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Non-linear planar oscillation of a satellite in elliptical orbit under the influence of solar radiation pressure (II)

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Abstract. This paper is the continuation of our paper (I) (non-resonance case). In the present paper, parametric and main resonance have been studied by BKM method; and also the half width of the chaotic separatrix has been estimated by Chirikov's criterion. Through surface of section method, it has been observed that the solar radiation pressures, the eccentricity of the orbit and mass-distribution of the satellite play an important role to change the regular motion into the chaotic one. Interestingly, the phenomenon of period-3 which implies chaos, has been witnessed in this mathematical model.

Key words : celestial mechanics—solar system

1. Introduction

Goldreich & Peale (1966) obtained a pendulum-like equation for each spin-orbit state by writing the equation of motion of the planar oscillation of a satellite in terms of resonance variables and eliminating the higher frequency non-resonant terms through averaging. They found that $(B - A)/C << 1$, (A, B, C are the principal moments of inertia of the satellite at its centre of mass) and eccentricity of the orbit (e) determine the strength of each resonance. Wisdom, Peale & Mignard (1984) have studied the same spin-orbit coupling in the motion of Hyperion for those cases where averaging fails. Chirikov (1979) predicts the presence of large chaotic zone through the resonance overlap criterion.

Recently Bhatnagar *et al.* (1992 I) have studied the planar oscillation of a satellite under the influence of the solar radiation pressure; and have obtained the perturbed pendulum-like equation. Through Melnikov's method they have shown the non-integrability of the non-linear rotational equation of motion. Also by taking the solar radiation pressure parameter (ε) of the order of eccentricity of the orbit (e), by BKM method they have observed that the amplitude of the oscillation remains constant up to the second order of approximation. Finally they have also observed that the parametric and main resonance occur at $\omega \cong \pm 1$, $\omega \simeq \pm 1/2$ and $\omega = 1/K$; $K \in I - \{0\}$.

Here, in sequel to that we have studied the parametric and main resonances at $\omega \simeq \pm 1$, $\omega \simeq \pm 1/2$ and $\omega \simeq \pm 1/K$ by BKM method. Through Chirikov's criterion we have estimated the half width of the chaotic separatrix. Through surface of section method we have observed that by varying the solar radiation pressure parameter some of the regular trajectories are captured into the chaotic zone. Eccentricity of the orbit and mass-distribution of the satellite also seem to be effective to change the regular motion into irregular one. In this mathematical model we have observed the phenomenon of the period-3 which implies chaos.

2. Equations of motion

Let us consider a rigid satellite moving in an elliptic orbit (semi-major axis a, eccentricity e) around the Earth under the influence of the solar radiation pressure $F_\mu = F_g(1-q)$, F_g being solar gravitational attraction force. In general $q \simeq 1$ and so $0 < 1 - q << 1$. The satellite is assumed to be triaxial ellipsoid with principal moments of inertia about the spin axis which is regarded as one of the principal axes. The torque caused by the solar radiation pressure is assumed to be perpendicular to the orbital plane (centre of resultant pressure lies on x'-axis, figure 1 of paper I).

Let the instantaneous radius be r, the true anomaly α, the orientation of the satellite's long axis ϕ. Then $\phi - \alpha = \theta$, measures the orientation of the satellite's long axis relative to the satellite's radius vector. The equation of motion of the planar oscillation of the satellite is

$$\frac{d^2\phi}{dt^2} + \frac{n^2}{2r^3} \sin 2(\phi - \alpha) + \varepsilon \sin \phi = 0 \qquad \ldots (2.1)$$

where

$$n^2 = 3\left(\frac{B-A}{C}\right); \qquad \frac{l}{r} = 1 + e\cos\alpha$$

and ε is proportional to solar radiation pressure torque. The above equation reduces to equation (2.1.1) of our paper (I) if we take the true anomaly α as the independent variable.

3. Resonant planar oscillations of a satellite

Taking $2\theta = \eta$, $\omega = n$, $\alpha = \upsilon$, $\varepsilon = \varepsilon_1 e$, equation (2.1) can be written as in Bhatnagar *et al.* (1994)

$$\frac{d^2\eta}{d\upsilon^2} + \omega^2\eta = 4e\sin\upsilon + 2e\sin\upsilon\frac{d\eta}{d\upsilon} - e\cos\upsilon\frac{d^2\eta}{d\upsilon^2}$$
$$-2\varepsilon_1 e \sin\left(\upsilon + \frac{\eta}{2}\right) + \omega^2(\eta - \sin\eta). \qquad \ldots (3.1)$$

In equation (3.1) the non-linearity $(\eta - \sin\eta)$ is sufficiently weak and therefore it can also be taken of the order of e. So, by taking $\omega^2 = \alpha e$, equation (3.1) becomes

$$\frac{d^2\eta}{d\upsilon^2} + \omega^2\eta = e\left[4 \sin \upsilon + 2 \sin \eta \frac{d\eta}{d\upsilon} - \cos \upsilon \frac{d^2\eta}{d\upsilon^2}\right.$$

$$\left. -2\varepsilon_1 \sin\left(\upsilon + \frac{\eta}{2}\right) + \alpha\left(\eta - \sin \eta\right)\right]. \qquad \ldots (3.2)$$

In paper (I), while studying the non-resonance case, we observed that the system experiences resonances at $\omega = \pm 1$, $\omega = 1/K$, $K \in I - \{0\}$ and the parametric resonance at $\omega = \pm 1/2$.

Now, we propose to study the asymptotic solution near the resonances $\omega = \pm 1$, $\omega = \pm 1/K$ and $\omega = \pm 1/2$ making use of the BKM method. The resonances $\omega = \pm 1$, $\omega = \pm 1/2$ are being studied separately as the study cannot be deduced from $\omega = 1/K$. For $e = 0$, the generating solutions are

$$\eta = a \cos \psi, \qquad \psi = \frac{\upsilon}{K} + \theta \qquad \ldots (3.3)$$

where, amplitude a and phase θ are determined by the following equations :

$$\frac{da}{d\upsilon} = e A_1(a, \theta)$$

$$\frac{d\theta}{d\upsilon} = \omega - \frac{1}{K} + eB_1(a, \theta)$$

$$\frac{d\psi}{d\upsilon} = \frac{1}{K} + \frac{d\theta}{d\upsilon} = \omega + eB_1(a, \theta) \qquad \ldots (3.4)$$

