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Non-linear stability of a self-gravitating fluid column in the presence of a magnetic field

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I must express my deep sense of gratitude to the organizers for inviting me to the International Workshop on Binary Stars and Stellar Atmospheres being held at one of the oldest seats of astronomical activity in the country. This assumes a very special significance as this workshop commemorates the 60th birthday of Professor Krishna Damodar Abhyankar, one of our most dedicated and distinguished astronomers. I first came in contact with Krishna in 1958 at Berkeley; he had finished his Ph.D. at Berkeley and stayed there till June 1959. I had just joined the University of California as a research associate after having completed my degree at Chicago. I was very fortunate in having kept contact with Krishna over the years and my esteem and admiration has always been the highest for Krishna. He is, perhaps, the most well rounded astronomer in the country to-day. I wish him good health for many more years to come; this will enrich the nation by the astronomers he may get to train.

1. Introduction

The capillary instability of a circular jet has been the subject matter of great attention since the pioneering work in the 19th century by Savart, Plateau & Lord Rayleigh (1878, 1879). Neglecting the affect of the surrounding air, Rayleigh showed that only symmetrical surface disturbances ($m = 0$) with wavelengths larger than the circumference of the cylinder would grow. In fact the surface waves grow as e^{nt} where

$$n^2 = \frac{T}{\rho R^3} \frac{x I_1(x)}{I_0(x)} (1 - x^2), \qquad \ldots(1)$$

where $x = kR$; T is the surface tension; ρ the density of the liquid; R the undisturbed radius of the jet; k the wave number of the disturbance along the axis of the cylinder; I_0, I_1 are modified Bessel functions of the second kind which are regular on the axis of the cylinder. The dispersion curve (1) has the maximum growth rate at

$$x = 0.697, \ (dn/dx = 0). \qquad \ldots(2)$$

It is also clear from relation (1) that the cut-off lies at $x = 1$ *i.e.* all wave numbers $x < 1$ are unstable ($n^2 > 0$) and for $x > 1$, $n^2 < 0$ and the amplitude of the disturbance varies as $e^{-i\omega t}$, where we have set $n^2 = -\omega^2$, $\omega^2 > 0$.

The effect of viscosity on the capillary instability was examined by Weber (1931). He showed that within the framework of the linear theory, viscosity does not alter the criterion of stability as predicted by the inviscid theory. However, the viscous effects would cause the wavelength of the most unstable state to become larger than that predicted by the inviscid theory. Thus if k_* denotes the wave number of the distrubance having the maximum growth rate, we find that

$$k_{*\,\text{vis}} < k_{*\,\text{invis}}. \qquad \ldots(3)$$

The analogous problem within the framework of magnetohydrodynamics has enormous significance because of its applications. The effect of a uniform axial magnetic field on the capillary instability has been examined by Chandrasekhar in the approximation of infinite conductivity (frozen-in-field approximation), zero viscosity; The jet has been assumed to be incompressible in these studies. In this case we find that the disturbances vary as $e^{i\omega t}$, where

$$\omega^2 = \frac{x I_1(x)}{I_0(x)} (x^2 - 1) + \frac{A_0^2\, x}{I_0(x) K_1(x)}, \qquad \ldots(4)$$

where $A_0^2 = B_0^2/(4\pi\rho)$, and ω is measured in units of $(T/\rho R^3)^{1/2}$. Note that if $A_0 = 0$, $\omega^2 < 0$ for $x < 1$. However with $A_0 \neq 0$, it is clear from equation (4) that ω^2 need not be negative for $x < 1$. In fact it can be readily shown that for $A_0^2 > 1/2$, ω^2 is positive for all k's.

When the amplitude of oscillation $\eta(z, t)$ is sufficiently small, the problem admits solution within the framework of the linearized theory. The eigenvalue problem posed by the system of governing equations and the associated boundary conditions is one of Sturm-Liouville type and can be solved by the principle of superposition which allows us to construct any wave form by a sum of elementary sinusoidal waves. When the amplitude of the waves is not small enough, the various sinusoidal components interact with each other and they are no longer independent elements. One of the significant aspects of nonlinear waves is that the wave speed depends on the amplitude as well. This may cause the wave front to become steeper in the course of propagation (because the speed of the trough is different from that of the crest of the wave). It is therefore of considerable interest to examine the nonlinear amplitude modulation of a travelling wave in the presence of a magnetic field. In order to describe the nonlinear interactions of small but finite amplitude waves we use the derivative expansion method with multiple scales and assume that all the physical quantities have uniformly valid asymptotic expansion in powers of a small ordering parameter ϵ. By requiring that these formal expansions satisfy the exact governing equations and the boundary conditions for all values of ϵ, sets of linearized boundary value problems are obtained.

