Some Rotational Effects in Black Hole Spacetimes

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To my parents

Declaration

I hereby declare that this thesis, submitted to Bangalore University, Bangalore, for the award of a Ph.D. degree, is a result of the investigations carried out by me at Indian Institute of Astrophysics, Bangalore, under the supervision of Professor C. V. Vishveshwara. The results presented herein have not been subject to scrutiny, by any university or institute, for the award of a degree, diploma, associateship or fellowship whatsoever.

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April 5, 2000

Certificate

This is to certify that the thesis entitled 'Some Rotational Effects in Black Hole Spacetimes' submitted to the Bangalore University by K. Rajesh Nayak for the award of the degree of Doctor of Philosophy in the faculty of Science, is based on the results of the investigations carried out by his under my supervision and guidance, at the Indian Institute of Astrophysics. This thesis has not been submitted for the award of any degree, diploma, associateship, fellowship, etc. of any university or institute.

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Some Rotational Effects in Black Hole Spacetimes

Abstract:

We study some of the rotational effects such as gyroscopic precession, the general relativistic analogues of inertial forces and to gravito-electromagnetism in black hole spacetimes and establish interrelations among them. The phenomenon of gyroscopic precession is not only important from the conceptual point of view but also it has been proposed as a test to the general theory of relativity itself. We use the covariant Frenet-Serret formalism for gyroscopic precession as given by Iyer and Vishveshwara. Recently, there has been considerable interest in the general relativistic analogues of inertial forces, especially the centrifugal force and its reversal. We study inertial forces using the covariant formalism given by Abramowicz, Nurowski and Wex. The similarity between gravity and electromagnetism allows one to define the concept of gravito-electromagnetism. We use the properties of Killing vectors in order to define gravito-electromagnetic fields. We define gravito-electric and gravito-magnetic fields with respect to the irrotational congruence in an axially symmetric stationary spacetime. We establish the direct covariant relations between these rotational effects in an axially symmetric stationary spacetime. One of the important results which emerged from these relations is that of the simultaneous reversal of gyroscopic precession and centrifugal force in general static spacetimes. Previous studies indicated this by computing the centrifugal force and gyroscopic precession for specific examples. From direct relations we also show that neither centrifugal force nor gyroscopic precession reversal occurs at the photon orbits in stationary spacetimes. In order to get more physical insight, we also apply this formalism to the black hole spacetimes such as the Kerr-Newman and the Ernst spacetimes. In static spacetimes such as the Ernst,

Reissner-Nordstrom and Schwarzschild spacetimes we observe that reversal of both gyroscopic precession and centrifugal force occurs at the circular photon orbits. In case of stationary spacetimes such as the Kerr-Newman and Kerr spacetimes reversal of gyroscopic precession and centrifugal force occurs at different points in the spacetime.

The concept of inertial forces from Newtonian mechanics was generalized to the general theory of relativity in order to study the motion of test particles. In Newtonian mechanics the concept of potential which defines the force is used to express the gravitational field through the Laplace or the Poisson equations. If the fiducial test particle is moving along the Killing trajectories such as a circular orbit in axially symmetric stationary spacetimes, one observe the fact that the inertial forces are proportional to the gradient of a scalar function. We express the Einstein equations in axially symmetric stationary spacetimes in terms of inertial forces. For source free axially symmetric stationary spacetimes, we use the formalism given by Geroch in order to express field equations in terms of inertial forces. Hansen and Winicour extended the Geroch formalism for field equations with source terms. We use this formalism for expressing the field equations in terms of inertial forces.

Contents

Chapter 1

Introduction

1.1 Introduction

The existence of black holes is one of the most interesting predictions of the general theory of relativity. In contrast to the Newtonian solution to the gravitation field of a point mass, the Schwarzschild solution to the Einstein equations exhibits novel physical effects such as the existence of the event horizon and the circular null geodesic. Physical phenomena in the strong gravitational fields of black holes are not only interesting from the conceptual point of view but are also of astrophysical importance. There has been considerable amount of studies in the area of black hole physics over the past three decades. These include, mainly, the geometric structure and physical properties of black holes. In addition, rotation induces interesting as well as intriguing physical effects both in Newtonian mechanics and the general theory of relativity. In relativity, these effects are built into the structure of spacetime, such as that of the rotating black hole. In the case of the Kerr black hole, rotation separates the stationary limit from the event horizon and gives rise to the ergo-region. This leads to several interesting effects such as the Penrose process and super-radiance. In this thesis we study some of the rotational effects in stationary, axially symmetric spacetimes, in particular those of black holes. These are gyroscopic precession, the general relativistic analogues of inertial forces and gravito-electromagnetism. We investigate these effects in the general case of axially symmetric, stationary spacetimes and apply them to the black hole spacetimes as specific examples. We also establish the interrelations among these rotational effects. These formalisms are also applicable to the case of ultra compact objects whose radii are very close to the event horizon. We shall now give a brief outline of these effects.

1.2 Inertial Forces

Inertial forces give very simple physical descriptions of dynamical systems involving non-inertial frames in Newtonian mechanics. In order to apply Newton's laws within non-inertial frames such as rotating frames, one needs to add suitable force terms. These forces are called pseudo-forces or inertial forces. Inertial force terms are generated when Newton's laws are transformed from a global inertial frame to a non-inertial frame. Newtonian dynamics can be fully described using inertial frames. However, non-inertial frames, hence inertial forces cannot be avoided in many applications. The well known example of an inertial force is the centrifugal force, which one experiences in day-to-day life. In order to gain further insight, we consider a rotating frame as an example of non-inertial frames. The equation of motion of a test particle in a non-inertial frame can be decomposed into the following terms,

$$m\frac{dv}{dt} = m\frac{dv_0}{dt} + mr \times \dot{\omega} + 2mv \times \omega + m(\omega \times r) \times \omega.$$

where ω is the angular velocity of the frame. The first term on the right is the acceleration of the test particle in the rotating frame. Other terms correspond to pseudo-forces or inertial forces called, Euler force, Coriolis force, and centrifugal force respectively. The Euler force arises because of the change in the angular velocity of the rotating frame itself and vanishes in the case of uniformly rotating frames. The Coriolis force arises because of the relative motion of the test particle with respect to the rotating frame. When one considers the earth as an example of a rotating frame, the Coriolis force give rise to several interesting physical effects. These effects are important when the relative velocity of test particles are significant, such as in the case of ocean currents and wind flows. For a test particle moving along a meridian of the earth with a constant angular velocity, it can be shown that the Coriolis force acts along the eastward direction in the northern hemisphere, along the western direction in the southern hemisphere and vanishes on the equator. This phenomenon manifests itself in the case of an imaginary river flowing from the south pole to the north pole by exerting more pressure on the eastern bank in the northern hemisphere and on the western bank in the southern hemisphere. In the case of wind flow one of the effects of Coriolis force is known as Buys-Ballot law, according to which the direction of wind velocity is not along the pressure gradient but is deviated considerably to the right in the northern, to the left in the southern hemispheres respectively. The Coriolis force vanishes if the particle is at rest with respect to the accelerated frame. These are some of the physical effects which can be explained using the concept of inertial forces. However, since these forces are merely due to the reference system, by choosing suitably a generalized coordinate system one can avoid referring to these inertial forces.

One of the very important properties of inertial forces is that they are always proportional to the mass of the particle . The same is true of gravitational force as well. This striking similarity of the two gave rise to the equivalence principle, which states that the properties of the motion in a non-inertial system are the same as those in an inertial system in the presence of a gravitational field. Hence, in a local inertial frame the gravitational and inertial forces are indistinguishable. In the general theory of relativity, all forces are replaced by the curvature effects of the spacetime. However, the concept of forces is a very useful tool for analyzing the rotational effects in the general theory of relativity.

The general theory of relativity is a fully covariant theory, which remains invariant under arbitrary coordinate transformations. Because of this, it does not in allow general any global inertial frames in order to define the inertial forces. However, one can generalize the concept of Newtonian global rest frames even in general relativity in special cases. The concept of a global rest frame is associated with the existence of a timelike vector, which is hypersurface orthogonal. The three surface orthogonal to this timelike vector defines the space of simultaneity. This hypersurface orthogonal timelike vector along with the three-surface orthogonal to it defines the generalized Newtonian rest frame. For the case of an axially symmetric stationary spacetime, Greene, Schücking and Vishveshwara showed that such global rest frames can be uniquely defined[35].

Abramowicz, Carter and Lasota [1] first formulated the concept of inertial forces in static and stationary spacetimes using the optical reference frame. The optical reference geometry can be constructed by conformally rescaling the spacetime metric with a suitable conformal factor. The important result that emerged from their studies is the reversal of centrifugal force in static spacetimes at the photon orbits if such orbits exist, whereas in the case of Newtonian mechanics the direction of centrifugal force is always away from the axis. Abramowicz and Prasanna showed that this occurs at $r = 3m$ in the Schwarzschild spacetime. This paradoxical behavior at $r = 3m$ in the Schwarzschild spacetime was first observed by Abramowicz and Lasota[10, 11]. They pointed out that, at $r = 3m$, the thrust needed to keep a test particle in a circular orbit is independent of the velocity of the particle. Abramowicz, Nurowski and Wex [2] have presented a general formalism for inertial forces in an arbitrary spacetime.

They decomposed the acceleration of the test particle with respect to a global rest frame defined by the timelike hypersurface orthogonal vector field provided such a vector field exists. The details of this formalism are given in the second chapter. We adopt this covariant formalism for our study. In an arbitrary spacetime the three-space and time splitting is non-unique. However, one may note that all such splittings may not represent the generalized Newtonian global rest frame. Hence, a totally arbitrary decomposition is not useful for defining inertial forces. As mentioned before, in the case of an axially symmetric stationary spacetime, the global rest frame can be defined uniquely[35]. However, the splitting and identifying the various terms in the acceleration might differ depending on the physical nature of the problem or depending on the observers' frame. Various such possible splitting schemes have been discussed by Bini, Carini and Jantzen[16]. They have also studied these formalisms in the case of an axially symmetric stationary spacetime as an example[17].

De Felice has criticized the decomposition of spacetime into spatial and temporal parts. He has argued that the paradoxical behavior of the test particle below the radius of circular null geodesic in the Schwarzschild spacetime is due to more gravitational attraction than the centrifugal force reversal[24]. He has also defined what is called pre-horizon regime, a region in spacetime in which, when an increase in the angular velocity of the test particle orbiting on timelike non-geodesic spatially circular trajectories, causes more gravitational attraction than centrifugal repulsion[24, 25]. In static, spherically symmetric spacetimes, the pre-horizon regime occurs below the circular null geodesic. In the case of Kerr spacetime, the situation is more complex due to the existence of co- and counter- rotating null geodesics. This causes a shrinkage in the pre-horizon regime[26]. De Felice and Usseglio-Tomasset have investigated various definitions of inward and outward directions with respect to the center in the Schwarzschild spacetimes and the possibility of inferring the direction using a gyrocompass[27]. As mentioned before, in static spacetimes, for spatially circular orbits with radius less than the circular null geodesics, have the property that an increase in the angular velocity corresponds to an increase in the thrust in the outward direction. In the case of the Schwarzschild spacetime this occurs for radii which are too small ($r < 3m$) for direct observations. However, De Felice has suggested that, in the case of the Kerr spacetime, the above effect not only holds at small coordinate distances from the event horizon for co-rotating circular orbits, but also holds arbitrarily far away from the source on counter-rotating circular orbits with angular velocities which tend asymptotically to zero. Since this behavior has no Newtonian analogue, it can be taken as a possible test for the general theory of relativity[28].

An alternative scheme for decomposing the acceleration has been proposed by Semeràk[64]-[70]. In this approach also, the observers following timelike hypersurface orthogonal vector fields are used as fundamental observers. The transport law for such fundamental observer's tetrad is decomposed into a Fermi-Walker part and a spatial rotation part. The acceleration of a test particle as observed by a fundamental observer can be split into terms consisting of the gravitational (gravito-electric) part, the dragging (gravito-magnetic or Lense-Thirring) part, the Coriolis part, the centrifugal part (minus normal component of the particle's specific inertial resistance) and the tangent component of the particles' specific inertial resistance[64]. Thus defined, centrifugal force is always repulsive as in the case of Newtonian mechanics and in contrast to the definition given by Abramowicz *et. al.* [1, 2]. Semeràk has also related inertial forces to the concept to gravito-electro magnetism and of the gyroscopic precession[65].

Despite the fact that the general theory of relativity does not allow the concept of forces, the generalization of the Newtonian concept of forces is a useful tool for understanding the motion of test particles as well as for understanding the spacetime structure. Many of the advantages of inertial forces are described by Abramowicz *et. al* [6]-[9]. Alternative definitions of inertial forces and interrelations among them also helps to gain more insight into the physical phenomena. In an attempt to understand the connection between the general theory of relativity and the Mach principle, Prasanna and Iyer have introduced a new parameter called cumulative drag index[61, 62]. This parameter is defined for a particle in a circular orbit in an axially symmetric stationary spacetime. It was also shown that the behavior of this parameter is similar for both co- and counter- rotating orbits[61]. Initialy this parameter was defined on circular orbits on which the centrifugal force vanishes and later it was generalized to all circular orbits[62].

