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Formation of a binary in the general three-body problem

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Abstract. This paper reviews the role of triple encounters in the evolution of stellar systems. If a symmetric triple collision is perturbed, we obtain a family of asymmetric triple close approaches with arbitrary escape velocities and with the formation of binaries. We obtain two-parameter family of orbits; for certain values of the parameters two of the bodies form a binary and third escapes to infinity. The work of Szebehely is reviewed in detail for fixed values of the parameters with special reference to the application in stellar and galactic dynamics. The numerical technique and controls used are also mentioned without which no reliable numerical results can be obtained regarding the dynamical behaviour of multicomponent stellar systems.

Key words : three body problem—celestial mechanics

1. Introduction

Birkhoff (1922, 1927) conjectured that sufficiently close simultaneous asymmetric approaches occurring in the problem of three bodies result in the formation of a binary and the third one escaping to infinity. This conjecture has been supported by numerical evidence by Agekian (1967) and Szebehely (1967). The conjecture has been reformulated by Szebehely (1971, 1973) which is of fundamental importance in the global behaviour of three interacting gravitational masses.

In the three body problem, one encounters motions of different types under certain initial conditions. In fact the main problem is the partition of the phase space of the initial conditions. The region of phase space with bounded motion is mixed with escape regions according to Henon (1974). Some orbits amongst a large number of periodic orbits discovered in the general problem of three bodies by Henon (1974), Broucke (1974), Hadjidemetriou (1975), Standish (1970), and Szebehely (1970) showing linear stability are unstable.

Sundaman (1912) has shown that for a triple collision the total angular momentum c must vanish and for close approaches c should be sufficiently small. If asymmetric changes of the initial conditions are introduced and c is small, the equilateral configuration in the three body problem leads to escape instead of periodic orbits.

We can classify different types of motions with the help of total energy h and the moment of inertia $I = \Sigma\, m_i r_i^2$ about the origin.

2. Classification of possible motions

Case (a): $h > 0$ and $I \to \infty$ as $t \to \infty$

(i) Hyperbolic-parabolic. In this case $|r_{ij}| < \to t$ and the bodies move along hyperbolic/parabolic orbits and the motion is termed explosion.

(ii) Hyperbolic elliptic. In this case $|r_{12}| < a$ or $|r_{13}| < a$ or $|r_{23}| < a$; $|r_{13}|$ and $|r_{23}| \to t$. Two of the three bodies form a binary and the orbit is elliptic. The third body moves along a hyperbolic orbit and escapes to infinity.

Case (b): $h = 0$, $I \to \infty$

(i) Hyperbolic-elliptic. It is similar to the motion as mentioned in case (a) (ii).

(ii) Parabolic. In this case $|r_{ij}| \to t$, the bodies move along parabolic orbits and the motion is termed explosion.

Case (c): $h < 0$, I is bounded

(i) Interplay. In this case $|r_{ij}|$ remain bounded and the bodies repeatedly come close to each other.

(ii) Ejection. In this case two bodies form a binary and the third is ejected with elliptic relative velocity.

(iii) Revolution. In this case two bodies form a binary and the orbit of the third body surrounds them.

(iv) Equilibrium solutions. In this case the three bodies appear to be stationary in a rotating frame of reference, (Lagrange's straight line and equilateral triangle solutions).

(v) Periodic motions. In this case the motion of the three bodies are bounded, periodic and unstable.

(vi) In this case one of the three bodies recedes arbitrarily far away and returns. $I(t)$ is oscillatory.

Case (d): $h < 0$, $I \to \infty$

Hyperbolic/parabolic-elliptic: in this case $|r_{12}| < a$, $|r_{13}|$ and $|r_{23}| \to t$. Two of the bodies form a binary and the third escapes.

It may also be noted that (i) escape orbits are dense, (ii) interplay leads to either escape or ejection, (iii) Orbits near the equilibrium points are of interplay types. (iv) Some unstable periodic orbits leads to interplay. (v) Revolution leads to interplay.

3. Conditions of escape

Various conditions of escape, *i.e.* two of the bodies forming a binary and the third one escaping to infinity are available in the literature.

Suppose there is a system with already formed binary; m_2 moving relative to m_1 in an elliptic orbit and m_3 escaping. We define

$$E_b = \text{bounding energy} = -Gm_1 m_2/2a$$

$$E_e = \frac{1}{2} m_3 v_3^2 + \frac{m_1 + m_2}{2} v_{12}^2 - \frac{G(m_1 + m_2) m_3}{\rho}$$

$$h = E_b + E_e = \text{total energy}$$

a = semi-major axis of the elliptic orbit of m_2 relative to m_1
ρ = distance between the mass m_3 and the centre of mass of m_1 and m_2.
v_3 = velocity of m_3
v_{12} = velocity of m_2 relative to m_1.