The effect of a magnetic field on the gravitational instability of an infinite cylinder was examined by Chandrasekhar & Fermi (1953) who showed that a magnetic field has a stabilizing effect on the stability of the cylinder which was otherwise unstable for all $k < k_c = 1.0668$ (in units of the radius of the cylinder). The effect of finite amplitude perturbations on the capillary stability was examined by Yuen (1968), Wang (1968), Nayfeh (1970) and Kakutani *et al.* (1974). Tassoul & Aubin (1974) studied the finite amplitude disturbances in a self gravitating media and obtained a cut off wave number beyond which no stable flow pattern can be obtained. Malik & Singh (1979) studied the full amplitude modulation of a standing wave in the neighbourhood of the wave number $k = k_c$. We examine here the amplitude modulation of a progressive wave in the presence of an axial magnetic field for a self gravitating cylinder. (This work has been done jointly with R. K. Chhabra and is under publication in *Ap. Sp. Sci*). The nonlinear cut-off wave number obtained by Malik & Singh (1979) is recovered in the absence of the magnetic field ($A_0^2 = 0$).

2. The basic equations

We consider a self gravitating cylinder in the presence of uniform magnetic field along the axis of the cylinder. The fluid in the cylinder is assumed to be incompressible, inviscid, perfectly conducting, and of uniform density. We choose units such that the radius of the cylinder is $R = 1$. We restrict ourselves to axially symmetric disturbances of the cylinder in which the outer surface is distorted to $r = 1 + \eta(2z, t)$, where $\eta(z, t)$ denotes the elevation of the free surface measured from the unperturbed level $R = 1$. The equations valid in $r \leqslant 1 + \eta$ are then

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\,\mathbf{u} = -\nabla(\pi - V) + (\mathbf{h} \cdot \nabla)\mathbf{h}; \qquad \ldots(5)$$

$$\frac{\partial \mathbf{h}}{\partial t} = (\mathbf{h} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{h}, \qquad \ldots(6)$$

$$\nabla \cdot \mathbf{u} = 0, \;\; \nabla \cdot \mathbf{h} = 0, \qquad \ldots(7)$$

and

$$\nabla^2 V = -4\pi G \rho. \qquad \ldots(8)$$

Here $\mathbf{u}(r, z, t)$ is the velocity field and $\sqrt{4\pi\rho}\,\mathbf{h}(r, z, t)$ is the magnetic field, ρ being the constant fluid density. V is the internal gravitational potential; $\pi = p/\rho + (1/2)\,\mathbf{h}^2$; p is the pressure inside the cylinder.

The equations in the exterior region $r \geqslant 1 + \eta(z, t)$, are

$$\nabla^2 W = 0. \qquad \ldots(9)$$

$$\nabla \cdot \mathbf{h}^{(0)} = 0, \;\; \nabla \times \mathbf{h}^{(0)} = 0. \qquad \ldots(10)$$

The magnetic field $\mathbf{h}^{(0)}$ is expressible in terms of a potential ϕ, and we write

$$\mathbf{h}^{(0)} = A_0\, \mathbf{e}_z + \nabla\phi, \;\; \nabla^2 \phi = 0; \;\; r \geqslant 1 + \eta, \qquad \ldots(11)$$

where $A_0\sqrt{4\pi\rho}$ is the magnitude of the impressed magnetic field, A_0 being the Alfven velocity.

The boundary conditions at $r = 1 + \eta(z, t)$ are:

(i) The radial component of the velocity field must be compatible with the assumed form of the boundary, (ii) the normal component of $\mathbf{h}$ must be continuous, (iii) the normal component of the total stress must be continuous, (iv) the gravitational potential and its derivative must be continuous.