where $A_1(a, \theta)$, $B_1(a, \theta)$ are particular solutions periodic with respect to θ. Using (3.3) and (3.4), we find $d\eta/d\upsilon$ and $d^2\eta/d\upsilon^2$ and then substituting the values of η, $d\eta/d\upsilon$, $d^2\eta/d\upsilon^2$ in (3.2) and equating the coefficients of e, we get

$$\left\{\left(\omega - \frac{1}{K}\right)\frac{\partial A_1}{\partial\theta} - 2aB_1\omega\right\} \cos \psi - \left\{a\left(\omega - \frac{1}{K}\right)\frac{\partial B_1}{\partial\theta} + 2\omega A_1\right\} \sin \psi$$

$$= 4 \sin \upsilon - 2a\omega \sin \upsilon \sin \psi + a\omega^2 \cos \upsilon \cos \psi + \alpha a \cos \psi$$

$$- \alpha \sin (a \cos \psi) - 2\varepsilon_1 \sin\left(\upsilon + \frac{a}{2} \cos \psi\right) \qquad \ldots (3.5)$$

using Fourier expansions given by

$$\sin (a \cos \psi) = 2 \sum_{k=0}^{\infty} (-1)^k J_{2k+1}(a) \cos (2k + 1) \psi$$

$$\cos (a \cos \psi) = J_0(a/2) + 2 \sum_{k=1}^{\infty} (-1)^k J_{2k}(a) \cos 2k \psi$$

where, J_k, $k = 0, 1, 2, \ldots$ stand for Bessel's functions; in (3.5) and then comparing the coefficients of $\cos\psi$ and $\sin\psi$, we get the following cases :

Case a : When K = 1

$$(\omega-1)\frac{\partial A_1}{\partial\theta} - 2a\omega B_1 = \alpha[a-2J_1(a)] - \left[4 + 2\varepsilon_1\left(J_2\left(\frac{a}{2}\right) - J_0\left(\frac{a}{2}\right)\right)\right]\sin\theta,$$

$$a(\omega-1)\frac{\partial B_1}{\partial\theta} + 2\omega A_1 = -\left[4 - 2\varepsilon_1\left(J_0\left(\frac{a}{2}\right) + J_2\left(\frac{a}{2}\right)\right)\right]\cos\theta. \qquad \ldots (3.6)$$

Solving equations (3.6), we get

$$A_1 = \left[-\frac{4}{\omega+1} + \frac{2\varepsilon_1}{3\omega-1}J_2\left(\frac{a}{2}\right) + \frac{2\varepsilon_1 J_0}{\omega+1}\left(\frac{a}{2}\right)\right]\cos\theta,$$

$$B_1 = \frac{-\alpha}{2a\omega}\left((a-2J_1(a)\right) + \left[\frac{4}{a(\omega+1)}\ \frac{2\varepsilon_1}{a(3\omega-1)}J_2\left(\frac{a}{2}\right) - \frac{2\varepsilon_1}{a(\omega+1)}J_0\left(\frac{a}{2}\right)\right]\sin\theta.$$

Case b : When K = –1

$$(\omega+1)\frac{\partial A_1}{\partial\theta} - 2a\omega B_1 = \alpha[(a-2J_1(a)] + \left[4 + 2\varepsilon_1\left\{J_2\left(\frac{a}{2}\right) - J_0\left(\frac{a}{2}\right)\right\}\right]\sin\theta,$$

$$a(\omega+1)\frac{\partial B_1}{\partial\theta} + 2\omega A_1 = \left[4 + 2\varepsilon_1\left\{J_2\left(\frac{a}{2}\right) - J_0\left(\frac{a}{2}\right)\right\}\right]\cos\theta. \qquad \ldots (3.7)$$

Solving equations (3.7), we get

$$A_1 = \left[\frac{4}{\omega-1} - \frac{2\varepsilon_1 J_2(a/2)}{3\omega+1} - \frac{2\varepsilon_1 J_0(a/2)}{\omega-1}\right]\cos\theta,$$

$$B_1 = -\frac{\alpha}{2a\omega}[a-2J_1(a)] - \left[\frac{4}{a(\omega-1)} + \frac{2\varepsilon_1 J_2(a/2)}{a(3\omega+1)} - \frac{2\varepsilon_1 J_0(a/2)}{a(\omega-1)}\right]\sin\theta.$$

Case c : When K = 2

$$\left(\omega-\frac{1}{2}\right)\frac{\partial A_1}{\partial\theta} - 2a\omega B_1 = \alpha\left(a-2J_1(a)\right) + \left[a\omega\left(\frac{\omega}{2}-1\right)\right.$$
$$\left. + 2\varepsilon_1\left(J_3\left(\frac{a}{2}\right) - J_1\left(\frac{a}{2}\right)\right)\right]\cos 2\theta,$$

$$a\left(\omega-\frac{1}{2}\right)\frac{\partial B_1}{\partial\theta} + 2\omega A_1 = \left[2\varepsilon_1\left\{J_1\left(\frac{a}{2}\right) + J_3\left(\frac{a}{2}\right)\right\}\right.$$
$$\left. - a\omega\left(\frac{\omega}{2}-1\right)\right]\sin 2\theta. \qquad \ldots (3.8)$$

Solving equation (3.8), we have

$$A_1 = \left[-a\omega\left(\frac{\omega}{2}-1\right)+\frac{2\varepsilon_1 J_3(a/2)}{4\omega-1}+2\varepsilon_1 J_1\left(\frac{a}{2}\right)\right]\sin 2\theta,$$

$$B_1 = -\frac{\alpha}{2a\omega}\left[a-2J_1(a)\right]+\left[-\omega\left(\frac{\omega}{2}-1\right)-\frac{2\varepsilon_1 J_3(a/2)}{a(4\omega-1)}+\frac{2\varepsilon_1}{a}J_1\left(\frac{a}{2}\right)\right]\cos 2\theta.$$

Case d : When K = –2

$$\left(\omega+\frac{1}{2}\right)\frac{\partial A_1}{\partial\theta}-2a\omega B_1 = \alpha\left(a-2J_1(a)\right)$$

$$+\left[a\omega\left(\frac{\omega}{2}+1\right)+2\varepsilon_1\left(J_3\left(\frac{a}{2}\right)-J_1\left(\frac{a}{2}\right)\right)\right]\cos 2\theta,$$

$$a\left(\omega+\frac{1}{2}\right)\frac{\partial B_1}{\partial\theta}+2\omega A_1 = \left[2\varepsilon_1\left(J_3\left(\frac{a}{2}\right)+J_1\left(\frac{a}{2}\right)\right)\right.$$

$$\left.-\,a\omega_1\left(\frac{\omega}{2}+1\right)\right]\sin 2\theta. \qquad \ldots (3.9)$$