In the next section we introduce another important rotational effect,the concept of gravito-electromagnetism.

1.3 Gravito-electromagnetic Fields

Another important tool for investigating the rotational effects in the general theory of relativity is gravito-electromagnetism. The concept of gravito-electromagnetism arose due to the similarity between the general theory of relativity and the Maxwell theory of electrodynamics. The analogy between the electrostatic force and the gravitational force is well known as both the forces follow the inverse square law. If one compares the motion of a charged test particle with a test particle in a uniformly rotating frame, one can further identify the similarity between the magnetic field and inertial forces. This comparison becomes straight forward if one writes the Lagrangian for both motion of a charged test particle in an electromagnetic field and a test particle in a uniformly rotating frame. The generalized potential for a charged test particle in an electromagnetic field may be written as,

$$U_{EM} = q\phi - \frac{q}{c}\vec{A}\cdot\vec{v}. \tag{1.1}$$

In the above, ϕ and $\vec{A}$ are electric(scalar) and magnetic(vector) potentials respectively. The charge of the test particle is represented by q and $\vec{v}$ is its spatial velocity. Similarly the effective potential for a test particle in a uniformly rotating frame may be written as,

$$U_{ROT} = V - \frac{1}{2}m|\vec{\omega}\times\vec{r}|^2 - m\vec{v}\cdot(\vec{\omega}\times\vec{r}). \tag{1.2}$$

Here $\vec{\omega}$ is the angular velocity of the rotating frame, and $\vec{v}$ is the velocity of the test particle with respect to the rotating frame. The function V is the potential in the rotating frame. From the above effective potentials, one can draw the following analogies. The term $V - \frac{1}{2}m|\vec{\omega}\times\vec{r}|^2$ is similar to the electromagnetic scalar potential ϕ and the term $(\vec{\omega}\times\vec{r})$ is similar to the electromagnetic vector potential $\vec{A}$. The magnetic force acting on the charged test particle is similar to the Coriolis force $(\vec{v}\times\vec{\omega})$, acting on the test particle in a rotating frame. The above example illustrates the similarity between electromagnetism and inertial forces within the framework of Newtonian mechanics.

Faraday unified the concept of electric and magnetic fields, finally giving rise to the Maxwell theory of electromagnetism. The equivalence principle proposed by Einstein does not allow one to distinguish between the gravitational and the inertial forces locally. From the point of view of structure of the equations, both Einstein's equations and Maxwell's equations constitute a set of hyperbolic partial differential equations. Using the suitable gauge condition, Maxwell's equations can be written in the form of a wave equation. Similarly, in the linearized theory of gravity, the Einstein equations can also be cast in the form of a wave equation. The important implication of this is the existence of gravitational waves, which was predicted by Einstein. When the linearized field equations are applied to a slowly rotating mass configuration, one obtains several interesting results. The gravitational field produced by a massive sphere in the general theory of relativity is analogous to the electric field produced by an electrically charged sphere. When rotation is induced, in addition to the electric field, one also has the magnetic field generated whose strength is determined by the angular velocity. When a massive sphere is slowly rotated in the general theory of relativity, it generate a field similar to a magnetic field known as the gravito-magnetic field. This solution was first obtained by Lense and Thirring[51, 52, 53, 54]. In this case also, the strength of the gravito-magnetic field depends on the angular momentum of the rotating mass. In this example the $g_{\phi t}$ component of the metric tensor is the vector potential for the gravito-magnetic field. The details of this approach may be found in reference[23, 74]. As in the case of a charged spinning test particle in a magnetic filed, a spinning gyroscope would experience a torque when subjected to the gravito-magnetic field. Therefore, the gyroscopic would precess with a frequency proportional to the gravito-magnetic field. This phenomenon is known as the dragging of inertial frames and the precession is known as the Lense-Thirring precession. In this manner, gyroscopes can be used as a probe to investigate the gravito-electromagnetic fields. Several authors have discussed gravito-electromagnetism and its effect on gyroscopic precession within similar formalism[23, 74, 64, 29, 46].

Our approach to the gravito-magnetism is not based on the weak field or linearized field equations. Striking similarity between a constant electromagnetic field and the Killing vector field was demonstrated by Honing, Schücking and Vishveshwara[42]. We extend this analogy and define the gravito-electric and gravito-magnetic fields with respect to irrotational observers in an axially symmetric stationary spacetime. In similar fashion it is possible to define electric and magnetic fields for a given electromagnetic field tensor[13]. In the case of the gravito-electromagnetic field the derivative of the timelike Killing vector plays the role of the anti-symmetric field tensor[42, 43, 59]. The details of the formalism are given in chapter 3. In the next section we shall describe the gyroscopic precession.

1.4 Gyroscopic Precession

The phenomenon of gyroscopic precession is of interest in various situations in physics. In the context of Newtonian mechanics, a gyroscope can be idealized by a rapidly spinning rigid body, for example, a rapidly rotating symmetrical top. When subjected to the external torque or gravitational field, the gyroscope undergoes precession and nutation. One of the natural examples of gyroscopic precession is the precession of equinox. In this particular case the Earth itself acts like a gyroscope and precesses because of the torque exerted by the solar system. The Larmor precession is another example of the precession in the framework of electromagnetic theory. Here, a magnetic dipole of the charged particle acts like a gyroscope and precesses when subjected to a uniform magnetic field. The frequency of precession is proportional to the strength of the magnetic field. From the conservation of angular momentum one can show that, in Newtonian mechanics, precession can occur only in the presence of a torque.

In the special theory of relativity, the Thomas precession refers to the precession of the inertial compass along an arbitrary world line of an accelerated particle in the Minkowski space. When two successive pure Lorentz transformations are applied on an inertial frame it induces a spatial rotation of the frame. This is referred to as Thomas precession. One can show that the Thomas precession frequency can be written as $\omega = \frac{1}{2}(a \times v)$, where a is the acceleration of the particle and v is its spatial velocity. This indicates that the Thomas precession occurs only when the acceleration is non-zero. In other words when the trajectories are along geodesics precession is zero. This is also true for accelerated straight line motion as the precession frequency is the cross product of the acceleration and velocity. Thomas precession plays an important role in atomic physics, where this effect was first experimentally verified[73].

In the context of the general theory of relativity, gyroscopic precession involves kinematic effects, contributions from spacetime curvature and the effect of inertial frame dragging when the spacetime possesses inherent rotation. The gyroscopic precession has been proposed as a test of the theory itself. Since in curved spacetimes, the geodesics are in general not straight lines, the precession along the geodesics is non-zero. In the general theory of relativity the gyroscopes are mathematically idealized as spatial triads of Fermi-Walker transported tetrads transported along a worldline[56]. Precession of the Fermi-Walker frame with respect to any other frame transported along the same trajectory is physically realized as the gyroscopic precession. We adapt the Frenet-Serret formalism for gyroscopic precession developed by Iyer and Vishveshwara[43]. In this formalism, the precession of the Fermi-Walker frame is realized with respect to a Frenet-Serret tetrad. The Frenet-Serret frame is one of the most natural and intrinsic frames associated with an arbitrary curve[71, 76, 15]. Here a curve is associated at every point with the orthonormal Frenet-Serret tetrad. The members of the tetrad, of which the first is the unit tangent to the trajectory, satisfy the Frenet-Serret equations. Furthermore, the intrinsic geometry of the curve is uniquely determined by the Frenet-Serret scalars, namely the curvature (κ) and the first and second torsions (τ_1 and τ_2 respectively) defined along the trajectory. Such a description proves to be quite elegant when the world lines follow the directions of spacetime symmetries, or Killing vector fields, provided of course that the spacetime admits such symmetries. Honig, Schücking and Vishveshwara used the Frenet-Serret formalism to describe the motion of a charged test particle in a constant electromagnetic field[42]. In this case they showed that the Frenet-Serret parameters κ, τ_1 and τ_2 are constants along the trajectory. It was shown that the components of the Frenet-Serret triad satisfy a Lorentz like equation and can be expressed uniquely in terms of the applied field and four velocity of the test particle. Furthermore, they also showed that the Frenet-Serret formalism when applied to a timelike Killing trajectory in Riemannian spacetimes, yields similar results to that of the trajectory of a charged test particle in a homogeneous electromagnetic field. The vorticity of the Killing congruence, which represents the rotation of the connecting vector, was expressed in terms of the Frenet-Serret parameters. In the case of axially symmetric stationary spacetimes, Rindler and Perlick showed that the vorticity of the Killing congruence describes the gyroscopic precession[63]. They also observed that the gyroscopic precession reversal occurs at $r = 3m$ in the Schwarzschild spacetime. The Frenet-Serret formalism for gyroscopic precession was developed by Iyer and Vishveshwara[43]. In this elegant formalism the precession of a Fermi-Walker tetrad is computed with respect to a Frenet-Serret tetrad, when both are transported along a given trajectory. They have given a comprehensive treatment of gyroscopic precession in axially symmetric stationary spacetimes making use of the Frenet-Serret formalism. In this case, two of the Frenet-Serret parameters, namely the torsions τ_1 and τ_2, are directly related to the precession. Several interesting results emerge when the above considerations are applied to black hole spacetimes.

1.5 Plan of the Thesis

The main objective of the present thesis is to study the rotational effects such as inertial forces, gravito-electromagnetism and gyroscopic precession within the framework of the general theory of relativity. We investigate these rotational effects and establish interrelations among them in general, stationary, axially symmetric spacetimes. We take black hole spacetimes such as the Kerr-Newman and Ernst solutions as specific examples to study these rotational effects.

In the second chapter we briefly describe the necessary formalism needed for our studies. First we describe the Frenet-Serret description for gyroscopic precession which was formulated by Iyer and Vishveshwara[43]. For inertial forces we use the covariant formalism given by Abramowicz, Nurowski and Wex[2]. As mentioned before, for gravito-electromagnetism we use the formalism given by Honig Schücking and Vishveshwara[42]. In the second chapter we show how the acceleration of a test particle can be decomposed into electric and magnetic force terms using the example given by Landau and Lifshitz[50].

In the third chapter, we apply these formalisms to circular quasi-Killing trajectories, which we shall define later, in an axially symmetric stationary spacetime. We also establish direct relations between the gyroscopic precession and inertial forces by expressing the Frenet-Serret parameters τ_1 and τ_2 in terms of inertial forces. We also prove an important theorem concerning the simultaneous reversal of gyroscopic precession and centrifugal force at the circular null geodesics in static spacetimes. In the case of stationary spacetimes we show that the reversal of neither centrifugal force nor gyroscopic precession occurs at the photon orbits.

We define the gravito-electric and gravito-magnetic fields with respect to a global rest frame. This definition is very useful in establishing the direct relations between inertial forces and gravito-electromagnetic fields. We also define gravito-electric and gravito-magnetic fields with respect to the comoving frame of the test particle. Using this definition one observes the *one-to-one* correspondence between a constant electromagnetic field[42] and the gravito-electromagnetic field.

In the fourth chapter we apply these formalisms to some of the black hole spacetimes. In the case of Ernst spacetime, which represents a Schwarzschild black hole in a constant magnetic field, we show that gyroscopic precession and centrifugal force reversal occurs at both of the circular null geodesics, which in fact, is the generic property of all static spacetimes. The Schwarzschild solution and Melvin universe are treated as special cases of the Ernst solution. We take the Kerr-Newman spacetime as a typical example of a stationary, axially symmetric spacetime, which represents a charged Kerr black hole. In this case we show that the reversal of neither centrifugal force nor the gyroscopic precession occurs at the circular null geodesics. The Kerr and Reissner-Nordstrom solutions are treated as special cases of the Kerr-Newman solution.

In the fifth chapter, we relate inertial forces and gravito-electromagnetic fields to the Einstein equations. For this purpose, we project Einstien's equations on to the two-manifold orthogonal to the two-surface formed by the Killing vector fields, using the Geroch formalism[32, 33]. We recast the Geroch formalism in terms of potential functions whose gradients are proportional to inertial forces. In the case of field equations with the source term we use the formalism developed by Hansen and Winicour[38]. Finally we end the thesis with the sixth chapter which comprises a few concluding remarks.