These boundary conditions can be written as (at $r = 1 + \eta(z, t)$)

$$u_r = \frac{\partial \eta}{\partial t} + u_z \frac{\partial \eta}{\partial z}, \qquad \ldots(12)$$

$$h_r^{(i)} - \eta_z\, h_z^{(i)} = h_r^{(0)} - \eta_z h_z^{(0)}, \qquad \ldots(13)$$

$$\Pi = p \quad \rho + \frac{1}{2}\mathbf{h}^2 = \frac{1}{2}\mathbf{h}^{(0)2}, \qquad \ldots (14)$$

$$V = W, \text{ and } \frac{\partial V}{\partial r} = \frac{\partial W}{\partial r}. \qquad \ldots (15)$$

We wish to investigate motions which are finite perturbations of the steady state. We use the derivative expansion method with multiple scales and assume that all physical quantities have uniformally valid asymptotic expansions in powers of a small ordering parameter ϵ. By requiring that these formal expansions satisfy the exact governing equations and the boundary conditions for all values of ϵ sets of linearized boundary value problems are obtained. We introduce the slow scales in space and time as

$$z_n = \epsilon^n z, \; t_n = \epsilon^n t, \; n = 0, 1, 2, 3. \qquad \ldots (16)$$

In order to describe nonlinear interactions of small but finite amplitude waves, we write for any variable.

$$f(r, z, t) = \sum_{n=1}^{N+1} \epsilon^n f_n(r, z_0, z_1, \ldots, z_N, t_0, t_1, \ldots, t_N) + O(\epsilon^{(N+2)}). \qquad \ldots (17)$$

where f is any of the variables u_i, h_i, v, w, ϕ, or η (z, t). While writing the expansion for η it will be noted that η depends only on z and t and not on r. Also for the derivative, we write

$$\frac{\partial}{\partial \alpha} = \sum_{n=0}^{N+1} \epsilon^n \frac{\partial}{\partial \alpha_n} + O(\epsilon^{(N+2)}) \qquad \ldots (18)$$

where α is any of the variable t or z. Substituting these expansions into equations (5)-(15) and comparing the coefficients of $\epsilon^n (n = 1, 2, 3)$, we obtain the following system of equations.

(i) *First order equations*

$$L_0(\mathbf{u}_1, \mathbf{h}_1, \Pi_1 - V_1) = 0,$$

$$M_0(\mathbf{h}_1, \mathbf{u}_1) = 0,$$

$$\underset{\sim}{\nabla}_0 \cdot \mathbf{u}_1 = 0, \; \underset{\sim}{\nabla}_0 \cdot \mathbf{h}_1 = 0, \qquad \ldots (19)$$

$$DX_1 = 0, \text{ where } X_1 = V_1, \; W_1, \text{ or } \phi_1.$$

The boundary conditions, reduced to $r = 1$ are

$$\frac{\partial \eta_1}{\partial t_0} = u_{1r},$$

$$A(h_{1r}, h^{(0)}_{1r}) = 0,$$

$$Y(\Pi_1, \eta_{1+} h^{(0)}_{1z}) = 0,$$

$$T(V_1, W_1, \eta_1) = 0,$$

$$A(V_1, W_1) = 0. \qquad \ldots (20)$$

where the various operators L_i, M_i etc. are defined in appendix A.

(ii) *Second order equations*

$$L_0(\mathbf{u}_2, \mathbf{h}_2, \Pi_2 - V_2) = -M(\mathbf{u}_1, \mathbf{h}_1) + N_0(\mathbf{h}_1, \mathbf{u}_1) - \mathbf{e}_z \frac{\partial}{\partial Z_1}(\Pi_1 - V_1),$$

$$M_0(\mathbf{h}_2, \mathbf{u}_2) = -M_1(\mathbf{h}_1, \mathbf{u}_1) + E_0^-(\mathbf{h}_1, \mathbf{u}_1),$$

$$\underset{\sim}{\nabla}_0 \cdot \mathbf{u}_2 + \frac{\partial u_{1z}}{\partial z_1} = 0,$$

$$\underset{\sim}{\nabla}_0 \cdot \mathbf{h}_2 + \frac{\partial h_{1z}}{\partial z_1} = 0,$$

$$DX_2 = 0, \text{ where } X_2 = V_2, W_2, \text{ or } \phi_2. \qquad \ldots (21)$$