Solving (3.9), we get

$$A_1 = \left[\frac{2\varepsilon_1}{4\omega+1}J_3\left(\frac{a}{2}\right)-2\varepsilon_1 J_1\left(\frac{a}{2}\right)+a\omega\left(\frac{\omega}{2}+2\right)\right]\sin 2\theta,$$

$$B_1 = \left[\frac{-2\varepsilon_1}{a(4\omega+1)}J_3\left(\frac{a}{2}\right)-\frac{2\varepsilon_1}{a}J_1\left(\frac{a}{2}\right)+\frac{\omega}{2}(\omega+2)\right]\cos 2\theta-\frac{\alpha}{2a\omega}\left[a-2J_1(a)\right].$$

Case e : When K = 2m + 1, m # 0, or –1

$$\left(\omega-\frac{1}{K}\right)\frac{\partial A_1}{\partial\theta}-2a\omega B_1 = \alpha\left[a-2J_1(a)\right]+(-1)^{(k-1)/2}\,2\varepsilon_1 J_{k-1}\left(\frac{a}{2}\right)\sin k\theta,$$

$$a\left(\omega-\frac{1}{K}\right)\frac{\partial B_1}{\partial\theta}+2\omega A_1 = 2\varepsilon_1(-1)^{(k-1)/2}\cos k\theta\, J_{k-1}\left(\frac{a}{2}\right). \qquad \ldots (3.10)$$

Solving equations (3.10), we get

$$A_1 = \frac{2\varepsilon_1(-1)^{(k-1)/2}J_{k-1}(a/2)\cos k\theta}{(2\omega-K\omega+1)},$$

$$B_1 = \frac{-2\varepsilon_1(-1)^{(k-1)/2}J_{k-1}(a/2)\sin k\theta}{a(2\omega-K\omega+1)}-\frac{\alpha}{2a\omega}\left(a-2J_1(a)\right).$$

Case f : When $K = 2m$, $m \# 1$, or -1

$$\left(\omega - \frac{1}{K}\right)\frac{\partial A_1}{\partial \theta} - 2a\omega B_1 = \alpha[a - 2J_1(a)] - 2\varepsilon_1(-1)^{k/2} J_{k+1}\left(\frac{a}{2}\right)\cos k\theta,$$

$$a\left(\omega - \frac{1}{K}\right)\frac{\partial B_1}{\partial \theta} + 2\omega A_1 = \mp 2\varepsilon_1(-1)^{k/2} J_{k+1}\left(\frac{a}{2}\right)\sin k\theta. \quad \ldots (3.11)$$

Solving equations (3.11), we obtain

$$A_1 = \frac{2\varepsilon_1(-1)^{k/2}\sin k\theta\, J_{k+1}(a/2)}{(2\omega \mp K\omega \pm 1)},$$

$$B_1 = \frac{2\varepsilon_1(-1)^{k/2} J_{k+1}(a/2)\cos k\theta}{a(2\omega \pm K\omega - 1)} - \frac{\alpha}{2a\omega}\left(a - 2J_1(a)\right).$$

Substituting the values of A_1, B_1, for different cases in equations (3.4), we obtain two non-integrable differential equations. However, one can visualize the shape of the resonant curves for different cases by numerical integration of these equations.

4. Estimation of resonance width

In (2.1) if the units are so chosen that the orbital period of the satellite is 2π and its semi-major axis is 1, then the dimensionless time is equal to mean longitude. As r and α are 2π-periodic in time, second term in (2.1) can be written as Fourier-like Poisson series, Wisdom *et al.* (1984).

$$\frac{d^2\upsilon}{dt^2} + \frac{\omega_0^2}{2}\sum_{m=-\infty}^{\infty} H\left(\frac{m}{2}, e\right)\sin(2\upsilon - mt) + \varepsilon\sin\upsilon = 0.$$

Here we have changed the symbols as

$$\phi = \upsilon, \qquad n = \omega_0.$$

The coefficients $H[(m/2), e]$ are proportional to exp $[2(|(m/2)| - 1)]$ and tabulated by Caley (1859) and Goldriech & Peales (1966); when e is small $H[(m/2), e] \cong (1/2)e$, 1, $(7/2)e$ for $m/2 = 1/2, 1, 3/2$ respectively. The half integer $m/2$ is denoted by p. The resonance is witnessed for $|(d\upsilon/dt) - p| << 1/2$. In such a situation it is advantageous to rewrite the equation of motion in terms of slowly varying resonance variable $r_p = \upsilon - pt$

$$\frac{d^2 r_p}{dt^2} + \frac{\omega_0^2}{2}H(p, e)\sin 2r_p + \frac{\omega_0^2}{2}\sum_{n\#0}^{\infty} H\left(p + \frac{n}{2}, e\right)\sin(2\gamma_p - nt)$$
$$+ \varepsilon\sin(r_p + pt) = 0.$$

When ω_0 is sufficiently small, the above equation is approximately

$$\frac{d^2 r_p}{dt^2} + \frac{\omega_0^2}{2} H(p, e) \sin 2r_p + \varepsilon \sin (r_p + pt) = 0. \quad \ldots (4.1)$$

Equation (4.1) can be studied under the cases $\varepsilon = 0$ and $\varepsilon \# 0$. The case $\varepsilon = 0$ has been studied by Wisdom *et al.* (1984). When $\varepsilon \# 0$, equation (4.1) represents the motion of a disturbed pendulum given by

$$\frac{d^2 x}{dt^2} + f'(x) = mg'(x, t) \quad \ldots (4.2)$$

where

$$f'(x) = K_1^2 \sin x,$$

$$K_1^2 = \omega_0^2 H(p, e) << 1, \; m = -2\varepsilon << 1, \; X = 2\gamma_p$$

and

$$g'(x, t) = \sin\left(\frac{x}{2} + pt\right).$$

The unperturbed part of (4.2) is

$$\frac{d^2 x}{dt^2} + f'(x) = 0.$$

For this equation we have

$$\left(\frac{dx}{dt}\right)^2 = C + 2K_1^2 \cos x$$

where C is a constant of integration. There are three categories of motion depending upon $C >=< 2K_1^2$.