Chapter 2

Gyroscopic Precession, Inertial Forces, and Gravito-electromagnetism

2.1 Introduction

In the first chapter we outlined general considerations about the rotational effects such as the phenomenon of gyroscopic precession, inertial forces and gravito-electromagnetism. In the present chapter, we study these effects within the frame work of the general theory of relativity. In section 2 we present the Frenet-Serret description of gyroscopic precession as given by Iyer and Vishveshwara[43]. In a previous approach, Rindler and Perlick estimated gyroscopic precession in the Schwarzschild and the Kerr space-times by computing the vorticity of the congruence along which the gyroscopes are transported[63]. But in the Frenet-Serret formalism, the geometric properties of a curve defined by the Frenet-Serret parameters are related to physical phenomenon such as the precession of a gyroscope in a covariant manner. In the general theory of relativity, a gyroscope is mathematically idealized as a frame obeying the Fermi-Walker transport law[56, 72]. In section 2.2.1 we give a brief description of Fermi-Walker transport and its physical significance as a set of gyroscopes and we also compute the precession of a gyroscope with respect to an arbitrary frame transported along the same trajectory. In section 2.2.2 we describe the Frenet-Serret formalism in three and four dimensions. In the Frenet-Serret formalism the geometric properties of a curve are described by scalar parameters such as κ the curvature and τ_1, and τ_2 the torsions. Considerable simplification occurs when the formalism is applied to the Killing trajectories, as these trajectories are of great importance in the case of black hole spacetimes[42]. In section 2.2.3 we show that the Frenet-Serret parameters κ, τ_1 and τ_2 are constant along the quasi-Killing trajectories. In section 2.2.4, we compute the precession of a Frenet-Serret triad with respect to a Fermi-Walker triad, which gives the Frenet-Serret description of gyroscopic precession. We show that the phenomenon of gyroscopic precession is described essentially by the Frenet-Serret parameters τ_1 and τ_2. In section 2.2.5 we relate the Frenet-Serret description of gyroscopic precession to the vorticity of the quasi-Killing congruence. In the case of the Killing trajectories, one can show that the Frenet-Serret precession is equal to the vorticity of the congruence.

Abramowicz, Carter and Lasota for the first time, formulated the concept of inertial forces in static and stationary spacetimes[1]. In this formalism, they adapted so called optical reference geometry, by multiplying the spacetime metric with a suitable conformal factor. In optical reference geometry, the acceleration of a test particle can be decomposed into various inertial force terms. One of their important results is that the centrifugal force reversal occurs at the photon orbits in the case of static spacetimes. In the optical reference geometry formalism, the three-space is constructed orthogonal to the timelike Killing vector, which is in general not hyper surface orthogonal. As mentioned in the last chapter, in order to define the inertial forces, one needs global Newtonian-like rest frames. In section 2.3.2 we discuss the concept of a global rest frame in the general theory of relativity. Abramowicz, Nurowski and Wex generalized the optical reference geometry by incorporating the concept of a global rest frame, to give a covariant description of inertial forces in an arbitrary spacetime[2].

In section 2.4 we briefly outline the formalism for describing the gravito-electromagneti by showing the similarities between the forces acting on a test particle in a stationary spacetime and a charged test particle in an electromagnetic field. We end the chapter with section 2.5 comprising the concluding remarks

2.2 Gyroscopic Precession

2.2.1 Fermi-Walker Frame and Gyroscopic Precession

In this section we describe the Fermi-walker transported frame and its importance in the rotational effects such as gyroscopic precession. In order to study the rotational effects inherent to a spacetime, one needs to construct a reference frame, which does not have any intrinsic rotation of its own. Since the general theory of relativity does not allow, in general, any global inertial frames as in the Newtonian theory, one needs to construct a system of tetrads along a worldline. These frames can then be related to each other by successive Lorentz transformations along the trajectory. Since Lorentz transformation on a frame is equivalent to rotation in spacetime, when one constructs a system of tetrads along a worldline with an arbitrary acceleration, it implies rotation of the tangent vector in spacetime. This rotation of the tangent vector is inevitable. One defines the concept of a non-rotating frame of reference by a frame which allows pseudo-rotation only in the timelike plane spanned by the velocity and the acceleration vectors, and does not allow any rotation in the three-space. The Fermi-Walker frame is one of such frames of reference and it can be constructed as follows.

In Newtonian mechanics, a rotating vector v^i is represented by the following equation

$$\frac{d}{dt}v_\alpha = (\omega \times v)_\alpha = \epsilon_{\alpha\beta\gamma}\omega^\beta v^\gamma, \tag{2.1}$$

where ω is the angular velocity. Similarly, in a four dimensional spacetime, rotation of a vector can be expressed by

$$\frac{D}{D\tau}v_a = \Omega_{ab}v^b \tag{2.2}$$

where $\Omega_{ab} = -\Omega_{ba}$, which would take the form $\epsilon_{ijk}\omega^j$ in the non-relativistic case. Here, we define the rotation in a plane orthogonal to the vector ω^j. We adopt the convention that Latin indices $a, b, \ldots = 0-3$ and Greek indices $\alpha, \beta, \ldots = 1-3$ and the metric signature is $(+,-,-,-)$. Geometrized units with $c = G = 1$ are chosen.

In our case, we would like to confine rotation to the timelike plane spanned by the velocity and the acceleration. In this case, the Ω_{ab} can be uniquely expressed as,

$$\Omega^{pq}_{(FW)} = a^p u^q - u^p a^q \tag{2.3}$$

where, u^p is the four velocity and a^q is the four acceleration along the trajectory. One can easily show that the velocity vector trivially satisfies the transport equations (2.2) and (2.3). Also, rotation vanishes for a spacelike vector, if it is orthogonal to both acceleration and velocity, *i.e*

$$\frac{d}{d\tau}w^a = 0, \text{ if } \; w^a \cdot u_a = a^p \cdot w_p = 0 \tag{2.4}$$

A vector which satisfies the following equation

$$\frac{d}{d\tau}w^q = (u^p a^q - u^p a^q)w_p \tag{2.5}$$

is said to be Fermi-Walker transported. A frame or a tetrad $\{f_{(i)}\}$, whose components follow the Fermi-Walker transport law, with $f^a_{(0)}$ along the four velocity of the worldline, is referred to as the Fermi-Walker frame. The spatial triad of a Fermi-walker frame does not undergo any rotation. This can be physically realized as a set of three gyroscopes each of which is aligned to a basis vector of the Fermi-Walker triad. If an observer chooses an arbitrary frame different from that of a Fermi-Walker frame along an arbitrary worldline, then his spatial triad will undergo precession with respect to the Fermi-Walker frame[56]. Let us assume that the transport law for an arbitrary observer is given by,

$$\frac{d}{d\tau}(e^a_{(i)}) = \Omega^a{}_b e^b_{(i)} \tag{2.6}$$

where $\Omega_{ab} = -\Omega_{ba}$, which can be uniquely decomposed into a Fermi-Walker part and a spatial rotation part as follows,

$$\Omega^{ab} = \Omega^{ab}_{(FW)} + \Omega^{ab}_{(SR)} \tag{2.7}$$

$$\Omega^{ab}_{(FW)} = a^a u^b - a^b u^a \tag{2.8}$$

$$\Omega^{ab}_{(SR)} = u_c \omega_d \epsilon^{cdab} \tag{2.9}$$

In the above, ω^a is a vector orthogonal to the four velocity u^a. *i.e.*

$$\omega^a u_a = 0. \tag{2.10}$$

From the above equations it is straightforward to show that the precession frequency ω^a can be expressed as,

$$\omega_p = \frac{1}{2}\sqrt{-g}\epsilon_{pqab}\Omega^{ab}u^q \tag{2.11}$$

If $\{f_{(\alpha)}\}$ is the Fermi-Walker triad, then,

$$\frac{d}{d\tau}f^a_{(i)} = \Omega^a_b f^b_{(i)} \tag{2.12}$$

From equation (2.6) and (2.12) we get,

$$\frac{d}{d\tau}\left(e^a_{(\alpha)} - f^a_{(\alpha)}\right) = \Omega^{ab}_{(SR)} e_{(\alpha)b} \tag{2.13}$$

or

$$\frac{d}{d\tau}(e_{(\alpha)} - f_{(\alpha)}) = \omega \times e_{(\alpha)}. \tag{2.14}$$

This can be physically interpreted to as the spatial triad $\{e_{(\alpha)}\}$ precessing with respect to the Fermi-Walker triad, with a frequency ω. In the Frenet-Serret description of gyroscopic precession, one uses the Frenet-Serret transported frame $\{e_{(i)}\}$ in order to compute the precession frequency given by equation (2.11). The Frenet-Serret formalism is described in the next section.

2.2.2 Frenet-Serret Formalism

The Frenet-Serret(FS) formalism is one of the elegant methods for investigating the geometric properties of curves. In this formalism, curves are studied by assigning an orthonormal frame called the 'Frenet-Serret frame' at each point. Rates of changes of these frames are expressed by the Frenet-Serret formulae in term of scalar parameters called the Frenet-Serret parameters. These parameters along with the Frenet-Serret frames describe the fundamental geometric properties of the curve.

We shall first illustrate the Frenet-Serret formalism in three-dimensional space[71, 76, 15]. Let $\gamma(s)$ be a three curve in space with unit tangent $\vec{e}_{(1)} = \gamma'(s)$. If the acceleration $\gamma''(s)$ is not zero along the curve, one can construct a unique circle at each point on the curve with radius $1/|\gamma''(s)|$. This circle is called the osculating circle. The center of the circle lies along the vector $\gamma''(s)$ which is orthogonal to the tangent vector. The unit vector $\vec{e}_{(2)}$ along $\gamma''(s)$ is called the principle normal of the curve, which satisfies the equation

$$\frac{d\vec{e}_{(1)}}{ds} = \kappa\vec{e}_{(2)} \tag{2.15}$$

where κ is called the curvature. The plane described by the vectors $\vec{e}_{(1)}$ and $\vec{e}_{(2)}$ is called the osculating plane.

The unit vector $\vec{e}_{(3)}$ normal to the osculating plane, called the binormal vector, is given by

$$\vec{e}_{(3)} = \vec{e}_{(1)} \times \vec{e}_{(2)} \tag{2.16}$$

One defines the torsion $\tau(s)$ at each point on the curve as from the equation

$$\frac{d\vec{e}_{(2)}}{ds} = -\kappa\vec{e}_{(1)} + \tau\vec{e}_{(3)}. \tag{2.17}$$

Further, the binormal vector satisfies the equation

$$\frac{d\vec{e}_{(3)}}{ds} = -\tau\vec{e}_{(2)}. \tag{2.18}$$

Equations (2.15), (2.17) and (2.18) are called the Frenet-Serret equations, which describe the geometry of the curve in space. In this case the geometry of a curve is described by an orthonormal frame consisting of unit vectors $\vec{e}_{(1)}, \vec{e}_{(2)}$ and $\vec{e}_{(3)}$, and the scalar parameters κ, the curvature and τ the torsion.

From the above equations one can clearly see that for all curves confined to a plane, the torsion τ is identically zero. When $\tau = 0$ and $\kappa = constant$, the curve represents a circle with radius $1/\kappa$. A helix is characterized by $\tau = constant$ and $\kappa = constant$

The Frenet-Serret formalism can be extended to four dimensional spacetime[42, 72]. If $e_{(0)}$ is the unit timelike tangent vector along a trajectory, the Frenet-Serret equations in general are given by

$$\dot{e}^a_{(0)} = \kappa e^a_{(1)}, \tag{2.19}$$

$$\dot{e}^a_{(1)} = \kappa e^a_{(0)} + \tau_1 e^a_{(2)}, \tag{2.20}$$

$$\dot{e}^a_{(2)} = -\tau_1 e^a_{(1)} + \tau_2 e^a_{(3)}, \tag{2.21}$$

$$\dot{e}^a_{(3)} = -\tau_2 e^a_{(2)}. \tag{2.22}$$

The parameters κ,τ_1 and τ_2 are the curvature, the first and the second torsion while $e^a_{(i)}$ form an orthonormal tetrad. These six quantities not only describe the geometry of the worldline completely, but also elucidate the interlink between the physical quantities and geometric properties. If one specifies that the unit vector $e_{(0)}$ is the four velocity of a test particle, then we have

$$\dot{e}^{p}_{(0)} = e^{p}_{(0);b} e^{b}_{(0)} \equiv a^{p}. \tag{2.23}$$

Where a^p is the four acceleration of the test particle. The Frenet-Serret parameter κ can be expressed as,

$$\kappa^2 \;=\; -\dot{e}^{p}_{(0)} \dot{e}_{p(0)} = -a^p \cdot a_p \tag{2.24}$$

Which clearly shows that the parameter κ is the magnitude of the four acceleration. Next we shall show that the parameters τ_1 and τ_2 are directly related to the phenomenon of gyroscopic precession.

2.2.3 The Frenet-Serret Parameters and Gyroscopic Precession

In the last section we have seen that the Frenet-Serret parameter κ is the magnitude of the four-acceleration along the trajectory. In this section, we shall give the physical interpretation of the Frenet-Serret parameters τ_1 and τ_2, by relating them to the phenomenon of gyroscopic precession.