The boundary conditions (on $r = 1$) are

$$F_t \eta_2 - u_{2r} = \eta_1 \frac{\partial u_{1r}}{\partial r} - U_{1z} \frac{\partial \eta_1}{\partial z_0},$$

$$A(h_{2r}, h_{2r}^{(0)}) = \eta_1 \frac{\partial}{\partial r} A(h_{1r}, h_{1r}^{(0)}) + A(h_{1z}, h_{1z}^{(0)}) \frac{\partial \eta_1}{\partial z_0},$$

$$Y(\Pi_2, \eta_2, h_{2z}^{(0)}) = \frac{1}{2}\eta_1^2 \frac{\partial^2 \Pi_0}{\partial r^2} + \frac{1}{2}\mathbf{h}_1^{(0)2} + \eta_1 \left(A_0 \frac{\partial h_{1z}^{(0)}}{\partial r} - \frac{\partial \Pi_1}{\partial r} \right),$$

$$T(V_2, W_2, \eta_2) = \eta_1 \left(\frac{\partial^2 W_1}{\partial r^2} - \frac{\partial^2 V_1}{\partial r^2} \right) - 2\pi G \rho \eta_1^2,$$

$$A(V_2, W_2) = \eta_1 \left(\frac{\partial W_1}{\partial r} - \frac{\partial V_1}{\partial r} \right). \qquad \ldots (22)$$

(iii) *Third order equations*

$$L_0(\mathbf{u}_3, \mathbf{h}_3, \Pi_3 - V_3) = -M_1(\mathbf{u}_2, \mathbf{h}_2) - M_2(\mathbf{u}_1, \mathbf{h}_1) + E_0^+(\mathbf{h}_1, \mathbf{h}_2)$$
$$- E_0^+(\mathbf{u}_1, \mathbf{u}_2) + D_1(\mathbf{h}_1, \mathbf{u}_1) - \mathbf{e}_z \frac{\partial}{\partial z_1}(\Pi_2 - V_2) + \frac{\partial}{\partial z_2}(\Pi_1 - V_1),$$

$$M_0(\mathbf{h}_3, \mathbf{u}_3) = -M_1(\mathbf{h}_2, \mathbf{u}_2) - M_2(\mathbf{h}_1, \mathbf{u}_1) + E_0^-(\mathbf{h}_1, \mathbf{u}_2)$$
$$+ E_0^-(\mathbf{h}^2, \mathbf{u}_1) + D_1(\mathbf{u}_1, \mathbf{h}_1),$$

$$\underset{\sim}{\nabla}_0 \cdot \mathbf{u}_3 + \frac{\partial u_{1z}}{\partial z_2} + \frac{\partial u_{2z}}{\partial z_1} = 0, \qquad \ldots (23)$$

$$\underset{\sim}{\nabla}_0 \cdot \mathbf{h}_3 + \frac{\partial h_{1z}}{\partial z_2} + \frac{\partial h_{2z}}{\partial z_1} = 0,$$

$$DX_3 = 0, \text{ where } X_3 = V_3, W_3, \text{ or } \phi_3.$$

The boundary conditions (on $r = 1$) are

$$F_t \eta_3 - u_{3r} = K(u_{2r}, u_{1r}) - u_{1z} F_z \eta_2 - u_{2z} \frac{\partial \eta_1}{\partial z_0},$$

$$A(h_{3r}, h_{3r}^{(0)}) = -K(h_{2r}, h_{1r}) + K(h_{2r}^{(0)}, h_{1r}^{(0)}) + A(h_{1z}, h_{1z}^{(0)})F_z\eta_2 + \frac{\partial\eta_1}{\partial z_0}\left\{A(h_{2z}, h_{2z}^{(0)}) + \eta_1\frac{\partial}{\partial r}A(h_{1z}, h_{1z}^{(0)})\right\},$$