Category (i) : $C > 2K_1^2$

In this case unperturbed solution is

$$x = l + C_1 \sin l + O(C_1^2)$$

where

$$l = nt + \varepsilon_1, \quad C_1 = \frac{K_1^2}{n^2}$$

and

$$\frac{1}{n} = \frac{1}{2\pi} \int_0^{2\pi} \frac{dx}{(C + 2K_1^2 \cos x)^{1/2}}$$

C_1 and ε_1 are arbitrary constants and l is an argument. Here we may observe that dx/dt # 0 and the motion is said to be of revolution.

In case of the perturbed equation by the theory of variation of parameters (Brown & Shook 1964), we have

$$\frac{dC_1}{dt} = \frac{m}{\nu} \frac{\partial x}{\partial l} g'(x, t),$$

$$\frac{dl}{dt} = n - \frac{m}{\nu} \frac{\partial x}{\partial C_1} g'(x, t), \quad \ldots (4.3)$$

where

$$\nu = \frac{\partial}{\partial C_1}\left[n - \frac{\partial x}{\partial l}\right]\frac{\partial x}{\partial l} - n\frac{\partial^2 x}{\partial l^2}\frac{\partial x}{\partial C_1} \cong -\frac{n}{2C_1}.$$

So, we get

$$\frac{dC_1}{dt} = -\frac{2mK_1^2}{n^3}\sin\left(\frac{x}{2} + pt\right) \cong 0.$$

Since both m and K_1 are small quantities, the term mK_1^2 is of the third order and therefore rejected.

Thus $\qquad \dfrac{dC_1}{dt} \cong 0.$

or $\qquad C_1$ = const. up to second order of approximation.

and

$$\frac{dl}{dt} \cong n + \frac{2mC_1}{n}\sin l \sin\left(\frac{x}{2} + pt\right)$$

$$\frac{d^2 l}{dt^2} \cong m \sin\left\{\frac{-K_1\varepsilon_1}{n} + l\left(\frac{1}{2} + \frac{K_1}{n}\right)\right\}.$$

In the first approximation taking $n = n_0$, and rejecting $K_1\varepsilon_1$ being a term of the second order, the above equation can be written as

$$\frac{d^2 l}{dt^2} \cong m \sin l\left\{\frac{1}{2} + \frac{K_1}{n_0}\right\}$$

If we take $l[(1/2) + (K_1/n_0)] = x$, then the above equation becomes

$$\frac{d^2x}{dt^2} + \varepsilon \sin x = 0 \qquad \ldots (4.4)$$

or $$\left(\frac{dx}{dt}\right)^2 = C_2 + 2K_2^2 \cos x.$$

Equation (4.4) describes the motion of a pendulum. We get again three types of motions. Type (i) is that in which dx/dt is never zero. Type (ii) is that in which $dx/dt = 0$ at $X = 0$ or π. For type (i) our solution is

$$x = Nt + \varepsilon_2 + \frac{K_2^2}{N^2} + \sin (Nt + \varepsilon_2)$$

where $K_2^2 = \varepsilon$,

$$\frac{1}{N} = \frac{1}{2\pi} \int_0^{2\pi} \frac{dx}{(C_2 + 2K_2 \cos x)^{1/2}}$$

C_2 and ε_2 are arbitrary constants.

In the first approximation $N = N_0$ and the solution is

$$X = X_0 + \frac{K_2^2}{N^2} \sin X_0,$$

where

$$X_0 = N_0 t + \varepsilon_2.$$

This is the case of revolution.

For type (ii) the solution is

$$X = \lambda \sin (p't + \lambda_0)$$

where

$$p' = \sqrt{\varepsilon}$$

λ and λ_0 being arbitrary constants. This is the case of libration.

Type (iii) occurs when $C_2 = 2K_2^2 = 2\varepsilon$, the solution is

$$x + \pi = 4 \tan^{-1} \exp (\sqrt{\varepsilon}\, t + \alpha_0)$$

where α_0 is an arbitrary constant and the other have a particular value.

When $t \rightarrow \pm\infty$, $x = \pm\pi$ and at both places $dx/dt = 0$ and all higher derivatives of x approach to zero. Near this point x one of the limits tends to $\pm\pi$, t tends to become an indeterminate function of x. This is the case of infinite period separatrix which is asymptotic backward and forward to unstable equilibrium. Thus the results of type (i), type (ii), and type (iii) enable us to conclude that the Solar radiation pressure plays a significant role. It may change a revolution to libration (type ii) or to infinite period separatrix (type iii).

Category (ii) : $C < 2K_1^2$

In this case unperturbed solution is

$$x = C_1 \sin l + O(C_1^3),$$

where

$$l = nt + \varepsilon_1,\ n = K_1\left[1 - \frac{1}{16}c_1^2 + O(C_1^3)\right],$$

C_1 and ε_1 are arbitrary constants.

In case of the perturbed equation, again using the theory of variation of parameters, we get

$$\frac{dC_1}{dt} \cong \frac{m}{K_1}\cos l \sin\left(\frac{x}{2} + pt\right)$$

$$\frac{dl}{dt} \cong \frac{-m}{K_1C_1}\sin l \sin\left(\frac{x}{2} + pt\right)$$

$$\frac{d^2l}{dt^2} \cong \frac{-mp}{K_1C_1}\sin l \cos\left(\frac{l - \varepsilon_1}{n}\right)$$

in the first approximation, if $n = n_0$ and $C_1 = C_0$, then

$$\frac{d^2l}{dt^2} \cong \frac{-mp}{K_1C_0}\sin l \cos\left(\frac{l - \varepsilon_1}{n_0}\right). \quad \ldots (4.5)$$

As a special case, let us assume that $(l - \varepsilon_1)/n_0 = n_1\pi$, $n_1 \in I$. When n_1 is odd then (4.5) becomes

$$\frac{d^2l}{dt^2} + K_3^2 \sin l = 0, \quad \text{where } K_3^2 = \frac{-mp}{K_1C_0} > 0,$$

which is again the equation of disturbed pendulum. As in previous case this equation also gives us revolution, libration and infinite period separatrix motion.

On the other hand if n_1 is even then,

$$\frac{d^2l}{dt^2} \cong K_3^2 \sin l$$

when l is small, the solution of the above equation is

$$l = e^{K_3 t} + e^{-K_3 t}.$$

Category (iii) : $C = 2K_1^2$

The unperturbed solution is

$$x + \pi = 4 \tan^{-1} e^{K_1 t + \alpha_0}$$

where α_0 is arbitrary constant and other having a specific value. This is the case of infinite period separatrix which is asymptotic forward and backward to unstable equilibrium. We are mainly concerned in this category of motion. In this category the nature of the unperturbed solution does not change by taking into account the solar radiation pressure.