Let us consider a Fermi-Walker tetrad $\{f^a_{(i)}\}$ and a Frenet-Serret tetrad $\{e^a_{(i)}\}$ are being transported along an arbitrary trajectory. Let u^a be the four velocity and a^a be the four-acceleration along the trajectory. Then we have the following,

$$e^{a}_{(0)} \;=\; f^{a}_{(0)} \;=\; u^a \tag{2.25}$$

Since the four-velocity u^a is an unit vector, without loss of generality, we can take

$$\frac{d}{d\tau} f^{a}_{(0)} \;\equiv\; a^a \;=\; \kappa f^{a}_{(1)} \tag{2.26}$$

Where κ is the magnitude of the four-acceleration along the trajectory. Using the above, the Fermi-Walker transport law, as given in the equation (2.3) takes the following form

$$\Omega^{pq}_{(FW)} = \kappa\left(f^{p}_{(1)}f^{q}_{(0)} - f^{p}_{(0)}f^{q}_{(1)}\right) \tag{2.27}$$

and we have ,

$$\frac{d}{d\tau}f^{a}_{(i)} = \Omega_{(FW)(i)}{}^{(j)}\, f^{a}_{(j)} \tag{2.28}$$

In the above,

$$\Omega_{(FW)(i)}{}^{(j)} = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ \kappa & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{2.29}$$

Here, $(i)\cdots(j)$ are tetrad indices. We take the orthonormality condition for tetrad components[22],

$$e^{a}_{(i)}e_{(j)b} = \eta_{(a)(b)} \tag{2.30}$$

where

$$\eta_{(a)(b)} = Diag\,[1, -1, -1, -1] \tag{2.31}$$

In a similar fashion, the Frenet-Serret transport law given by the equations (2.19-2.22) can be written as,

$$\frac{d}{d\tau}e^{a}_{(i)} = \Omega_{(FS)(i)}{}^{(j)}e^{a}_{(j)} \tag{2.32}$$

where

$$\Omega_{(FS)(i)}{}^{(j)} = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ \kappa & 0 & \tau_1 & 0 \\ 0 & -\tau_1 & 0 & \tau_2 \\ 0 & 0 & -\tau_2 & 0 \end{bmatrix} \tag{2.33}$$

We now decompose the $\Omega_{(FS)}$ into a Fermi-Walker term $\Omega_{(FW)}$ and a spatial rotation term $\Omega_{(SR)}$ as given in equation (2.7),

$$\Omega_{(SR)(i)}{}^{(j)} = \Omega_{(FW)(i)}{}^{(j)} + \Omega_{(FS)(i)}{}^{(j)} \tag{2.34}$$

where

$$\Omega_{(SR)(i)}{}^{(j)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \tau_1 & 0 \\ 0 & -\tau_1 & 0 & \tau_2 \\ 0 & 0 & -\tau_2 & 0 \end{bmatrix} \tag{2.35}$$

From the above, for the Frenet-Serret frame, the equation (2.14) can be written as,

$$\frac{d}{d\tau}\left(e^{(\alpha)} - f^{(\alpha)}\right) = \eta^{(\alpha)(\beta)(\gamma)} e_{(\beta)} \omega_{(\gamma)} \tag{2.36}$$

$$= (\omega \times e)^{(\alpha)} \tag{2.37}$$

where $\eta^{(\alpha)(\beta)(\gamma)}$ are the tetrad components of the completely antisymmetric tensor.

The tetrad components of the precession frequency can be obtained from the equations (2.13) and (2.35) and can be written as[34],

$$\omega^{(\alpha)} = [\tau_2,\ 0,\ \tau_1] \tag{2.38}$$

$$\omega_{(\alpha)} = [-\tau_2,\ 0,\ -\tau_1] \tag{2.39}$$

The components of the precession frequency in the coordinate basis, can be expressed as,

$$\omega^a = \omega^{(i)} e^a_{(i)} \tag{2.40}$$

$$= \tau_1 e^a_{(3)} + \tau_2 e^a_{(1)} \tag{2.41}$$

As mentioned before, ω^a represents the precession frequency of the Frenet-Serret triad with respect to a Fermi-Walker triad. Since a Fermi-Walker transported frame is physically realized as a set of gyroscopes, the precession frequency ω^a represents the precession of the Frenet-Serret triad with respect to a set of gyroscopes. In other words a gyroscope would precess with a frequency $-\omega^a$ with respect to a Frenet-Serret frame. Here we have shown that the Frenet-Serret parameters τ_1 and τ_2 are directly related to the phenomenon of gyroscopic precession. Considerable simplification occurs when the trajectories involved are along the Killing vector fields. Next, we shall apply the Frenet-Serret formalism to the Killing trajectory in four dimensional spacetimes.

2.2.4 Application to Quasi-Killing Trajectories

In this section, we apply the Frenet-Serret formalism to the case of a spacetime admitting Killing vector fields. For example, an axially symmetric stationary spacetime such as the Kerr spacetime, admits a timelike Killing vector field ξ^a and a spacelike Killing vector field η^a, which generate closed circular orbits around the axis of symmetry. Furthermore, the static spherically symmetric Schwarzschild spacetime admits a timelike Killing vector ξ^a and three rotational Killing vectors corresponding to spherical symmetry. These Killing vectors satisfy the Killing equation,

$$\mathcal{L}_\xi g_{ab} = \xi_{(a;b)} = 0. \tag{2.42}$$

When the Frenet-Serret formalism is applied to the Killing trajectories of spacetimes, many interesting features emerge. These considerations apply to a single trajectory in any specific example. However, additional geometric insight may be gained by identifying the trajectory as a member of one or more congruences generated by combining different Killing vectors. For this purpose, the Frenet-Serret formalism is applied to quasi-Killing trajectories. In the discussion given below we closely follow the reference [43].

Consider a spacetime that admits a timelike Killing vector ξ^a and a set of spacelike

Killing vectors $\eta_{(A)}$ (A=1,2,...m). Then a quasi-Killing vector may be defined as

$$\chi^a \equiv \xi^a + \omega_{(A)}\eta^a_{(A)}, \tag{2.43}$$

where (A) is summed over. The Lie derivative of the functions $\omega_{(A)}$ with respect to χ^a is assumed to vanish,

$$\mathcal{L}_\chi \omega_{(A)} = 0. \tag{2.44}$$

A congruence of quasi-Killing trajectories is generated by the integral curves of χ^a. As a special case, we obtain a Killing congruence when $\omega_{(A)}$ are constants. Assuming χ^a to be timelike, we may define the four velocity of a particle following χ^a by

$$e^a_{(0)} \equiv u^a \equiv e^\psi \chi^a, \tag{2.45}$$

so that

$$e^{-2\psi} = \chi^a\chi_a, \quad \psi_{,a}\chi^a = 0 \tag{2.46}$$

and

$$\dot{e}^a_{(0)} \equiv e^a_{(0);b}e^b_{(0)} = F^a{}_b e^b_{(0)}, \tag{2.47}$$

it follows from equation (2.42) that

$$F_{ab} \equiv e^\psi \left(\xi_{a;b} + \omega_{(A)}\eta_{(A)a;b}\right). \tag{2.48}$$

The derivative of $\omega_{(A)}$ drops out of the equation. The Killing equation (2.42) and the equation $\xi_{a;b;c} \equiv R_{abcd}\xi^d$ satisfied by any Killing vector lead to

$$F_{ab} = -F_{ba} \quad and \quad \dot{F}_{ab} = 0. \tag{2.49}$$

When the Frenet-Serret formalism is applied to a quasi-Killing congruence important simplifications occur. It can be shown that κ, τ_1 and τ_2 are constants and that each of $e^a_{(i)}$ satisfies a Lorentz-like equation[42], which can be summarized as follows,

From equation(2.47) and (2.19) we have.

$$\dot{e}^a_{(0)} = F^a_b e^b_{(0)} = \kappa e^a_{(1)} \tag{2.50}$$

Taking the derivative on both sides and using equation (2.49), we get

$$\dot{\kappa} e^a_{(1)} + \kappa \dot{e}^a_{(1)} = \kappa F^a_b e^b_{(1)} \tag{2.51}$$

Contracting the above equation with $e_{(1)a}$, gives

$$\dot{\kappa} = 0, \tag{2.52}$$

since, $e_{(1)a}$ being a unit vector

$$\dot{e}^a_{(1)} e_{(1)a} = 0 \tag{2.53}$$

and

$$F_{ab} e^a_{(1)} e^b_{(1)} = 0 \tag{2.54}$$

by the antisymmetry of F_{ab}. From equations (2.52) and (2.51) we have,

$$\dot{e}^a_{(1)} = F^a_b e^b_{(1)}, \tag{2.55}$$

which shows that $e^a_{(1)}$ also satisfies the Lorentz like equation(2.47). Similarly it is easy to show that the Frenet-Serret parameters τ_1 and τ_2 are also constant along the worldline and that $e^a_{(2)}$, $e^a_{(3)}$ also satisfy the Lorentz like equation (2.47)(for more details see reference[42]). To summarize,

$$\dot{\kappa} = \dot{\tau}_1 = \dot{\tau}_2 = 0 \tag{2.56}$$

and

$$\dot{e}^a_{(i)} = F^a_b e^b_{(i)}. \tag{2.57}$$

Further, κ, τ_1, τ_2 and $e^a_{(\alpha)}$ can be expressed in terms of $e^a_{(0)}$ and $F^n{}_{ab} \equiv F_a{}^{a_1} F_{a_1}{}^{a_2} \cdots F_{a_{n-1}b}$.

$$\kappa^2 = F^2_{ab} e^a_{(0)} e^b_{(0)} \tag{2.58}$$

$$\tau_1^2 = \kappa^2 - \frac{F^4_{ab} e^a_{(0)} e^b_{(0)}}{\kappa^2} \tag{2.59}$$

$$\tau_2^2 = \frac{F^6_{ab} e^a_{(0)} e^b_{(0)}}{\kappa^2 \tau_1^2} - \frac{\left(\kappa^2 - \tau_1^2\right)^2}{\tau_1^2} \tag{2.60}$$

$$e^a_{(1)} = \frac{1}{\kappa} F^a{}_b e^b_{(0)} \tag{2.61}$$

$$e^a_{(2)} = \frac{1}{\kappa\tau_1}\left[F^{2a}{}_b - \kappa^2\delta^a_b\right] e^b_{(0)} \tag{2.62}$$

$$e^a_{(3)} = \frac{1}{\kappa\tau_1\tau_2}\left[F^{3a}{}_b + \left(\tau_1^2 - \kappa^2\right) F^a{}_b\right] e^b_{(0)} \tag{2.63}$$

The above equations were first derived by Honig, Schücking and Vishveshwara [42] to describe charged particle motion in a homogeneous electromagnetic field. Interestingly, they are identical to those that arise in the case of quasi-Killing trajectories[43]. In the next section we shall compute the precession of the Frenet-Serret frame $e^a_{(i)}$ with respect to the Fermi-Walker frame.

2.2.5 Frenet-Serret Description of Gyroscopic Precession: Along Killing Trajectories

The Frenet-Serret formalism offers a covariant method for treating gyroscopic precession. It turns out to be a convenient and elegant description of the phenomenon when the worldlines along which the gyroscopes are transported, follow spacetime symmetry directions or Killing vector fields. In fact, in most cases of interest, orbits corresponding to such worldlines are considered for simplicity. For instance, this is the case when one studies circular orbits in black hole spacetimes. As we have already shown that, the torsions τ_1 and τ_2 are directly related to gyroscopic precession, transported along an arbitrary worldline. Here we shall prove some of the identities corresponding to the Killing trajectories.

Let us consider an inertial tetrad $\left(e^a_{(0)}, f^a_{(\alpha)}\right)$ which undergoes Fermi-Walker transport along the worldline. The triad $f_{(\alpha)}$, as we have mentioned before, may be physically realized by a set of three mutually orthogonal gyroscopes. Here we show that the angular velocity of the Frenet-Serret triad $e^a_{(\alpha)}$ with respect to the Fermi-Walker triad $f^a_{(\alpha)}$ can be directly related to the Frenet-Serret parameters τ_1 and τ_2. This result has been pointed out by Honig, Schücking and Vishveshwara[42]. We briefly outline the details of the derivation.