$$Y(\Pi_3, \eta_3, h_{3z}^{(0)}) = -K(\Pi_2, \Pi_1) + A_0 K(h_{2z}^{(0)}, h_{1z}^{(0)}) - \eta_1\eta_2\frac{\partial^2\Pi_0}{\partial r^2} - \frac{1}{6}\eta_1^3\frac{\partial^3\Pi_0}{\partial r^3} + \frac{1}{2}\eta_1\frac{\partial \mathbf{h}_1^{(0)2}}{\partial r} + A_0\eta_1\frac{\partial h_{2z}^{(0)}}{\partial r} + \mathbf{h}_1^{(0)}\ \mathbf{h}_2^{(0)},$$

$$T(V_3, W_3, \eta_3) = \eta_1\left(\frac{\partial^2 W_2}{\partial r^2} - \frac{\partial^2 V_2}{\partial r^2}\right) - 4\pi G\rho\eta_1\eta_2 + 2\pi G\rho\eta_1^3. + \eta_2\left(\frac{\partial^2 W_1}{\partial r^2} - \frac{\partial^2 V_1}{\partial r^2}\right) + \frac{1}{2}\eta_1^2\left(\frac{\partial^3 W_1}{\partial r^3} - \frac{\partial^3 V_1}{\partial r^3}\right),$$

$$R(V_3, W_3) = \eta_1\frac{\partial}{\partial z_0}\left(\frac{\partial W_2}{\partial r} - \frac{\partial V_2}{\partial r}\right) + \frac{1}{2}\eta_1^2\frac{\partial}{\partial z_0}\left(\frac{\partial^2 W_1}{\partial r^2} - \frac{\partial^2 V_1}{\partial r^2}\right) + \eta_2\frac{\partial}{\partial z_0}\left(\frac{\partial W_1}{\partial r} - \frac{\partial V_1}{\partial r}\right) + \eta_1\frac{\partial}{\partial z_1}\left(\frac{\partial W_1}{\partial r} - \frac{\partial V_1}{\partial r}\right). \quad \ldots (24)$$

3. Solutions

(i) *Zero order solutions*

We may recall that the zero order solutiqns are

$$V_0 = -\pi G\rho r^2,\ W_0 = -2\pi G\rho\ \ln(r),$$
$$\Pi_0 = p_0/\rho + (1/2)A_0^2;\ p_0/\rho = \pi G\rho(R^2 - r^2). \quad \ldots (25)$$

(ii) *First order solution*

We seek solutions of the first order equations (19) - (20) in the form of a progressive harmonic wave. Assuming all field quantities to be proportional to exp $(i\psi)$, where $\psi = kz_0 - \omega t_0$, we obtain the following real solutions.

$$\eta_1 = A e^{i\psi} + \text{c.c.},$$

$$u_{1r} = -\frac{i\omega I_1(kr)}{I_1(k)}(A e^{i\psi} - \text{c.c.}),$$

$$u_{1z} = \omega\frac{I_0(kr)}{I_1(k)}(A e^{i\psi} + \text{c.c.}),$$

$$h_{1r} = \frac{ikA_0 I_1(kr)}{I_1(k)}(A e^{i\psi} - \text{c.c.}),$$

$$h_{1z} = -\frac{kA_0 I_0(kr)}{I_1(k)}(A e^{i\psi} + \text{c.c.}), \quad \ldots (26)$$

$$\Pi_1 - V_1 = \frac{\omega I_0(kr)}{\alpha I_1(k)}(A\, e^{i\psi} + \text{c.c.}),$$

$$V_1 = 4\pi G \rho K_0(k) I_0(kr)(Ae^{i\psi} + \text{c.c.}),$$

$$W_1 = 4\pi G \rho I_0(k) K_0(kr)(Ae^{i\psi} + \text{c.c.}),$$

$$\phi_1 = -\frac{iK_0(kr)}{K_0(k)}(Be^{i\psi} - \text{c.c.}),$$

where

$$\alpha = \frac{\omega k}{\omega^2 - k^2 A_0^2},$$

$$B = A\, A_0 \frac{K_0(k)}{K_1(k)}, \qquad \ldots(27)$$

and c.c. denotes the complex conjugate.