Near the infinite period separatrix broadened by the high frequency term into narrow chaotic band (Chirikov 1979), for small ω_0, the half width of the chaotic separatrix is given by :

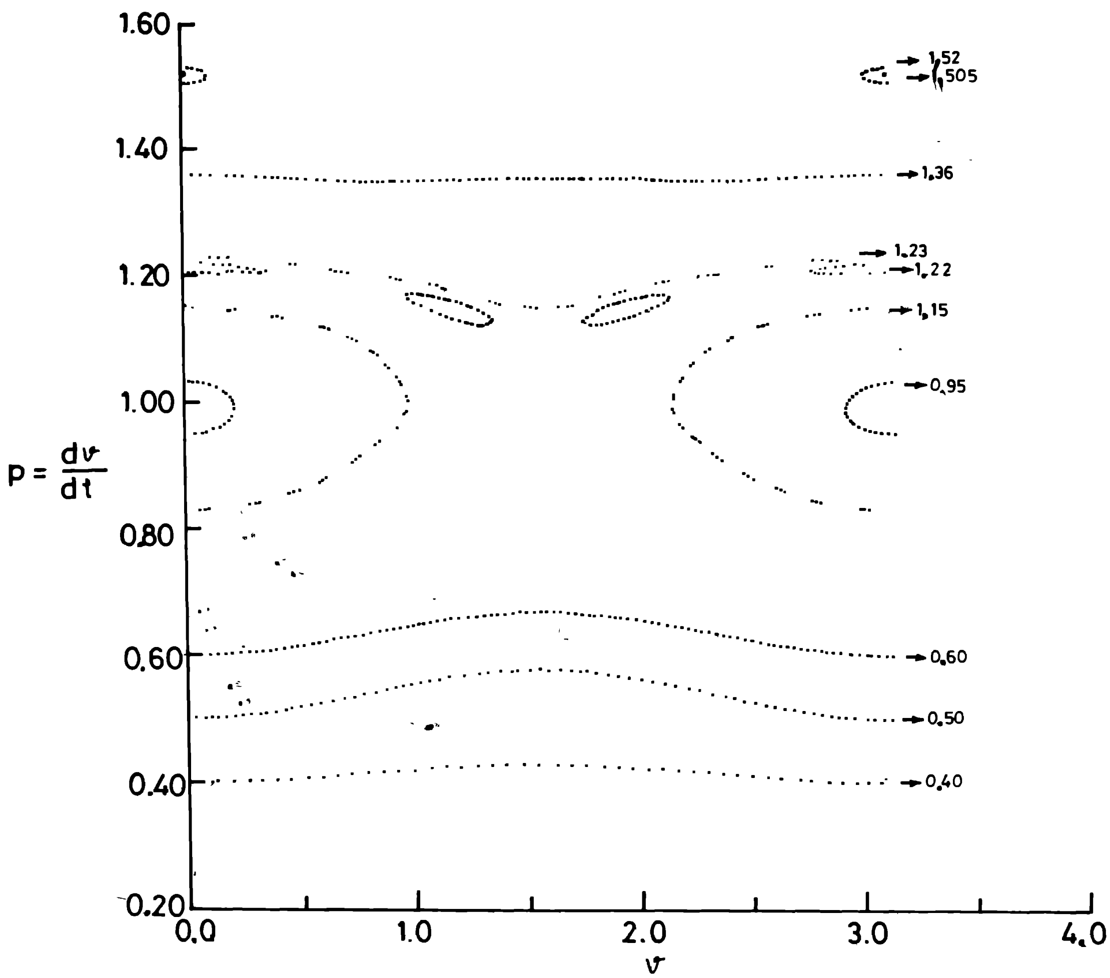

Figure 1. Surface of section for $\varepsilon_1 = 0$, $\omega = 0.2$, $e = 0\ 1$.

$$w_p = \frac{I_p - I_p^s}{I_p^s} = 4\pi\varepsilon_0\lambda_0^3\, e^{-\pi\lambda_0/2}$$

where ε_0 is the ratio of the coefficients of the nearest high frequency perturbing term to the coefficients of the perturbed term and $\lambda_0 = \Omega/\omega$ is the ratio of the frequency difference between the resonant term (Ω) to the frequency of the small amplitude libration (ω).

For the synchronous spin orbit state perturbed by the solar radiation pressure term, $\lambda_0 = 1/\omega_0$, $\varepsilon_0 = 2\varepsilon/\omega_0{}^2 H(1, e)$

$$w_1 = \frac{I_1 - I_1^s}{I_1^s} = 4\pi\varepsilon_0\lambda_0^3\, e^{-\pi\lambda_0/2} = \frac{8\pi\varepsilon}{\omega_0^5\, e^{\pi/2\omega_0}},$$

w_1, increases both with ε and ω_0. An estimate of ω_0 at which the wide spread chaotic behaviour can be observed is given by using the Chirikov's overlap criterion. This criterion states that when the sum of two unperturbed half widths equals the separation of resonance centres,