As given in the section 2.2.1, the precession frequency of an arbitrary spatial triad with respect to a Fermi-Walker triad is given by equation(2.11) *i.e.*

$$\omega^a = \frac{1}{2\sqrt{-g}}\epsilon^{abcd}\Omega_{cd}u_b \tag{2.64}$$

Considering equation (2.47) and (2.48) for a Killing trajectory we have,

$$\Omega_{ab} = F_{ab} = e^{\psi}(\xi_{a;b} + \omega_{(A)}\eta^{(A)}_{a;b}) \tag{2.65}$$

The tensor $\hat{F}_{ab}$, which is the dual of F_{ab} is defined by

$$\hat{F}_{ab} = \frac{1}{2}\sqrt{-g}\epsilon_{abcd}F^{cd} \tag{2.66}$$

and

$$\hat{F}^{ab} = \frac{1}{2\sqrt{-g}}\epsilon^{abcd}F_{cd}. \tag{2.67}$$

We define the scalar parameters

$$\alpha \equiv \frac{1}{2}F^a_bF^b_a \tag{2.68}$$

$$\eta \equiv F^a_b\hat{F}^b_a \tag{2.69}$$

The following identities are useful for simplifying the expression for the parameters τ_1 and τ_2 from equations (2.59) and (2.60).

$$(F^3)^a_b - \alpha F^a_b - \beta\hat{F}^a_b = 0 \tag{2.70}$$

$$(F^4)^a_b - \alpha(F^2)^a_b a - \beta^2\delta^a_b = 0 \tag{2.71}$$

$$(F^6)^a_b - (\alpha^2+\beta^2)(F^2)^a_b - \alpha\beta^2\delta^a_b = 0 \tag{2.72}$$

These equations are valid for any arbitrary second rank anti-symmetric tensor F_{ab}[42]. From equation (2.59) and (2.71) we have

$$\tau_1^2 = \kappa^2 - \alpha - \frac{\beta^2}{\kappa^2} \tag{2.73}$$

the equation (2.60) can be simplified using the identity (2.72) and we obtain

$$\tau_2^2 = \frac{\beta^2}{\kappa^2} \tag{2.74}$$

Using the above equation, equation (2.73) can be rewritten as

$$\kappa^2 - \tau_1^2 - \tau_2^2 = \alpha \tag{2.75}$$

From equations (2.63), (2.71), (2.74) and (2.75) we can show that

$$\omega^a = \hat{F}_{ab} e^b_{(0)} = \tau_2 e^a_{(1)} + \tau_1 e^a_{(3)}. \tag{2.76}$$

This establishes the important physical formula which directly relates the Frenet-Serret parameters τ_1 and τ_2 to the phenomenon of gyroscopic precession.

In other words the gyroscopes precess with respect to the Frenet-Serret frame at a rate given by $\Omega_{(g)} = -\omega_{FS}$. Furthermore, in case of the Killing congruence, ω_{FS} is identical to the vorticity of the congruence, which will be shown next.

2.2.6 Vorticity and Gyroscopic Precession

The concept of vorticity is another important geometrical notion associated with a trajectory as a member of a suitably chosen congruence of curves. It measures the twisting of the congruence. In this section, we study the relation between gyroscopic precession and vorticity. Here we show that the Frenet-Serret precession for a trajectory belonging to a Killing congruence is equal to the vorticity of the congruence.

As it was shown earlier (2.76), the precession frequency is given by

$$\omega^a = \hat{F}^{ab} u_b \tag{2.77}$$

where

$$\hat{F}^{ab} = \frac{1}{2\sqrt{-g}} \epsilon^{abcd} F_{cd} \tag{2.78}$$

From equation (2.47) and (2.48), we have

$$\omega^a = \frac{e^\psi}{2\sqrt{-g}}\epsilon^{abcd}\left(\xi_{c;d} + \omega_{(A)}\eta^{(A)}_{c;d}\right)u_b \tag{2.79}$$

since ω is constant from equation (2.46), we have

$$\omega^a = \frac{1}{2\sqrt{-g}}\epsilon^{abcd}u_{c;d}u_b \tag{2.80}$$

which is the vorticity of the congruence. Therefore, the gyroscopic precession for a Killing trajectory is determined by the vorticity of the Killing congruence. This is not true in the case of quasi-Killing trajectories. The vorticity of a congruence is defined as

$$\begin{aligned}
\Omega^a &\equiv \frac{1}{2\sqrt{-g}}\epsilon^{abcd}e_{(0)b}e_{(0)c;d} && (2.81)\\
&= \frac{1}{2\sqrt{-g}}\epsilon^{abcd}e_{(0)b}[F_{cd} + e^\psi\omega_{(A),d}\eta_{(A)c}] && (2.82)\\
&= \omega^a_{(FS)} + \tilde{D}^{ab}e_{(0)b} && (2.83)
\end{aligned}$$

where

$$\begin{aligned}
\tilde{D}^{ab} &= \frac{1}{2\sqrt{-g}}\epsilon^{abcd}D_{cd}\\
D_{cd} &\equiv e^\psi\omega_{(A)[,d}\eta^{(A)}_{c]}\\
A_{[ab]} &\equiv \frac{1}{2}(A_{ab} - A_{ba})
\end{aligned} \tag{2.84}$$

As is well known, physically, vorticity Ω^a represents the angular velocity of the connecting vector with respect to an orthonormal spatial frame Fermi-Walker transported along the congruence. On the other hand, Frenet-Serret rotation $\omega^a_{(FS)}$ represents the precession of the intrinsic Frenet-Serret frame with respect to the non-rotating Fermi-Walker frame. In general, for example in the quasi-Killing case, the two are not the same. Therefore the gyroscopic precession along a quasi-Killing trajectory differs from the rotation of the connecting vector of the corresponding quasi-Killing congruence.

2.3 Inertial Forces

2.3.1 Optical Reference Geometry

Abramowicz, Cater and Lasota(ACL) formulated the optical reference geometry to study the dynamics of test particles in stationary and static spacetimes[1]. In this formalism, the three-geometry orthogonal to the timelike Killing vector is conformally rescaled. Here we briefly describe this approach.

The metric for a stationary spacetime can be written as

$$ds^2 = \Phi(dt - 2\alpha_\mu dx^\mu)^2 - dl^2 \tag{2.85}$$

Here, dl^2 is the line element on the quotient space orthogonal to the timelike Killing vector ξ^a and is given by,

$$dl^2 = \gamma_{\mu\nu} dx^\mu dx^\nu = (-g_{\mu\nu} + 4\Phi\alpha_\mu\alpha_\nu) \tag{2.86}$$

where,

$$g_{00} = \Phi, \tag{2.87}$$

$$g_{0\mu} = -2\Phi\alpha_\mu, \tag{2.88}$$

$$\gamma_{\mu\nu} = -g_{\mu\nu} + 4\Phi\alpha_\mu\alpha_\nu. \tag{2.89}$$

The dynamics of a test particle with four momentum P^a is described by the equation

$$mf_a = P_{a;b}P^b \tag{2.90}$$

$$= P_{a,b}P^b - \frac{1}{2}g_{bc,a}P^bP^c \tag{2.91}$$

where f_a is the force acting on the test particle. The mass of the test particle m is given by

$$m^2 = g_{ab}P^aP^b. \tag{2.92}$$

The energy of the test particle can be defined as,

$$\mathcal{E} = P_a\xi^a = P_0 = \Phi(P^0 - 2P^\mu\alpha_\mu) \tag{2.93}$$

where ξ^a is timelike Killing vector. The energy $\mathcal{E}$ is conserved along a geodesic. In order to study the equation of motion in the quotient space, we define the three-momentum components p_i in the quotient space as follows,

$$p_\mu = \gamma_{\mu\nu}P^\nu \tag{2.94}$$

The equation of motion projected on the quotient space can be decomposed as follows,

$$mf_0 = P^\mu\partial_\mu\mathcal{E} \tag{2.95}$$

and

$$m(f_\mu - 2f_0\alpha_\mu) = P^\nu\left(\partial_\nu p_\mu - \frac{1}{2}P^\rho\partial_\mu\gamma_{\nu\rho}\right) + 2\mathcal{E}\ P^\nu\left(\partial_\mu\alpha_\nu\right) + \frac{1}{2}\left(\frac{\mathcal{E}}{\Phi}\right)^2\partial_\mu\Phi \tag{2.96}$$

In order to identify the above projected acceleration into various inertial force terms, ACL have introduced a new conformally modified metric.

$$\tilde{dl}^2 = \tilde{g}_{\mu\nu}dx^\mu dx^\nu \tag{2.97}$$

on the quotient space,

$$dl^2 = \Phi\tilde{dl}^2. \tag{2.98}$$

With the above conformal modification, the spacetime metric takes the form

$$ds^2 = \Phi[(dt - 2\alpha_\mu dx^\mu)^2 - \tilde{dl}^2] \tag{2.99}$$

The three-momentum of the test particle in the optical reference frame is given by,

$$p_\mu = \tilde{g}_{\mu\nu}p^\nu \tag{2.100}$$

and the contravariant three-momentum can be written as

$$p^{\mu} = \Phi P^{\mu} \tag{2.101}$$

With the above definitions the four acceleration of test particles, when projected onto the conformal spacetime, splits into terms that may be identified as inertial forces.

$$m\Phi\left(f_{\mu} - 2\alpha_{\mu} f_0\right) = p^{\nu}\left(\partial_{\nu} p_{\mu} - \frac{1}{2} p^{\rho} \partial_{\mu} \tilde{g}_{\nu\rho}\right) + \frac{1}{2} m^2 \partial_{\mu}\Phi + 2\mathcal{E}\ p^{\nu}\left(\partial_{\mu}\alpha_{\nu} - \partial_{\nu}\alpha_{\mu}\right), \tag{2.102}$$

which can be written as

$$m\Phi\left(f_{\mu} - 2\alpha_{\mu} f_0\right) = p^{\nu}\tilde{\nabla}_{\nu} p_{\mu} + \frac{1}{2} m^2 \partial_{\mu}\Phi + 2\mathcal{E}\ p^{\nu}\omega_{\nu\mu}, \tag{2.103}$$

where

$$\omega_{\nu\mu} = \partial_{\mu}\alpha_{\nu} - \partial_{\nu}\alpha_{\mu}. \tag{2.104}$$

Here, $\tilde{\nabla}_{\nu}$ is the covariant derivative operator in the optical reference geometry. In the above equation, the term $p^{\nu}\tilde{\nabla}_{\nu} p_{\mu}$ represents centrifugal acceleration in the three-space. The velocity independent second term $\frac{1}{2} m^2 \partial_{\mu}\Phi$, can be identified with Newtonian gravitational force. The last term $2\mathcal{E}\ p^{\nu}\omega_{\nu\mu}$ is the Lense-Thirring-Coriolis force which is a manifestation of frame dragging in stationary spacetimes.

In the case of static spacetimes, one can show that

$$\omega_{\mu\nu} = 0, \tag{2.105}$$

because of the fact that the timelike Killing vector ξ^a is hypersurface orthogonal or

$$\alpha_{\mu} = 0. \tag{2.106}$$

In which case the equation of motion simplifies to

$$m\Phi f_{\mu} = p^{\nu}\tilde{\nabla}_{\nu} p_{\mu} + \frac{1}{2} m^2 \partial_{\mu}\Phi. \tag{2.107}$$

The important result emerging from the above formalism is that, in the case of static spacetimes, the four-dimensional null geodesic, which is characterized by

$$m = 0, \quad f_a = 0 \tag{2.108}$$

satisfies the quotient-space geodesic equation,

$$p^{\nu}\tilde{\nabla}_{\nu}p_{\mu} = 0. \tag{2.109}$$

In other words, the centrifugal force reversal occurs when the four-dimensional trajectories are null geodesics.

As we have seen in this formalism, the quotient-three-space is defined with respect to the timelike Killing vector ξ^a which in general, does not define the surface of simultaneity. In the next section, we define the concept of global rest frames in the general theory of relativity, which are close to the Newtonian global inertial frames. This formalism was followed by the covariant definition of inertial forces, by Abramowicz, Nurowski and Wexi[2], which is described later in this chapter.

2.3.2 The Global Rest Frame

In this section, we describe the concept of global rest frames in the general theory of relativity. The general theory of relativity does not allow the construction of global inertial frames, as in the case of the Newtonian theory. However, one can construct spatial frame of reference, which are possible generalizations of the Newtonian non-rotating rest frames. The concept of a rest frame is closely connected with the existence of time symmetry, since all the rest observer clocks are synchronized with respect to a world-time. A consideration of rest frames is given in [75], which we shall outline below.

The rest frame in a flat spacetime is adapted to the inertial observer following a worldline along time t. This is the direction of the timelike Killing vector ξ^a. The four velocity of the observers at different spatial points are orthogonal to the hyperspace $t = constant$. The four velocity is given by,

$$u^a = e^{\psi}\xi^a; \;\; u^a u_a = 1; \;\; e^{\psi} = (\xi^a \xi_a)^{-\frac{1}{2}} \tag{2.110}$$

And

$$\xi^a = \delta^a_0 \;\text{ and }\; \xi_a = t_{,a} \tag{2.111}$$

Therefore 't' is the synchronous time for the rest observers. As in the case of Newtonian inertial frames, all the rest observers with four-velocity u^a, follow geodesic motion. These rest observers constitute an irrotational congruence. If we define the vorticity of this congruence, or that of the vector field ξ^a by

$$\omega^a_\xi \equiv \frac{1}{\sqrt{-g}}\varepsilon^{abcd}\xi_b\xi_{c;d}; \;\; \omega^a = e^{2\psi}\omega^a_\xi, \tag{2.112}$$

then

$$\omega^a_\xi = 0. \tag{2.113}$$

The concept of a global rest frame is directly extended to a static spacetime, which admits a hypersurface orthogonal timelike Killing vector field ξ^a. The four velocity u^a given in equation(2.110) which defines the rest observers, no longer follow geodesic motion. However the vorticity,

$$\omega_\xi = 0 = e^{-2\psi}\omega_u. \tag{2.114}$$

Therefore, the four velocities form an irrotational congruence. Once again 't' is the common synchronous time for the rest observers in the static spacetime.