On employing the boundary conditions one obtains the dispersion relation

$$\omega^2 = -\frac{4\pi G \rho k I_1(k)}{I_0(k)}\left(I_0(k) K_0(k) - \frac{1}{2}\right) + \frac{k A_0^2}{K_1(k) I_0(k)}. \qquad \ldots(28)$$

The linear dispersion relation (28) was obtained by Chandrasekhar & Fermi (1953). The wavelength at which instability occurs will be determined by the roots of the transcendental equation

$$\frac{k I_1(k)}{I_0(k)}\left(I_0(k) K_0(k) - \frac{1}{2}\right) = \left(\frac{H}{H_G}\right)^2 \frac{K}{K_1(k) I_0(k)}, \qquad \ldots(29)$$

where H_G is defined by

$$H_G = (4\pi \rho)\, G^{1/2}. \qquad \ldots(30)$$

It is clear that equation (29) allows, for an assigned value of (H/H_G), ;a single positive root ($k_c \neq 0$). We find that $\omega^2 < 0$ for $0 < k < k_c$. The cylinder, is, therefore, unstable for all varicose deformations with wavenumber $k < k_c$, where k_c now depends on the strength of the prevalent magnetic field through H/H_G. The travelling wave solutions are possible only for $k > k_c$. Our aim is to study the amplitude modulation of the travelling waves for $k > k_c$.

4. Higher order solutions

Substituting the first order solutions (26) into equation (21), we obtain the second order equations and the associated boundary conditions. On substituting the first order solutions into these equations, one can solve these to obtain the second order quantities for the velocity, magnetic field, and other physical variables. When one applies the boundary conditions, one obtains the equation

$$\frac{\partial A}{\partial t_1} + V_g \frac{\partial A}{\partial z_1} = 0, \qquad \ldots(31)$$

and its complex conjugate relation. The group velocity of the waves V_g is given by

$$V_g = \frac{k I_1(k)}{2\,\omega I_0(k)}\Big[-\frac{\omega^2}{k} + \frac{\omega}{\alpha}\frac{I_0^2(k)}{I_1^2(k)} + \frac{2A_0^2}{k I_1(k) K_1(k)} + \frac{k A_0^2 K_0^2(k)}{K_1^2(k)} + 4\pi G\rho\,(K_1(k) I_0(k) - K_0(k) I_1(k))\Big]. \qquad \ldots(32)$$

Equation (31) implies that the wave moves with the group velocity V_g in the second order approximation. This means that the amplitude A depends on the slow variables z_1, t_1, through the combinations $(z_1 - V_g t_1)$. A significant feature of these solutions is the presence of the second harmonic resonance characterized by the vanishing of the denominator in the expressions for the physical quantitites. Therefore, the analysis given here is not valid in the vicinity of such a resonance.

We now substitute the first and second order solutions into the third order equations (23). In writing these equations we shall retain only terms containing the constants and the first harmonic in the inhomogeneous part of the equations. This is due to the fact that it is only these terms that give us the evolution of the envelope of the monochromatic wave.

In order to obtain the non-linear Schrodinger equation, we seek a solution of these equations in the form

$$f_3 = a_3 e^{i\psi} + b_3 + \text{c.c.} \qquad \ldots(33)$$

where f_3 is any of the physical variables. The terms independent of ψ then lead to certain equations for the constants etc. On applying the boundary conditions, these equations lead to the nonlinear Schrödinger equation for A of the form:

$$i\left[\frac{\partial A}{\partial t_2} + V_g \frac{\partial A}{\partial z_2}\right] + P\frac{\partial^2 A}{\partial z_1^2} = Q A^2 \bar{A}. \qquad \ldots(34)$$

where P and Q are functions of the quantities describing the equilibrium and the first order variables, etc.

5. Discussion

It is well known that the solutions of equation (34) are stable or unstable against modulations accordingly as $PQ > 0$ or $PQ < 0$. The values of group velocity rate P, the interaction parameter Q are computed for various values of k and $(H/H_G)^2$. In the absence of magnetic field ($A_0^2 = 0$), we have modulational instability for $k > 1.382$ (in addition to linear instability for $k < 1.067$). This result has already been obtained by Malik & Singh (1980) for disturbances which are both solenoidal and irrotational. The present result is more general and does not restrict itself to irrotational disturbances, hence the discrepancy in the critical wave number which is here 1.382 as compared to 1.197 obtained by Malik & Singh (1980). This must be due to the fact that relaxation of the restriction of irrotational flow allows more general flows and hence greater stability. A similar disparity has also been observed by Lardner & Trehan (1983) in their calculations in the magnetohydrodynamic stability of a liquid jet.