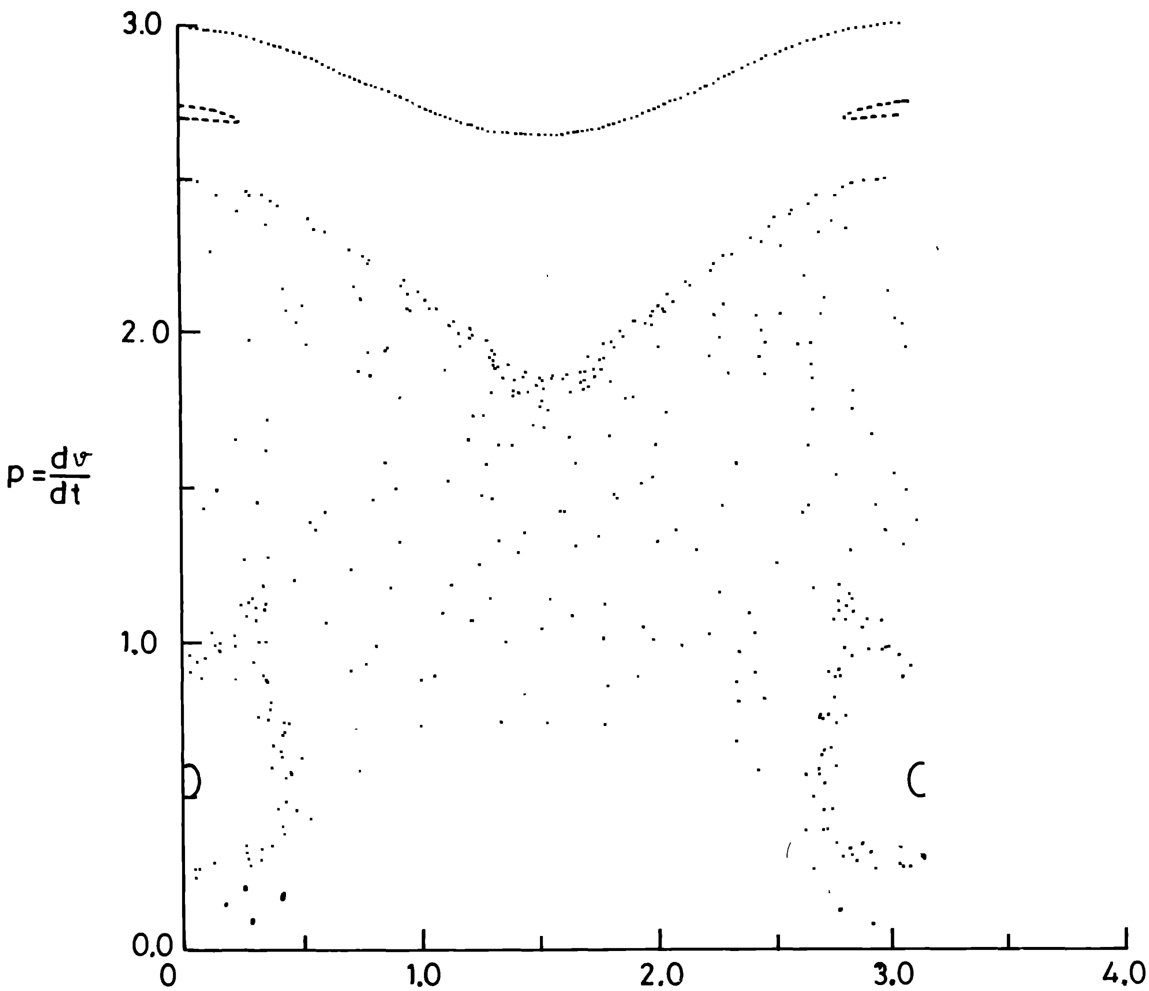

Figure 2. Surface of section for $\varepsilon_1 = 0\ 0$, $\omega_0 = 0.89$, $e = 0.1$.

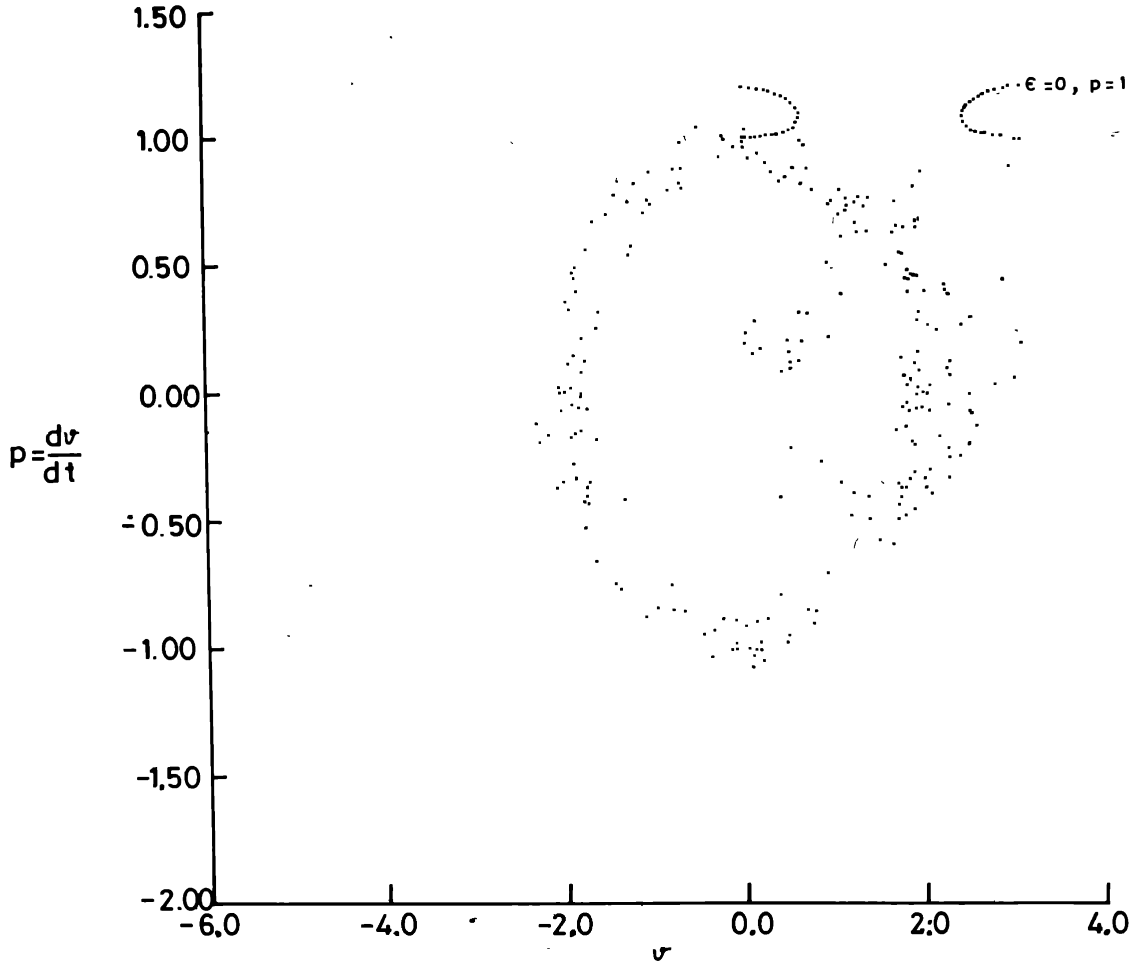

Figure 3. Surface of section for ε_1 = 0.0, 0.1, 0.2, 0.3, 0.4, 0.5; ω_0 = 0.2; e = 0.1

large scale chaos occurs. In the spin orbit problem the two resonances with the largest widths are $p = 1$ and $p = 3/2$ states. Criterion becomes

$$\omega_0^{R0}\sqrt{H(1, e)} + \omega_0^{R0}\sqrt{H(3/2, e)} = \frac{1}{2}$$

or

$$\omega_0^{R0} = \frac{1}{2 + \sqrt{14e}}.$$

For $e = 0.3$ large scale chaotic behaviour is expected when $\omega_0 > 0.25$. This is confirmed in (Fig. 5). This figure is studied in detail in section 5.