In the case of stationary axially symmetric spacetimes, such as those of the Kerr spacetime, the timelike Killing vector field is no longer irrotational and hence the Killing observers following ξ a no longer define the global rest frame. Nevertheless, considering the vector field

$$\zeta^a = \xi^a - \frac{(\xi^b\eta_b)}{(\eta^c\eta_c)}\eta^a \tag{2.115}$$

We notice,

$$\zeta^a \eta_a = 0 \tag{2.116}$$

so that ζ^a is the projection of ξ^a orthogonal to η^a. Furthermore, it is easy to show that the vorticity of the ζ^a - congruence

$$\omega_\zeta^a = 0 \tag{2.117}$$

This was first noticed by Bardeen[14], who called the frames adapted to ζ^a as locally non-rotating frames (LNRF). It was recognized that the physical phenomena in the Kerr spacetime could be studied in a significant manner when referred to LNRF. The observers with four velocity

$$u^a = (\zeta^b \zeta_b)^{-\frac{1}{2}} \zeta^a \tag{2.118}$$

are in fact the 'rest' observers and the frames adapted to them form the global rest frame since ζ_a is in fact hypersurface orthogonal :

$$\zeta_a = \left[\xi^b \xi_b - \frac{(\xi^b \eta_b)}{(\eta^c \eta_c)}\right] t_{,a} \tag{2.119}$$

As before $'t'$ is the synchronous time for these observers.

Properties of the global rest frames were studied in detail and generalized to arbitrary stationary, axisymmetric spacetimes by Greene, Schucking and Vishveshwara[35]. They showed that if the Killing fields ξ^a and η^a satisfied orthogonal transitivity, as in the Kerr spacetime, χ^a became null on the event horizon similar to ξ^a in the Schwarzschild spacetime. Furthermore, $t = constant$ can be shown to be maximal surfaces. Physical phenomena can be studied meaningfully in the global rest frames, especially since extended systems can be defined only on spatial surfaces of simultaneity, like $t = constant$. These global rest frames are used to define the inertial forces in the general theory of relativity.

2.3.3 Covariant Definition of Inertial Forces

In the previous section, we have described how the optical reference geometry can be used to identify the inertial forces in stationary and static spacetimes. As mentioned earlier, the three-surface orthogonal to the timelike Killing vector ξ^a in stationary spacetimes, in not hypersurface orthogonal. In this section we discusse the general covariant formalism given by Abramowicz *et. al.* [2] in which the general relativistic analogies of inertial forces are defined with respect to a global rest frame in an arbitrary spacetime.

An arbitrary spacetime metric can be expressed as

$$g = e^{-2\phi}\left(\frac{\partial}{\partial t}\right)^2 + 2g^{t\mu}\left(\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x^\mu}\right) + g^{\mu\nu}\left(\frac{\partial}{\partial x^\mu}\right)\left(\frac{\partial}{\partial x^\nu}\right) \tag{2.120}$$

with ϕ satisfying the condition,

$$\nabla^t\phi = g^{ti}\nabla_i\phi = 0. \tag{2.121}$$

One can define the one form

$$n_i dx^i = e^{\phi}dt. \tag{2.122}$$

The above equations suggest that,

$$n^k n_k = 1, \quad n_{[i}\nabla_j n_{k]} = 0, \quad \dot{n}_k \equiv n^i\nabla_i n_k = \nabla_k\phi \tag{2.123}$$

The vector field n^a defines the global rest frame. As one can see, when the spacetime is static the vector field n^a is a unit vector along the timelike Killing vector ξ^a. In the case of stationary axially symmetric spacetimes, the vector field n^a is along the irrotational congruence ζ^a. In an arbitrary spacetime, the vector field n^a can be defined uniquely from equation (2.123). However, locally, each particular choice of n^i uniquely defines a foliation of the spacetime into slices, each of which represents three-space at a particular instant of time. The metric for the three-space orthogonal to n^a is defined as

$$h_{ik} = g_{ik} - n_k n_k, \quad h^i_k = \delta^i_k - n^i n_k \tag{2.124}$$

Similar to the optical reference geometry given by Abramowicz, Carter and Lasota, the metric h_{ab} is conformally adjusted to define $\tilde{h}_{ab}$ as

$$\tilde{h}_{ik} = e^{-2\phi} h_{ik} \tag{2.125}$$

Let $\tilde{\tau}^i$ and $\tilde{\tau}_i$ denote the contravariant and covariant components of the unit vector along τ^i in the conformal geometry $\tilde{h}_{ik}$, which are given by

$$\tilde{\tau}^i = e^{\phi}\tau^i, \quad \tilde{\tau}_i = e^{-\phi}\tau_i \tag{2.126}$$

The dynamics of a test particle with an arbitrary four velocity can be studied as follows. First the four velocity u^a is split uniquely with respect to the unit timelike vector n^a as.

$$u^a = \gamma\left(n^a + v\tau^a\right) \tag{2.127}$$

Here τ^a is the unit vector orthogonal to n_a, along which the spatial three velocity v of the particle is aligned and γ is the normalization factor that makes $u^a u_a = 1$.

The acceleration of the test particle with the above four-velocity u^a (2.127)

$$a_k = u_{k;b}u^b \equiv u^i \nabla_i u_k \tag{2.128}$$

can be expressed as,

$$a_k = n_k\dot{\gamma} + \tau_k\dot{(\gamma v)} - \gamma^2\nabla_k\phi + \gamma^2 v\left(\tau^i\nabla_i\tau_k + n^i\nabla_i\tau_k\right) + \gamma^2 v^2 \tau^i \nabla_i \tau_k \tag{2.129}$$

We define the quantity

$$\mathcal{E} = e^{\phi}\gamma \tag{2.130}$$

which is the energy of the test particle when the spacetime is stationary. Then we have,

$$\dot{\gamma} = \gamma_{,b}n^b \tag{2.131}$$

and,

$$\begin{aligned}\dot{(\gamma v)} &= (\gamma v)_{,b} n^b \\ &= e^{-\phi} u^i \nabla_i (\mathcal{E} v) - \gamma^2 v^2 \tau^i \nabla_i \phi \end{aligned} \qquad (2.132)$$

If $\tilde{\nabla}_i$ is the covariant derivative operator in the space orthogonal to n^a, then one can show that

$$\tilde{\tau}^i \tilde{\nabla}_i \tilde{\tau}_k = \tau^i \nabla_i \tau_k - \tau^i \tau_k \nabla_i \phi - \nabla_k \phi \qquad (2.133)$$

Substituting the above expressions in equation (2.129), the expression for the acceleration can be written as

$$a_k = n_k \dot{\gamma} + \tilde{\tau}_k u^i \nabla_i (\mathcal{E} v) + \gamma^2 v \left(\tau^i \nabla_i n_k + n^i \nabla_i \tau_k \right) + \gamma^2 v^2 \tilde{\tau}^i \tilde{\nabla}_i \tilde{\tau}_k - \nabla_k \phi \qquad (2.134)$$

Using the fact that

$$\tau^i \nabla_i n_j = -\tau^i \nabla_j n_i - \tau^i n_j n^k \nabla_k n_i \qquad (2.135)$$

we obtain,

$$a_j = \nabla_j \phi + \dot{\gamma} n_j + \tilde{\tau}_j u^i \nabla_i (\mathcal{E} v) - \gamma^2 v n^i (\nabla_i \tau_j - \nabla_j \tau_i) - \gamma^2 v \tau^i n_j n^l \nabla_l n_i + \gamma^2 v^2 \tilde{\tau}^i \tilde{\nabla}_i \tilde{\tau}_j \qquad (2.136)$$

Projecting this equation on to the space orthogonal to n^a with the projection operator,

$$h^i_j = \delta^i_j - n^i n_j \qquad (2.137)$$

we get,

$$\begin{aligned} a^{\perp}_k &= h^j_k a_j \\ &= -\nabla_k \phi + \tilde{\tau}_k u^i \nabla_i (\mathcal{E} v) - \gamma^2 v n^i (\nabla_i \tau_k - \nabla_k \tau_i) + \gamma^2 v^2 \tilde{\tau}^i \tilde{\nabla}_i \tilde{\tau}_k \end{aligned} \qquad (2.138)$$

The various terms of the equation (2.138) can be identified with the inertial forces as follows,

$$a^{\perp}_k = G_k + Z_k + C_k + E_k \qquad (2.139)$$

where,

Gravitational force

$$G_k = \phi_{,k} \tag{2.140}$$

Centrifugal force

$$Z_k = -(\gamma v)\,\tilde{\tau}^i\tilde{\nabla}_i\tilde{\tau}_k \tag{2.141}$$

Euler force

$$E_k = -\dot{V}\tilde{\tau}_k \tag{2.142}$$

Coriolis-Lense-Thirring force

$$C_k = \gamma^2 v X_k \tag{2.143}$$

with,

$$\begin{aligned}\dot{V} &= (ve^{\phi}\gamma)_{,i}u^i \\ X_k &= n^i(\tau_{k;i} - \tau_{i;k}) \\ \phi_{,k} &= -n^i n_{k,i}\end{aligned} \tag{2.144}$$

This is the covariant formalism of inertial forces.

In the next chapter, we shall apply this formalism to a particle moving in a circular trajectory in stationary spacetimes. Though this formalism is general and applicable to an arbitrary spacetime, we shall not apply it to a general stationary spacetime but to the special case of stationary spacetimes with axial symmetry. The prime reason for this is that the vector field n^a is non-unique in a general stationary spacetime. Greene, Schücking and Vishveshwara showed that the irrotational congruence which defines the vector field n^a is unique in a stationary axially symmetric spacetime[35]. The details of these results are presented in the next chapter.

2.4 Gravito-electromagnetism

In the last chapter we have seen the strong analogy between gravitation and electromagnetism. In this section we shall highlight this striking similarity in the case of stationary spacetimes. We follow the example given by Landau and Lifshitz[50] to illustrate the electromagnetic analogy in a stationary spacetime. Here, we show that the acceleration of a test particle in a stationary spacetime can be split into terms analogous to forces acting on a charged test particle in an external electromagnetic field.

As given in section 2.3.1 we shall split a stationary metric with respect to the timelike Killing vector ξ^a. In this case we write the metric as follows

$$ds^2 = h\left(dt - g_\alpha dx^\alpha\right)^2 - dl^2 \tag{2.145}$$

where

$$\begin{aligned} dl^2 &= \gamma_{\alpha\beta} dx^\alpha dx^\beta \\ &= \left(-g_{\alpha\beta} + \frac{g_{0\alpha} g_{0\beta}}{g_{00}}\right) \end{aligned} \tag{2.146}$$

and

$$h = g_{00} \tag{2.147}$$

$$g_\alpha = -\frac{g_{0\alpha}}{g_{00}} \tag{2.148}$$

We study the motion of a test particle moving with a four velocity u^a by splitting the acceleration into various gravito-electromagnetic terms. The four velocity of a test particle given by the equation

$$u^i = \frac{d\,x^i}{ds} \tag{2.149}$$

can be decomposed into spatial and temporal parts as follows.

$$u^a = \left(u^0,\ u^\alpha\right) \tag{2.150}$$

where,

$$u^0 = \frac{1}{\sqrt{h}\sqrt{1-v^2}} + \frac{g_\alpha v^\alpha}{\sqrt{1-v^2}} \tag{2.151}$$

$$u^\alpha = \frac{v^\alpha}{\sqrt{1-v^2}} \tag{2.152}$$

Here, v^α is the spatial three-velocity in the three-space orthogonal to the timelike Killing vector field and is defined by the metric $\gamma_{\alpha\beta}$ as given in the equation (2.146). Also we have,

$$v_\alpha = \gamma_{\alpha\beta} v^\beta \tag{2.153}$$

$$v^2 = v_\alpha v^\alpha \tag{2.154}$$

In order to compute the acceleration of the test particle, we split the Christoffel symbols Γ^a_{bc} as given in reference[50].

$$\Gamma^\alpha_{00} = \frac{1}{2}h^{;\alpha}, \tag{2.155}$$

$$\Gamma^\alpha_{0\beta} = \frac{h}{2}\left(g^\alpha{}_{;\beta} - g^{;\alpha}{}_\beta\right) - \frac{1}{2}g_\beta h^{;\alpha} \tag{2.156}$$

$$\Gamma^\alpha_{\beta\gamma} = \lambda^\alpha_{\beta\gamma} + \frac{h}{2}\left[g_\beta\left(g^{;\alpha}_\gamma - g^\alpha{}_{;\gamma}\right) + g_\gamma\left(g^{;\alpha}_\beta - g^\alpha{}_{;\beta}\right)\right] + \frac{1}{2}g_\beta g_\gamma h^{;\alpha} \tag{2.157}$$

where $\lambda^\alpha_{\beta\gamma}$ are the three-dimensional Christoffel symbols defined by the metric $\gamma_{\alpha\beta}$.