The stability regions in the $k - (H/H_G)^2$ plane are shown in figure 1. The solid curves labelled P or Q are stability boundaries across each of which P or Q changes sign.

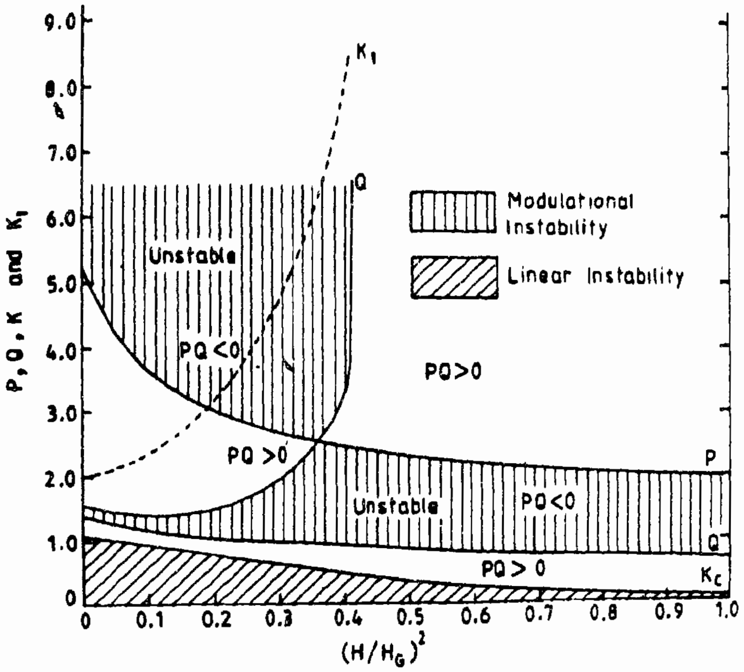

Figure 1. Instability regions in the $k-(H/H_G)^2$ plane The dashed curve indicates the second harmonic resonance

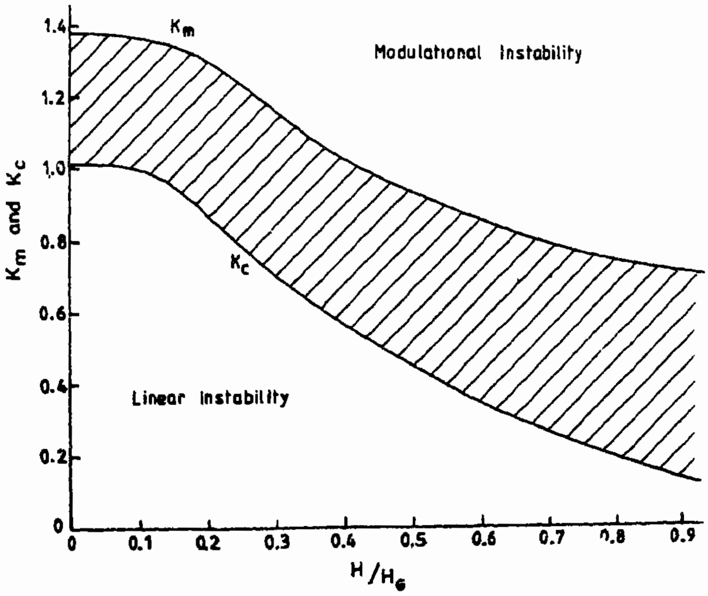

Figure 2. Stable region (shaded) in the $k-(H/H_G)$ plane

The shaded regions are those of modulational instability ($PQ < 0$). The solid curve labelled k_c gives the linear cut off wave number. The region below this curve is the region of linear instability. The dashed curve labelled K_l gives the second harmonic resonance number, in the neighbourhood of this curve the solution is not valid.

It is also clear from figure 1 that for any $(H/H_G)^2$ there are two bands of k values which give instability. When $(H/H_G)^2 = 0.365$, the upper instability band shrinks to the single value $k = 2.52$. It means that the wavelength of the unstable disturbance can be accurately controlled by the magnetic field. We have shown in figure 2 the linear stability boundary and the modulational stability boundary, the stable region is the shaded one.