Conditions of escape

(A) (i) When $h \geqslant 0$, binary formation gives escape if $\dot{\rho} > 0$. It does not lead to ejection. (ii) When $h < 0$, for escape $|E_b| > |E_c|$. This is true if a is sufficiently small. If a is large, this leads to ejection.

(B) When $h < 0$. If at some time, t_0 : (i) $\rho(t_0) = \rho_0 > a$, (ii) $\dot{\rho}(t_0) = \dot{\rho}_0 > 0$, (iii) $\dot{\rho}_0^2 \geqslant b$, $a > 0$, $b > 0$ (+ve nos.) then $\rho \to \infty$ as $t \to \infty$ and m_1 and m_2 form a binary.

These conditions are sufficient. The estimated values of a and b are given by:

(i) Birkhoff

$$a = \frac{2M^2G}{3|h|}, \quad b = \frac{8MG}{\rho_0}, \quad M = m_1 + m_2 + m_3.$$

(ii) Standish

$$a = \frac{G(m_1m_2 + m_2m_3 + m_3m_1)}{|h|}$$

$$b = 2GM\left[\frac{1}{\rho_0} + \frac{g_1}{\mu} + \frac{a^2}{\rho_0^2(\rho_0 - a)}\right]$$

$$g_1 = \frac{m_1m_2}{\mu}, \quad \mu = m_1 + m_2.$$

(C) Before stating the condition of escape, it is worthwhile to know some of the properties of $I(t)$. Let $I_c = C^2/2|h|$. From the graph of $I(T)$, figure 1, we may observe that

(i) Region AB : $I \leqslant I_c$, $\ddot{I} > 0$, I_1 is a proper minimum, $I_1 = 0$ at E,

(ii) Region BC : $\dot{I}_2 = 0$ at C, $I_{min} = I_1 < I_2$

$$I_3 = \frac{I_c^2}{I_1} > I_c,$$

(iii) $$I_c \leqslant \frac{I_c^2}{I_1} \leqslant I^2,$$

(iv) I cannot approach zero or a constant value.

Two of the bodies will form a binary and the third will escape if $I_{min} \leqslant S_1^2$

where $$S_1 = \frac{I_c\sqrt{I_0}}{P + I_0 + I_c}, \quad I_0 = a_0^2(g_1 + g_2)$$

$$P = \frac{1}{8|h|}(A + A' I^{3/4}), \quad I_c = \frac{C^2}{2|h|}$$

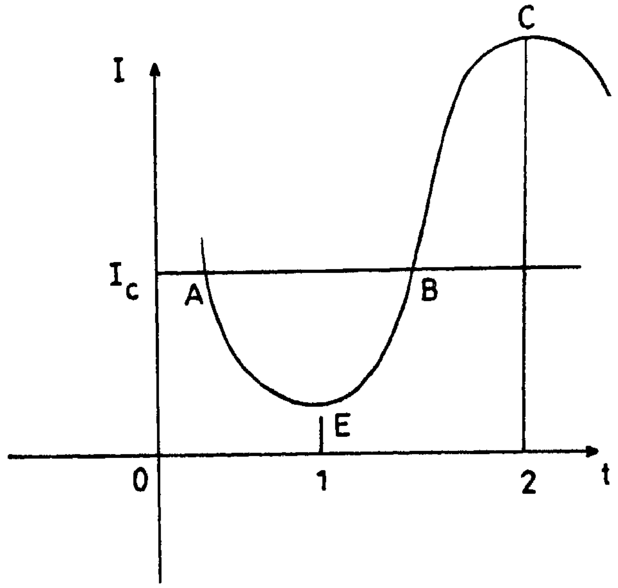


Figure 1. Moment of inertia $I(t)$.

$$a_0 = \frac{G\Sigma}{|h|}, \quad g_1 = \frac{m_1 m_2}{\mu}, \quad g_2 = \frac{\mu m_3}{M}$$

$$|h| = T - F, \quad A = \sqrt{2g_1 g_2 G\Sigma}, \quad A' = 4\sqrt{2MGg_2^{3/2}}$$

$$C^2 = [\sum_{i=1}^{3} m_i(\bar{r}_i \times \dot{\bar{r}}_i)]^2, \quad \Sigma = m_1m_2 + m_2m_3 + m_3m_1$$

$$\mu = m_1 + m_{2+} \quad M = m_1 + m_2 + m_3, \quad T = \frac{1}{2}\Sigma m_i \dot{\bar{r}}_i^2.$$

Now we study a mathematical model in which asymmetric changes of the initial conditions are introduced and c is small. We will see that the equilateral configuration in the three-body problem leads to escape under certain initial conditions.