5. The spin orbit phase space

It is known that most of the Hamiltonian systems give regular and irregular trajectories. Henon & Heiles (1964) have shown that the phase space is divided into two regions in which trajectories behave chaotically or quasi-periodically. One of the best methods to show whether a trajectory is chaotic or quasi-periodic is through the surface of section method. In our spin orbit problem which is 2π periodic in dimensionless time, we have drawn surface of section by looking at the trajectories stroboscopically with period 2π. The section has been drawn with $d\upsilon/dt$ versus υ at every periapse passage. In the case of quasi-periodic trajectory the points are contained in smooth curves while for the chaotic trajectories they appear to fill up the area in the phase space in random manner. Since the orientation denoted by υ is equivalent to the orientation denoted by $\pi + \upsilon$. We have therefore restricted the interval from 0 to π. The spin orbit states that are determined in the previous section are states where a resonance variable $\gamma_p = \upsilon - pt$ librates. For each of these states $d\upsilon/dt$ has an average value precisely equal to p and υ rotates through all values. If we take to the times of periapse passage, i.e. $t = 2\pi n$, then each γ_p, taken modulo π will be simply υ. This implies that a libration in γ_p will be a libration in υ on the surface of section. Consequently quasi-periodic successive points will trace a simple curve on the section near $d\upsilon/dt = p$. This will cover only a fraction of interval from 0 to π. In the case of non-resonant quasi-periodic trajectories all γ_p rotates and successive points on the surface of section will trace a simple curve which cover all values of υ. For small values of ω_0 resonance states will be separated from non-resonant states by a narrow chaotic zone. All these possibilities are shown in figure 1 to figure 5 for various values of ω_0, e and ε.

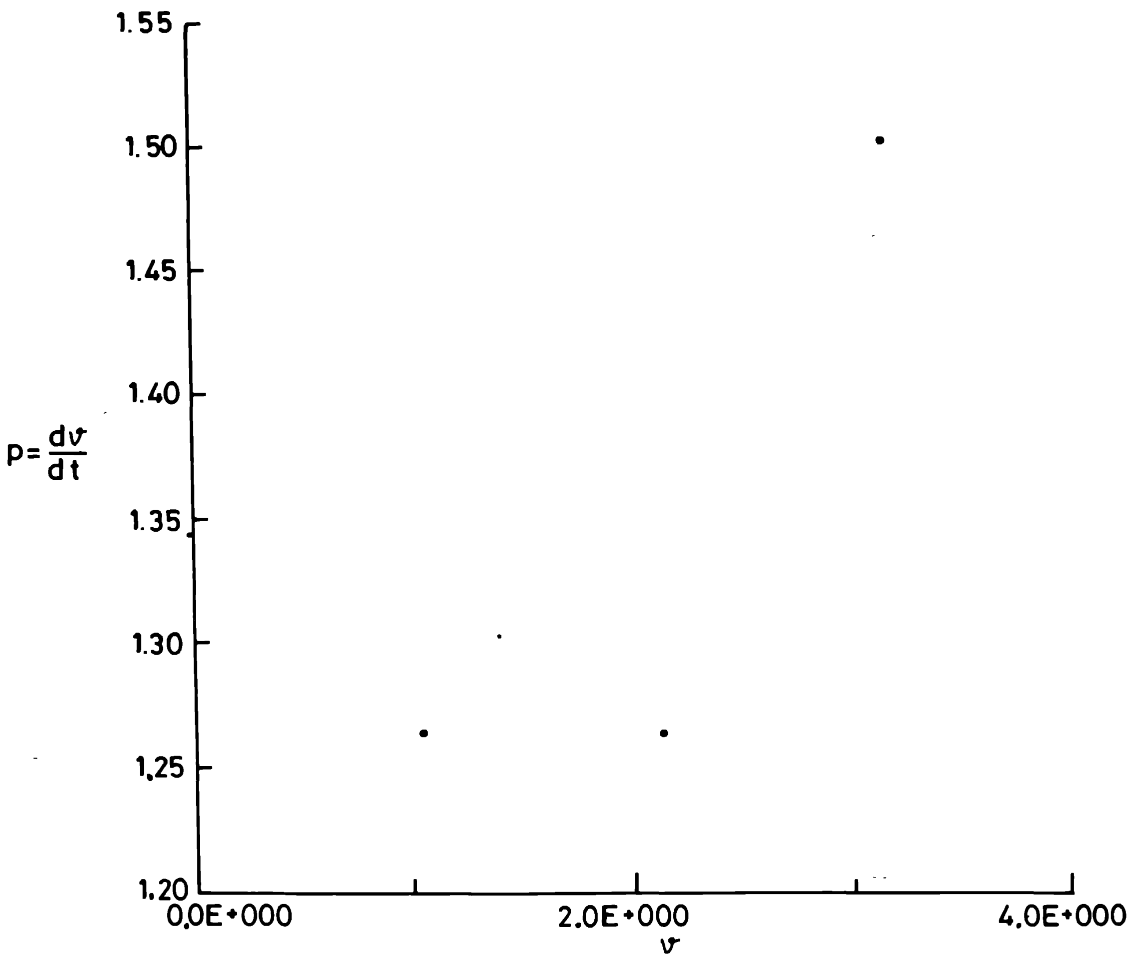

Figure 1. Surface of section for $\varepsilon_1 = 0.2$, $\omega_0 = 0.2$, $e = 0.1$, $p = 1.5$.

Figure 1 illustrates surface of section for $\varepsilon_1 = 0.0$, $\omega_0 = 0.2$, $e = 0.1$. Equation (2.1) has been numerically integrated for ten separate trajectories corresponding to $p = d\upsilon/dt = 0.40$, 0.50, 0.60, 0.95, 1.15, 1.22, 1.23, 1.36, 1.505, 1.52 and $d\upsilon/dt$ has been plotted versus υ at every periapse passage. Trajectories illustrate the chaotic separatrices surrounding each of these resonance states and some trajectories show that each of these chaotic zones is separated from the other by resonance quasi-periodic rotation trajectories. We have plotted 500 successive points for each quasi-periodic trajectory and 1000 points for each chaotic trajectory. This figure confirms results of Wisdom *et al.* (1984).