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Appendix A

We define the following operators for convenience:

$$L_1(\mathbf{X},\mathbf{Y},Z) = \frac{\partial \mathbf{X}}{\partial t_1} - A_0 \frac{\partial \mathbf{Y}}{\partial z_1} \underset{\sim}{\nabla}_1 \mathbf{Z}, \qquad \ldots (\mathrm{A1})$$

$$M_1(\mathbf{X},\mathbf{Y}) = \frac{\partial \mathbf{X}}{\partial t_1} - A_0 \frac{\partial \mathbf{Y}}{\partial z_1}, \qquad \ldots (\mathrm{A2})$$

$$N_1(\mathbf{X},\mathbf{Y}) = (\mathbf{X} \cdot \underset{\sim}{\nabla}_1)\,\mathbf{Y} - (\mathbf{Y} \cdot \underset{\sim}{\nabla}_1)\mathbf{Y} \qquad \ldots (\mathrm{A3})$$

$$E_1^{+}(\mathbf{X},\dot{\mathbf{Y}}) = (\mathbf{X} \cdot \underset{\sim}{\nabla}_1)\mathbf{Y} \pm (\mathbf{Y} \cdot \underset{\sim}{\nabla}_1)\mathbf{X}, \qquad \ldots (\mathrm{A4})$$

$$D_1(\mathbf{X},\mathbf{Y}) = h_{1z} \frac{\partial \mathbf{Y}}{\partial z_1} - u_{1z} \frac{\partial \mathbf{X}}{\partial z_1}, \qquad \ldots (\mathrm{A5})$$

$$F_\mu Z_1 = \frac{\partial Z_1}{\partial \mu_0} + \frac{\partial Z_{l-1}}{\partial \mu_1} + \ldots + \frac{\partial Z_1}{\partial \mu_{l-1}}, \qquad \ldots (\mathrm{A6})$$

$$K(Z,S) = \eta_1 \frac{\partial Z}{\partial r} + \eta_2 \frac{\partial S}{\partial r} + \frac{1}{2}\eta_1^2 \frac{\partial^2 S}{\partial r^2}, \qquad \ldots (\mathrm{A7})$$

$$Y(Z, S, p) = Z + S\frac{\partial \pi_0}{\partial r} - A_0 p, \qquad \text{..(A8)}$$

$$T(V_1, W_1, \eta_1) = \frac{\partial V_1}{\partial r} - \frac{\partial W_1}{\partial r} - 4\pi G \rho \eta_1, \qquad \text{. (A9)}$$

$$R(V_1, W_1) = \frac{\partial V_1}{\partial Z_0} - \frac{\partial W_1}{\partial Z_0}, \qquad \text{...(A10)}$$

$$A(Z, S) = Z - S, \qquad \text{..(A11)}$$

$$DX_i = \nabla_0^2 X_i + 2\frac{\partial^2 X_{i-1}}{\partial z_0 \partial z_1} + 2\frac{\partial^2 X_{i-2}}{\partial z_0 \partial z_2} + \frac{\partial^2 X_{i-2}}{\partial z_1^2}, \qquad \text{...(A12)}$$

where

$X_{-i} = 0;\ i = 1, 2, 3, \ldots\ldots$

$$\underset{\sim}{\nabla}_0 = \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_z \frac{\partial}{\partial z_0}. \qquad \text{...(A13)}$$

$$\nabla_0^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z_0^2}. \qquad \text{...(A14)}$$

Discussion

Bhatnagar : What happens after the instability sets in?
Trehan : When the instability sets in the amplitude of oscillations grows and the present analysis is not valid. The purpose of this calculation is to isolate the region of (linear) stability which is modulationally stable i.e. in the weakly nonlinear approximation.
Bhatnagar : Sometimes resonance occurs in the perturbed differing system. They overlap and KAM surfaces starts disintegrating, and when the last KAM surface disintegrates the chaos occurs. Please comment.
Trehan : The case of resonance is to be treated separately and has been done by Chhabra & Trehan (*Int. J. Engg Sci.*, in the press).