The equation of motion can be now written as,

$$\frac{du^\alpha}{ds} = -\Gamma^\alpha_{00}(u^0)^2 - 2\Gamma^\alpha_{0\beta}u^0 u^\beta - \Gamma^\alpha_{\beta\gamma}u^\beta u^\gamma. \tag{2.158}$$

Using the expression for the velocity from equation(2.150) we get

$$\frac{dv^\alpha}{ds\sqrt{1-v^2}} = -\frac{h^{;\alpha}}{2h(1-v^2)} - \frac{\sqrt{h}\left(g^\alpha{}_{;\beta} - g^{;\alpha}{}_\beta\right)v^\beta}{(1-v^2)} - \frac{\lambda^\alpha_{\beta\gamma}v^\beta v^\gamma}{(1-v^2)}. \tag{2.159}$$

One can simplify the above equation to

$$a_\alpha = \frac{1}{\sqrt{1-v^2}}\left\{-\frac{\partial}{\partial x^\alpha}\ln\sqrt{h} + \sqrt{h}\left(\frac{\partial g_\beta}{\partial x^\alpha} - \frac{\partial g_\alpha}{\partial x^\beta}\right)v^\beta\right\} \tag{2.160}$$

where a_α is the spatial acceleration of the test particle. In the three-dimensional notation one can write

$$a = \frac{1}{\sqrt{1-v^2}}\left\{-\nabla\sqrt{h} + \sqrt{h}v \times (curlg)\right\}. \tag{2.161}$$

We compare the above equation with the Lorentz force equation acting on a test particle in an electromagnetic field.

$$F_{EM} = qE + V \times B. \tag{2.162}$$

Here,E is the electric field given by

$$E = -\nabla\Phi_{electric} \tag{2.163}$$

and B is the magnetic field given by vector potential A,

$$B = curl A. \tag{2.164}$$

From the above comparison one can define gravito-electric force as,

$$E_G = -\frac{\nabla\sqrt{h}}{\sqrt{1-v^2}} \tag{2.165}$$

and the gravito-magnetic force as,

$$M_G = \frac{\sqrt{h}}{\sqrt{1-v^2}}v \times curlg. \tag{2.166}$$

In the above the three-vector g_α acts as a vector potential for the gravito-magnetic field. In the next chapter we shall define the gravito-electric and gravito-magnetic field in a simpler and covariant manner. In the formalism given above the gravito-electromagnetic fields are defined with respect to the timelike Killing vector in stationary spacetimes. In order to relate these fields to inertial forces one needs to define these fields with respect to the vector field n^a as in the case of inertial forces which will be considered in the next chapter.

2.5 Conclusions

In the present chapter we have considered the formalism necessary to establish the relations among gyroscopic precession, inertial forces and gravito-electromagnetism. In the next chapter we apply this formalism to circular trajectories in axially symmetric stationary spacetimes and establish interrelations among them. As mentioned earlier, Abramowicz, Carter and Lasota[1] showed that the centrifugal force reversal occurs at the photon orbits in static spacetimes, which was also shown by taking specific examples[3, 4, 60]. It was observed that the gyroscopic precession reversal also takes place at the photon orbits in static spacetimes, by taking specific examples such as the Schwarzschild spacetime[63, 43]. This simultaneous reversal of gyroscopic precession and centrifugal force[57] seems to indicate the possibility of direct relations between the two. These and other related questions are answered in the next chapter.

Chapter 3

Gyroscopic Precession, Inertial Forces and Gravito-electromagnetism: Covariant Connections

3.1 Introduction

In the last chapter we have given the covariant formalisms for gyroscopic precession, inertial forces and gravito-electromagnetism. When these formalisms are applied to specific examples, several interesting results emerge. Such examples would be of interest not only from the conceptual point of view but also for astrophysical applications. One of such interesting observations, is the reversal of the centrifugal force in static spacetimes. In their earlier studies, Abramowicz and coworkers indicate that the centrifugal force reversal occurs at the photon orbits in static spacetimes[3, 4]. This occurs in the Schwarzschild spacetime at $r = 3M$ where the circular null geodesic exists. It was noticed that the gyroscopic precession reversal also occurs in the Schwarzschild spacetime at $r = 3M$. This raises several interesting questions such as the existence of a direct relation between the phenomenon of gyroscopic precession and the centrifugal force in static spacetimes. In this chapter we try to answer these questions. We establish a covariant connection between these two phenomena and show that the reversal of both the gyroscopic precession and the centrifugal force, occur at the photon orbits in the case of static spacetimes. In the case of stationary spacetimes, the physical situation alters completely. Additional inertial forces such as the Coriolis-Lense-Thirring force and Euler force, can also exist. We therefore have, on the one hand, gyroscopic precession which is influenced not only by spacetime curvature but also by the rotation of spacetime itself. On the other hand, rotational effects enter also into the description of the general relativistic equivalent of inertial forces. Under these circumstances, we would like to study the possible reversal of both gyroscopic precession and inertial forces in stationary spacetimes. For this purpose, we also establish the direct relation between the phenomena of gyroscopic precession and inertial forces, *i.e.* centrifugal and Coriolis forces. From this we also prove the result that, in general, the simultaneous reversal of gyroscopic precession and centrifugal force does not occur in a stationary spacetime at the photon orbits.

This chapter is organized in the following manner. In section 3.2 we discuss the important properties of axially symmetric stationary spacetimes, which includes a brief study of global rest frames. In section 3.3 we compute the Frenet-Serret parameters κ, τ_1 and τ_2, and tetrad components for stationary observers. These results are extended to the circular orbits, using a rotating coordinate system. This approach is much simpler compared to the direct computation of Frenet-Serret parameters for circular quasi-Killing trajectories. These results can be specialized to static spacetimes by setting $\xi^a\eta_a = 0$. In section 3.4 we compute the inertial forces for a particle following a circular quasi-Killing trajectory. The Euler force is identically zero along a quasi-Killing trajectory. In the case of static spacetimes, which does not incorporate inertial frame dragging, the Coriolis-Lense-Thirring force is zero. In section 3.5 we establish covariant relations between inertial forces and Frenet-Serret parameters. In the case of simpler static spacetimes we show that the parameter τ_1 is proportional to the scalar product of the acceleration and the centrifugal force. The parameter τ_2 is also directly related to the centrifugal force. In the case of stationary spacetimes, the Coriolis-Lense-Thirring force is non zero and the centrifugal force in the expression for the parameters τ_1 and τ_2 is replaced by a combination of centrifugal and Coriolis-Lense-Thirring forces. In section 3.6 we establish direct relations between the phenomenon of gyroscopic precession and inertial forces, without using the Frenet-Serret formalism. Since the precession frequency is equal to the vorticity for Killing congruence, this establishes the relation between the vorticity of the Killing congruence and inertial forces. In section 3.7 we study the reversal of gyroscopic precession and the centrifugal force. We prove that, in the case of static spacetimes, the simultaneous reversal of gyroscopic precession and centrifugal forces takes place at the circular null geodesics. In the case of stationary spacetimes this does not happen. We also study the reversal of gyroscopic precession and centrifugal force, and their relation to the null geodesics. Abramowicz, Carter and Lasota[1] for the first time used optical reference geometry to study the motion of test particles in the general theory of relativity, where the three space orthogonal to the vector field n^a is conformally adjusted. We study gyroscopic precession and inertial forces in spacetimes which are conformal static spacetimes with the conformal factor corresponding to the optical reference geometry. In this conformal spacetime, the gravitational force is effectively removed. This gives rise to several interesting results, which are discussed in section 3.8. In section 3.9 we study the gravito-electromagnetic fields with respect to two different observers. First we define the gravito-electromagnetic fields with respect to the global rest observers and relate them to inertial forces. We also define the gravito-electromagnetic fields with respect to the four velocity (u^a) of a test particle and relate them to the Frenet-Serret parameters τ_1 and τ_2. Interestingly, these results are exactly similar to those pertaining to the of motion of a charged test particle moving in a constant electromagnetic field[42]. Finallv we end the chapter with a brief concluding section.

3.2 Axially Symmetric Stationary Spacetimes

In this section we summarize some of the important properties of axially symmetric stationary spacetimes, relevant to our studies. An axially symmetric stationary spacetime admits a timelike Killing vector ξ^a and a spacelike Killing vector η^a which generates closed circular orbits around the axis of symmetry. If the spacetime is asymptotically flat, ξ^a is a timelike unit vector at infinity and $(\xi^a\eta_a)/(\eta^b\eta_b)^{\frac{1}{2}}$ goes to zero at infinity. Furthermore, the Killing vectors ξ^a and η^a commute[19],

$$
\begin{aligned}
[\xi,\ \eta]^a &\equiv \pounds_\xi \eta^a \\
&= -\pounds_\eta \xi^a \\
&= \xi^b \eta^a_{;b} - \eta^b \xi^a_{;b} \ = \ 0,
\end{aligned}
\tag{3.1}
$$

where $\pounds_\xi \eta^a$ is the Lie derivative of the vector field ξ^a with respect to the vector field η^b. Assuming orthogonal transitivity, in coordinates ($x^0 \equiv t, \quad x^3 \equiv \phi$) adapted to the Killing vectors ξ^a and η^a respectively, the metric takes on its canonical form

$$
ds^2 \ = \ g_{00}dt^2 \ + \ 2g_{03}dtd\phi \ + \ g_{33}d\phi^2 \ + \ g_{11}dr^2 \ + \ g_{22}d\theta^2 \tag{3.2}
$$

with g_{ab} functions of $x^1 \equiv r$ and $x^2 \equiv \theta$ only. In this case, the spacetime is foliated by a two-parameter family of two-surfaces which are everywhere orthogonal to the two surfaces formed by the Killing vectors ξ^a and η^a. The condition for orthogonal transitivity is characterized by the Killing vector fields ξ^a and η^a satisfying the equation[49, 48],

$$\epsilon_{abcd}\eta^a\xi^b\xi^{c;d} = 0 = \epsilon_{abcd}\xi^a\eta^b\eta^{c;d} \tag{3.3}$$

As we have mentioned in the last chapter, such a spacetime admits a globally hypersurface orthogonal timelike vector field[14, 35],

$$\zeta^a = \xi^a + \omega_0\eta^a, \tag{3.4}$$

with

$$\zeta^a\zeta_a = \xi^a\xi_a + \omega_0\xi^a\eta_a.$$

The fundamental angular speed of the irrotational congruence is

$$\omega_0 = -(\xi^a\eta_a)/(\eta^b\eta_b). \tag{3.5}$$

The vector field ζ^a is the projection of the timelike Killing vector ξ^a orthogonal to η^a. Since ω_0 is not a constant, ζ^a is not a Killing vector. However, since $\pounds_\zeta\omega_0 = 0$, the vector field ζ^a forms a quasi-Killing congruence. The vorticity,

$$\omega^a_\zeta = \frac{1}{\sqrt{-g}}\epsilon^{abcd}\zeta_{b;c}\zeta_d = 0 \tag{3.6}$$

for this congruence, which implies that ζ^a is a locally irrotational congruence. Further more , observers who follow the worldlines along ζ^a do not rotate with respect to the neighbouring ones belonging to this congruence[14]. Thus, the dragging of inertial frames is eliminated. Irrotation is equivalent to local hypersurface orthogonality, *i.e.* in some neighbourhood about each point, the infinitesimal three surfaces orthogonal to ζ^a are surface- forming. But Greene, Schücking and Vishveshwara[35] showed that ζ^a is actually globally hypersurface orthogonal. Since ζ^a is a timelike global hypersurface orthogonal vector field, it defines a surface of simultaneity with each of the observers along ζ^a; their world-time clocks are synchronized by the hypersurface. The ζ^a frame is a generalization of the Newtonian non-rotating rest frame. In reference

[35], Greene, Schücking and Vishveshwara, show that for the existence and uniqueness of irrotational congruence, orthogonal transitivity is a sufficient but not a necessary condition. A weaker condition in comparison with equation (3.3),

$$\zeta_{[a}\eta_b\eta_{c,d]} = 0 \tag{3.7}$$

is a sufficient condition for the existence and uniqueness of i ζ^a. They also prove some of the important properties of irrotational congruence; it was shown that the ζ^a frame is well behaved down to the event horizon, where ζ^a becomes null. In the case of orthogonal transitivity ζ^a (if not zero) coincides with a Killing vector on the event horizon. This result also holds for the spacetime satisfying the weaker condition, viz. equation (3.7). In the case of static spacetimes where $\xi^a\eta_a = 0$ the vector field ζ^a coincides with the timelike Killing vector filed ξ^a.

In the following sections, we compute gyroscopic precession and inertial forces in the case of axially symmetric stationary spacetimes. For simplicity, we assume that the spacetime satisfies the orthogonal transitivity conditions as given in equation (3.3). As we have seen in chapter 2, in order to define the inertial forces in the general theory of relativity, one uses a rest frame which is hypersurface orthogonal. We use the observers along ζ^a in order to define the inertial forces.

3.3 Gyroscopic Precession in Axially Symmetric Stationary Spacetimes

In this section we compute the Frenet-Serret parameters and tetrads for a trajectory along the timelike Killing vector ξ^a in a stationary axially symmetric spacetime. These trajectories represent the worldlines of stationary observers; in the case of black hole spacetimes these observers are at rest with respect to the black hole. The four velocity for such observers is given by

$$u^a = e^{\psi}\xi^a, \tag{3.8}$$

where

$$e^{-2\psi} = \xi^a\xi_a = \mathcal{A}. \tag{3.9}$$

The above four velocity represents the stationary observer in the spacetime.