4. Mathematical model

There are three masses $m_1 = m_2 = m_3 = 1$ situated initially at the vertices of a traingle with unit sides. All initial velocities are parallel and inclined at an angle α to one of the side, say, $\overline{m_1m_2}$. The velocities of m_1 and m_2 are $v_0/2$ and of m_3 is v_0 in the oposite direction, ($v_0 \ll 1$), (figure 2).

Proceeding as in Szebehely (1974) we can show that when

(i) $\pi/6 \leqslant \alpha \leqslant 5\pi/6$, m_3 escapes,
(ii) $5\pi/6 < \alpha < 3\pi/2$, m_1 escapes,
(iii) $3\pi/2 < \alpha < 2\pi + \pi/6$, m_2 escapes,
(iv) $\alpha = 3\pi/2$, none escapes.

Special case $\alpha = 0$. This case has been studied by Szebehely (1974) in detail. The escape conditions are satisfied and escape does occur for sufficiently small perturbations. This follows from the fact that as $v_0 \to 0^+$, $I_{min} \to 0^+$ orbits of m_1, m_2, m_3 are given in figure 3

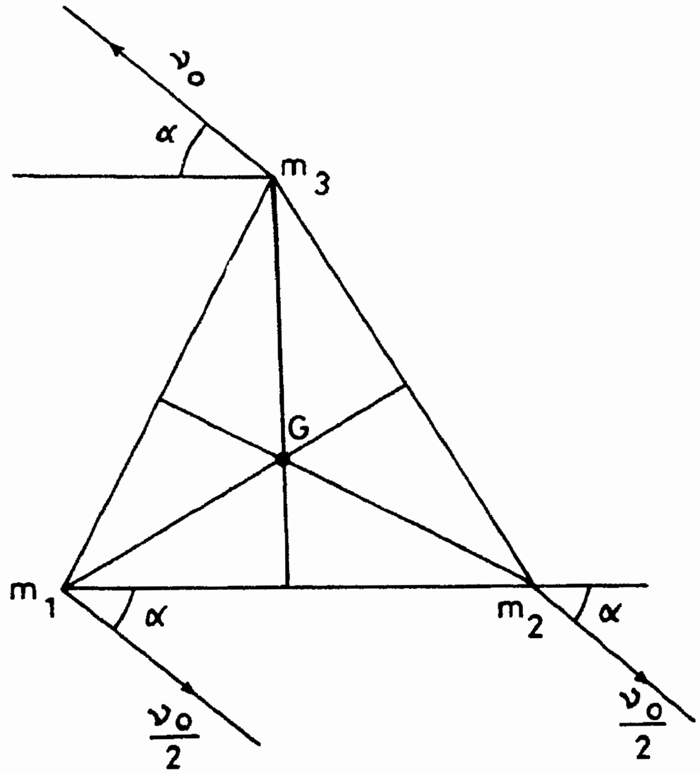
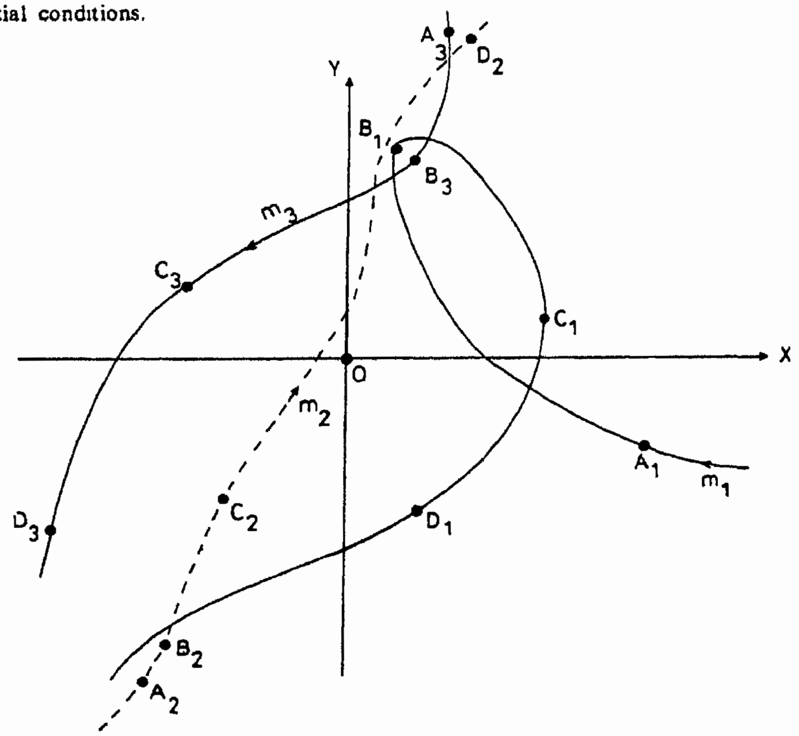


Figure 2. Initial conditions.



