-zero angle with the axis of rotation. The elegant formalism of Gel'fand functions, STF tensors is applied in obtaining the results.

Further, we have computed the response of a gravitational wave detector situated on Earth. Since the observation times for obtaining an appreciable signal-to-noise ratio are of the order of a few months, the response is calculated by taking into account the rotational and orbital motion of the Earth. Since these are not the only motions the Earth indulges in, other motions such as the Earth-moon motion may have to be included in computing the response; in fact those motions which have bearing on signal detection should be taken into account in computing the response. We have here analysed the response under a fairly general setting in which the wave direction and polarization, and the detector position and orientation are arbitrary. In this computation we have availed of the formalism based on expressing the detector and the wave as STF tensors and the response as their scalar product. Since the response of the detector is not isotropic, in fact quadrupole, the recorded response varies in amplitude as the detector changes the orientation with the motion of the Earth. Further, the response is Doppler shifted, the Doppler shift depends on the detector, wave parameters and the parameters governing the motion of the Earth. The data analysis problem is therefore quite complex and efforts are on way (Schutz, Kanti & Dhurandhar) for making inroads towards the solution to this very important problem. One way is to study the Fourier transform of the signal and see how this can be used in developing good algorithms. The 'stepping-around the sky' method developed by Schutz is another way to tackle the problem of the all-sky, all-frequency search for pulsars. Work in this direction is in progress (Jones' Ph.D. thesis).

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APPENDIX A

Spherical tensors

$$\mathcal{Y}^{22} = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\begin{bmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \mathcal{Y}^{2,-2} = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\begin{bmatrix} 1 & -i & 0 \\ -i & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{Y}^{21} = -\frac{1}{4}\sqrt{\frac{15}{2\pi}}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & i \\ 1 & i & 0 \end{bmatrix} \qquad \mathcal{Y}^{2,-1} = -\frac{1}{4}\sqrt{\frac{15}{2\pi}}\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & i \\ -1 & i & 0 \end{bmatrix}$$

$$\mathcal{Y}^{20} = \frac{1}{2}\sqrt{\frac{5}{4\pi}}\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\mathcal{Y}^{2m}_{il} n^i n^l = Y^{lm}(\theta, \phi)$$

where $\vec{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$.

APPENDIX B

Gel'fand functions $P^2_{mn}(\cos\theta)$

$m\rightarrow$ / $n\downarrow$	-2	-1	0	1	2
-2	$\frac{1}{4}(1+\cos\theta)^2$	$-\frac{i}{4}\sin\theta\,(1+\cos\theta)$	$-\frac{1}{2}\sqrt{\frac{3}{2}}\sin^2\theta$	$-\frac{i}{2}\sin\theta\,(\cos\theta-1)$	$\frac{1}{4}(\cos\theta-1)^2$
-1	$-\frac{i}{2}\sin\theta\,(1+\cos\theta)$	$\frac{1}{2}(2\cos^2\theta+\cos\theta-1)$	$-\sqrt{\frac{3}{2}}i\sin\theta\cos\theta$	$\frac{1}{2}(2\cos^2\theta+\cos\theta-1)$	$-\frac{i}{2}\sin\theta\,(\cos\theta-1)$
0	$-\frac{1}{2}\sqrt{\frac{3}{2}}\sin^2\theta$	$-\sqrt{\frac{3}{2}}i\sin\theta\cos\theta$	$\frac{1}{2}(3\cos^2\theta-1)$	$-\sqrt{\frac{3}{2}}i\sin\theta\cos\theta$	$-\frac{1}{2}\sqrt{\frac{3}{2}}\sin^2\theta$
1	$-\frac{i}{2}\sin\theta\,(\cos\theta-1)$	$\frac{1}{2}(2\cos^2\theta+\cos\theta-1)$	$-\sqrt{\frac{3}{2}}i\sin\theta\cos\theta$	$\frac{1}{2}(2\cos^2\theta+\cos\theta-1)$	$-\frac{i}{2}\sin\theta\,(1+\cos\theta)$
2	$\frac{1}{4}(\cos\theta-1)^2$	$-\frac{i}{2}\sin\theta\,(\cos\theta-1)$	$-\frac{1}{2}\sqrt{\frac{3}{2}}\sin^2\theta$	$-\frac{i}{2}\sin\theta\,(\cos\theta+1)$	$\frac{1}{4}(1+\cos\theta)^2$

The $P^2_{mn}(\cos\theta)$ obey the following symmetry properties :

$$P_{mn} = P_{nm} = P_{-m,-n} = P_{-n,-m}.$$

The Gel'fand functions are give by

$$T^2_{mn}(\theta, \phi, \psi) = e^{-im\phi} P^2_{mn}(\cos\theta) e^{-in\psi}.$$

APPENDIX C

Frequency modulation

In order to study frequency modulation of a monochromatic plane wave, one needs to calculate Doppler shift due to rotation and orbital motion of the Earth in the SSB frame. For this, we need to know relative velocity between the source and detector. The angles θ, ϕ give the detection of the incoming wave in the SSB frame. The radial vector r_{rot} in the SSB frame is given by

$$\vec{r}_{tot} = [A \cos \omega_{orb} t_b + R \sin \alpha \cos \omega_{rot} t_b, \quad A \sin \omega_{orb} t_b + R \sin \alpha \sin \omega_{rot} t_b \cos \kappa - R \cos \alpha \sin \kappa, \quad R \sin \alpha \sin \omega_{rot} t_b \sin \kappa + R \cos \alpha \cos \kappa], \qquad \text{... (C1)}$$

where A is distance from the centre of the SSB frame to the centre of the Earth, R the radius of the Earth, and unit vector in the direction of source $\vec{n}$ = (sin θ cos ϕ, sin θ sin ϕ, cos θ). Therefore, total Doppler shift due to rotation and orbital motion of the Earth in the SSB frame will be

$$\frac{\dot{\vec{r}}_{tot} \cdot \vec{n}}{c} = \left[\frac{A\omega_{orb}}{c} \sin \theta \sin (\phi - \omega_{orb} t_b) + \frac{R\omega_{rot}}{c} \sin \alpha [\sin \theta (\cos \omega_{rot} t_b \cos \kappa \sin \phi - \cos \phi \sin \omega_{rot} t_b) -+ \cos \omega_{rot} t_b \sin \kappa \cos \theta]\right], \qquad \text{... (C2)}$$

where t_b is the time measured by a clock located at centre of the SSB frame. The time t registered by the detector for the same phase of the wave is given by

$$t = t_b - \int_0^{t_b} \frac{\dot{\vec{r}}_{tot} \cdot \vec{n}}{c} dt - \frac{\vec{r}_{tot}(0) \cdot \vec{n}}{c} \qquad \text{... (C3a)}$$

$$= t_b - \frac{\Delta \vec{r}_{tot} \cdot \vec{n}}{c} - \frac{\vec{r}_{tot}(0) \cdot \vec{n}}{c} \qquad \text{... (C3b)}$$

where $\Delta \vec{r} = \vec{r}_{tot}(t_b) - \vec{r}_{tot}(0)$ is the change in the radial vector from initial to current position at time t_b. The last term, which is a constant, can be eliminated by choosing the zero of the detector time appropriately.