As ω_0 is increased both the resonance widths and the widths of the chaotic separatrices also increase. In figure 2 we have drawn surface of section for $\varepsilon_1 = 0.0$, $\omega_0 = 0.89$ and $e = 0.1$. It may be observed that chaotic region is very large surrounding all states from $p = 1/2$ to $p = 2$. This confirms the findings of Wisdom *et al.* (1984). Please, note the change in scale of figure 2 from figure 1. A total of ten trajectories of equation (2.1) are used to draw this figure. In both figure 1 and figure 2 we have ignored the effect of solar radiation pressure.

In figure 3 we have studied the effect of solar radiation pressure. Figure 3 illustrates 6 trajectories corresponding to $\varepsilon_1 = 0.0$, 0.1, 0.2, 0.3, 0.4 and 0.5, in each case we have taken

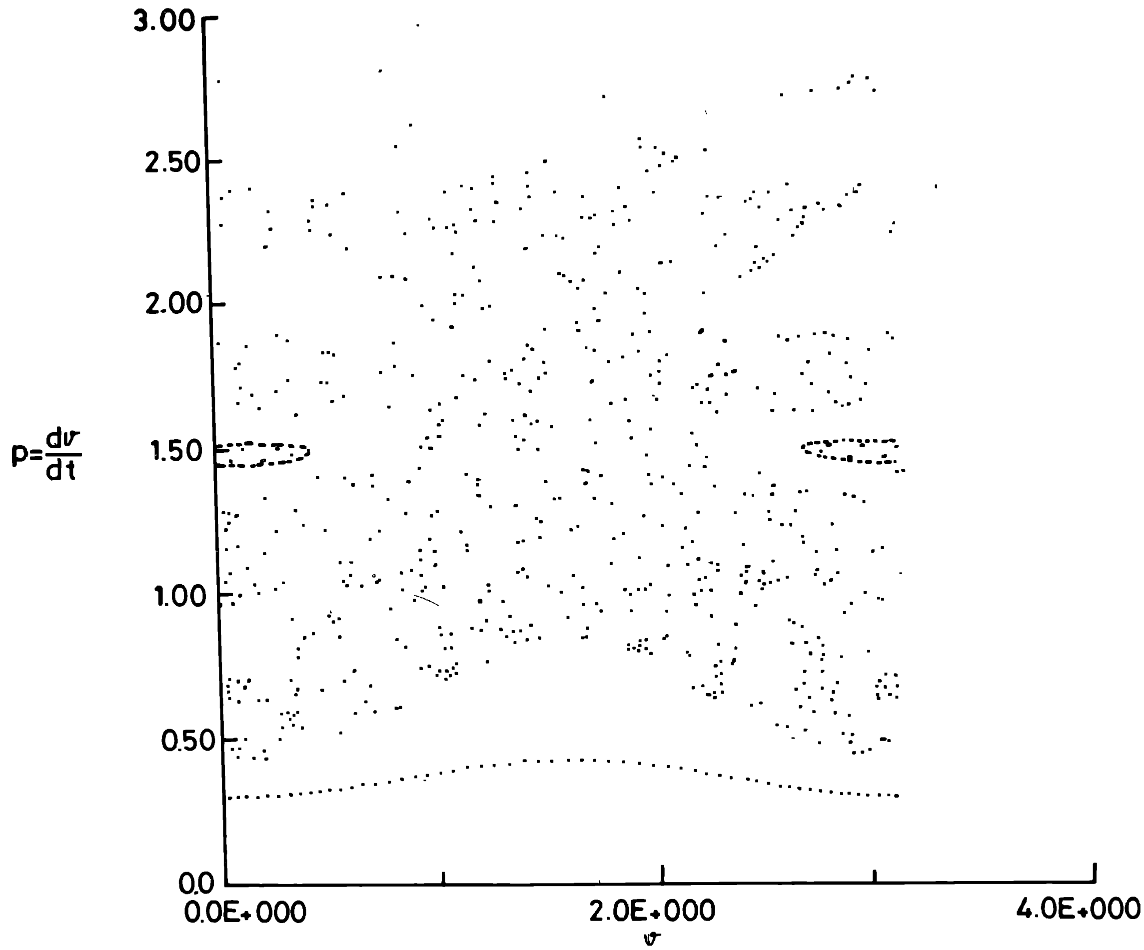

Figure 5. Surface of section for $\varepsilon_1 = 0.0$, $\omega_0 = 0.4$, $e = 0.3$, $p = 1.65$, 1.60, 1.50, 1.45, 1.20, 1.0, 0.80, 0.60, 0.50, 0.30

$\omega_0 = 0.2$ and $e = 0.1$. It is observed that as we increase the value of ε (solar radiation pressure parameter) the regular trajectory corresponding to $\varepsilon_1 = 0.0$ and $p = 1.0$ breaks up into chaotic zones. This indicates that solar radiation pressure plays an important role in changing a regular trajectory into chaotic one.

Figure 4 illustrates the surface of section corresponding to $p = 1.5$, $\varepsilon_1 = 0.3$, $\omega_0 = 0.2$, $e = 0.1$. This figure remarkably and explicitly indicates the period –3 which implies chaos.

As we increase the eccentricity both the resonance width and the widths of the chaotic separatrices grow. In figure 5 surface of section for $\varepsilon_1 = 0.0$, $\omega_0 = 0.4$, $e = 0.3$ has been drawn for ten trajectories similar to trajectories shown in figure 1. It shows that change in eccentricity may change the regular trajectory into a chaotic one.

6. Conclusions

From these studies we conclude that the solar radiation pressure plays a very significant role in changing the motion of revolution into the motion of libration and infinite period separatrix. We also observe that the regular motion changes into a chaotic one for some values of the radiation parameter ε_1 mass-distribution parameter ω_0, and the eccentricity e. Both the resonance width and the widths of the chaotic separatrices grow with the increase in the values of ω_0, e and ε_1. This means that regular mass distribution, elongated orbit and the solar radiation pressure on the satellite have significant effect on resonance overlap criterion which is in excellent agreement with the numerical results.

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