Figurte 3. Orbits near triple close approach ($\nu_0 = 0.001$; $\alpha = 0$)

for $v_0 = 0.001$. Because of the small initial velocities, the three bodies begin their motions with a contraction toward the centre of mass. The asymmetry is apparent when the bodies are at the points A_1, A_2, A_3. m_1 and m_3 experience a close approach when they are at the points B_1 and B_3 and B_2 is far away. At this instant $r_{13} = 1.5626 \times 10^{-4}$, $t_{13} = 0.641254$, $I = 5.134 \times 10^{-6}$. But this value of I is not the value for I_{min}, since m_2 and m_3 still move towards the centre of mass. Infact, I_{min} occurs at $t_m = 0.641288$, where $I_{min} = 2.919 \times 10^{-6}$. This occurs when the masses are at C_1, C_2, C_3. At t_m, $r_1 = 13.0 \times 10^{-4}$ $r_2 = 6.4 \times 10^{-4}$ and $r_3 = 9.0 \times 10^{-4}$.
After this time m_2 escapes and m_1 and m_3 form a binary.
Table 1 gives the positions of the masses at different timings. The graph of $I(t)$ is given in figure 4.

Table 1. Times corresponding to the points on the orbits in figure 1

Point	A_1	α_1	β_1	γ_1	B_1	δ_1	C_1	D_1
Tau	218	234	244	250	254	274	288	356

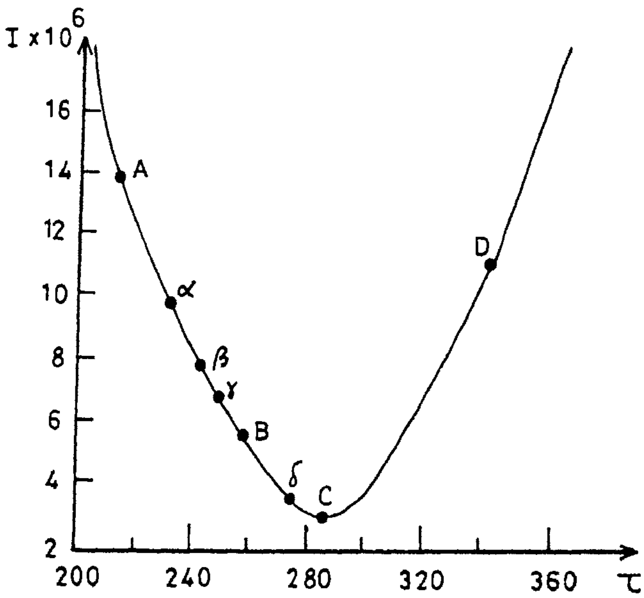


Figure 4. Moment of inertia at triple close approach ($\nu_0 = 0.001$, $\alpha = 0$, $t = 0.641 + \tau \times 10^{-6}$).

Szebehely (1974) has studied four models as applications of this analysis to stellar dynamics:

(i) Three stars of solar mass $M_m = M_\odot$,
(ii) Two models of white dwarfs,
(iii) Three galaxies of mass $M_m = 10^{10}$ $M_\odot$,
(iv) Three neutron stars.

In each case he has calculated velocity of escape of the third body and the semimajor axis of the binary.

5. Numerical method

While performing numerical computation, the following facts must be kept in mind. (i) The dynamical system of three bodies tend to a disruption or escape. (ii) Such an escape cannot occur for $h < 0$ without a triple close approach. (iii) The triple collision is not continuable analytically. (iv) Sufficient close approaches might invalidate the numerical integration unless they receive special attention.

Once reliable estimates are available, the rest is left to the operator of the computer. He may take the following two steps whenever unreliable results are obtained. (i) he may throw away his predictably unreliable results. He may retrace his steps and his integration with a smaller step size or with higher precision. (ii) He may select different initial conditions so that a critically low triple close approaches does not occur prior to the dissolution of his system.

At present we are numerically studying the various close approaches for different values of α and v_0. The results will be published as soon the study is completed.

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