After straight forward but long calculations, the expressions for the Frenet-Serret parameters can be expressed as[43],

$$\kappa^2 = -g^{ab}a_a a_b \tag{3.10}$$

$$\tau_1^2 = \left[g^{ab}a_a d_b\right]^2 \tag{3.11}$$

$$\tau_2^2 = \left[\frac{\varepsilon^{abcd}}{\sqrt{-g}}n_a\tau_b a_c d_d\right]^2 \tag{3.12}$$

and the Frenet-Serret basis vectors can be written as,

$$\begin{aligned}
e^a_{(0)} &= \frac{1}{\sqrt{\mathcal{A}}}(1,0,0,0)\\
e^a_{(1)} &= -\frac{1}{\kappa}(0,g^{11}a_1,g^{22}a_2,0)\\
e^a_{(2)} &= \frac{1}{\sqrt{\mathcal{A}}\sqrt{-\Delta_3}}(\mathcal{B},0,0,-\mathcal{A})\\
e^a_{(3)} &= \frac{\sqrt{g^{11}g^{22}}}{\kappa}(0,-a_2,a_1,0)
\end{aligned} \tag{3.13}$$

In the above,

$$\begin{aligned}
d_a &= \left(\frac{\mathcal{B}}{2\sqrt{-\Delta_3}\kappa}\right)\left[\frac{\mathcal{B}_a}{\mathcal{B}}-\frac{\mathcal{A}_a}{\mathcal{A}}\right] = \left(\frac{\mathcal{B}}{\sqrt{-\Delta_3}\kappa}\right)[b_a-a_a]\\
a_a &= \frac{\mathcal{A}_a}{2\mathcal{A}}\\
b_a &= \frac{\mathcal{B}_a}{2\mathcal{B}}\\
\mathcal{A} &= (\xi^a\xi_a), \quad \mathcal{B} = (\eta^a\xi_a)
\end{aligned} \tag{3.14}$$

$$\mathcal{A}_a = (\xi^b\xi_b)_{,a}, \quad \mathcal{B}_b = (\eta^b\xi_b)_{,a}; \quad a = 1, 2.$$
$$\Delta_3 = (\xi^a\xi_a)(\eta^b\eta_b) - (\eta^a\xi_a)^2 \tag{3.15}$$

where n^a is the unit vector along the irrotational congruence defined by ζ^a and τ^i is the unit vector along the rotational Killing vector η^a. We may note that all the above equations can be specialized to a static spacetime by setting $\xi^a\eta_a = 0$ and $\zeta^a \equiv \xi^a$. Since the quantity d_a in equation (3.14) is zero for static spacetimes, the gyroscopic precession is zero for static observers. But, for stationary observers the gyroscopic precession is non-zero. This is due to the effect of inertial frame dragging in stationary spacetimes.

The expressions for the Frenet-Serret parameters computed in this section for stationary observers can be easily generalized to the circular trajectories using the rotating coordinate systems. The rotating coordinate system approach was adapted by Rindler and Perlick[63] in order to compute the gyroscopic preccsion along circular tajectories. This method is far simpler than the actual computation of the Frenet-Serret parameters for circular trajectories. We discuss the rotating coordinate system approach to compute the Frenet-Serret parameters for circular orbits in the next section.

3.3.1 Gyroscopic Precession for Circular Orbits

In the last section, we have computed the Frenet-Serret parameters and the tetrad components for stationary observers. In the present section, we extend the computation for a test particle moving in a circular orbit. The circular orbits around the axis of symmetry can be represented by Killing trajectories

$$\chi^a = \xi^a + \omega\eta^a \tag{3.16}$$

where ω is constant along each orbit. The four velocity of a particle along these trajectories can be written as follows,

$$u^a = e^{\psi}\chi^a = e^{\psi}(\xi^a + \omega\eta^a). \tag{3.17}$$

In order to compute the Frenet-Serret parameters for circular orbits given by the above four velocity, we adapt a rotating coordinate system appraoch given by Rindler and Perlick[63]. A stationary axially symmetric metric of the form (3.2), adapted to the Killing vector ξ and η is form invariant under the coordinate transformation,

$$\phi = \phi' + \omega t'; \quad t = t' \tag{3.18}$$

where ω is a constant. In the rotating coordinate system, the line element can be expressed as[43],

$$ds^2 = g_{0'0'}dt'^2 + 2g_{0'3'}dt'd\phi' + g_{3'3'}d\phi'^2 + g_{11}dr^2 + g_{22}d\theta^2 \tag{3.19}$$

where

$$\begin{aligned} g_{0'0'} &= g_{00} + 2\omega g_{03} + \omega^2 g_{33} \\ g_{0'3'} &= g_{03} + \omega g_{33} \\ g_{3'3'} &= g_{33}. \end{aligned} \tag{3.20}$$

Under the coordinate transformation (3.18), we note that $\xi^a + \omega\eta^a$ is also a Killing vector, provided ω is a constant. The metric (3.19) is given for a coordinate system adapted to the Killing vector field $\xi'^a = \xi^a + \omega\eta^a$ and $\eta' = \eta$. The Killing vector $\xi' = (1, 0, 0, 0)$ is timelike (for timelike circular orbits in the original coordinate system) and we can use equations (3.10), (3.11) and (3.12) to obtain κ, τ_1 and τ_2 along this worldline. However, ξ' corresponds to $\xi + \omega\eta$ in the unprimed coordinates so that we can compute κ, τ_1 and τ_2 along trajectories $\xi + \omega\eta$ by replacing $\xi^a\xi_a$ and $\xi^a\eta_a$ in equations (3.10), (3.11) and (3.12) by $\mathcal{A}$ and $\mathcal{B}$. More importantly the prescription also works in the case where ω is not a constant but only satisfies $\pounds_\chi \omega = 0$.

Thus along trajectories of $\xi + \omega\eta$, we have,

$$\kappa^2 = -g^{ab} a_a a_b \tag{3.21}$$

$$\tau_1^2 = \left[g^{ab} a_a d_b\right]^2 \tag{3.22}$$

$$\tau_2^2 = \left[\frac{\varepsilon^{abcd}}{\sqrt{-g}} n_a \tau_b a_c d_d\right]^2 \tag{3.23}$$

In the above,

$$\begin{aligned}
d_a &= \left(\frac{\mathcal{B}}{2\sqrt{-\Delta_3}\kappa}\right)\left[\frac{\mathcal{B}_a}{\mathcal{B}} - \frac{\mathcal{A}_a}{\mathcal{A}}\right] = \left(\frac{\mathcal{B}}{\sqrt{-\Delta_3}\kappa}\right)[b_a - a_a] \\
a_a &= \frac{\mathcal{A}_a}{2\mathcal{A}} \\
b_a &= \frac{\mathcal{B}_a}{2\mathcal{B}} \\
\mathcal{A} &= (\xi^a\xi_a) + 2\omega\,(\eta^a\xi_a) + \omega^2\,(\eta^a\eta_a) \\
\mathcal{B} &= (\eta^a\xi_a) + \omega\,(\eta^a\eta_a) \\
\mathcal{C} &= (\xi^a\xi_a) + \omega\,(\eta^a\xi_a) \\
\mathcal{A}_a &= (\xi^b\xi_b)_{,a} + 2\omega(\eta^b\xi_b)_{,a} + \omega^2(\eta^b\eta_b)_{,a}; \quad a = 1,2. \\
\mathcal{B}_b &= (\eta^a\xi_a)_{,b} + \omega(\eta^a\eta_a)_{,b}; \quad b = 1,2. \\
\Delta_3 &= (\xi^a\xi_a)(\eta^b\eta_b) - (\eta^a\xi_a)^2
\end{aligned} \tag{3.24}$$

The Frenet-Serret tetrad is obtained by a vector transformation and can be written as,

$$\begin{aligned}
e^a_{(0)} &= \frac{1}{\sqrt{\mathcal{A}}}(1, 0, 0, \omega) \\
e^a_{(1)} &= -\frac{1}{\kappa}(0, g^{11}a_1, g^{22}a_2, 0) \\
e^a_{(2)} &= \frac{1}{\sqrt{\mathcal{A}}\sqrt{-\Delta_3}}(\mathcal{B}, 0, 0, -\mathcal{C}) \\
e^a_{(3)} &= \frac{\sqrt{g^{11}g^{22}}}{\kappa}(0, -a_2, a_1, 0)
\end{aligned} \tag{3.25}$$

One can also check explicitly that the same expressions for κ, τ_1 and τ_2 are obtained by direct calculations for the four velocity along $\xi + \omega\eta$. As mentioned before, the above result can be specialized to static spacetimes by setting $\xi^a\eta_a = 0$.

The above equations for the Frenet-Serret parameters are presented in a different form than those given in reference [43] in such a way, that it is convenient to establish direct relations with inertial forces. In the next section, we compute the inertial forces for a test particle moving in a circular orbit.

3.4 Inertial Forces in Axially Symmetric Stationary Spacetimes

In the last chapter, we have described the formalism for inertial forces in the general theory of relativity. In this section, we compute the inertial forces for a test particle in a circular orbit, described by the quasi-Killing trajectory

$$u^a = e^\psi(\xi^a + \omega\eta^a) \tag{3.26}$$

As given in chapter 2, we decompose the velocity with respect to the rest frame described by the irrotational congruence,

$$u^a = e^\psi(\xi^a + \omega\eta^a) \equiv \gamma(n^a + v\tau^a) \tag{3.27}$$

here n^a is a unit vector along the irrotational vector field given in equation (3.4),

$$n^a = e^{-\phi}\zeta^a = e^{-\phi}\left(\xi^a - \frac{\xi^b\eta_b}{\eta^c\eta_c}\eta^a\right). \tag{3.28}$$

These form the fundamental global rest observers in an axially symmetric stationary spacetime. Then we have that τ^a is a unit vector orthogonal to n^a or τ^a is a unit vector along the three-velocity of the particle,

$$\tau^a = e^{-\alpha}\eta^a. \tag{3.29}$$

The spatial speed v and the normalizing factor γ can be expressed as,

$$\begin{aligned} \gamma &= e^{\psi+\phi} \\ v &= e^{-\phi+\alpha}(\omega-\omega_0) \end{aligned} \tag{3.30}$$

where ϕ, α and ψ can be written as follows

$$\phi = \frac{1}{2}\ln(\zeta^a\zeta_a), \tag{3.31}$$

$$\alpha = \frac{1}{2}\ln(-\eta^a\eta_a), \tag{3.32}$$

$$\psi = \frac{1}{2}\ln(\chi^a\chi_a). \tag{3.33}$$

From the above relations, we can write down the inertial forces from their definition as follows.

Gravitational force

$$G_k = \phi_{,k} \tag{3.34}$$

Centrifugal force

$$Z_k = \frac{1}{2}e^{2(\psi+\phi)}\tilde{\omega}^2\left(\frac{\eta^a\eta_a}{\zeta^b\zeta_b}\right)_{,k} \tag{3.35}$$

Coriolis-Lense-Thirring force

$$C_k = e^{2(\psi+\alpha)}\tilde{\omega}\left(\frac{\xi^a\eta_a}{\eta^b\eta_b}\right)_{,k} \tag{3.36}$$

where

$$\tilde{\omega} = (\omega-\omega_0) \tag{3.37}$$

In the case of quasi-Killing trajectories, it is easy to show that $\dot{V} = 0$ and hence the Euler force does not exist,

$$E_k = 0 \tag{3.38}$$

As in the case of Newtonian gravity, in the general theory of relativity also, the gravitational force can be expressed as the gradient of a scalar potential. For a particle following a quasi-Killing trajectory, inertial forces are proportional to gradients of functions.

3.4.1 Specialization to Static Spacetimes:

In a static spacetime, the global timelike Killing vector ξ^a is itself hypersurface orthogonal. The unit vector n^a is now aligned along ξ^a,

$$n^a = e^{-\phi}\xi^a. \tag{3.39}$$

Then we have the inertial forces as follows:

Gravitational force

$$G_k = \phi_{,k} \tag{3.40}$$

where $\phi = \frac{1}{2}ln(\xi^a\xi_a)$

Centrifugal force

$$Z_k = -\frac{\omega^2}{2}e^{2(\psi+\alpha)}\left[\ln\left(\frac{\eta^i\eta_i}{\xi^j\xi_j}\right)\right]_{,k} \tag{3.41}$$

And the Coriolis-Lense-Thirring force is identically zero,

$$C_k = 0, \tag{3.42}$$

this is because of the fact that $\xi^a\eta_a = 0$ in static spacetimes, which determines the dragging of inertial frames.

3.5 Covariant Connections

In the preceding section, we have derived expressions for τ_1 and τ_2 which give the gyroscopic precession rate in terms of the Killing vectors. Similarly, inertial forces in an arbitrary axisymmetric stationary spacetime have also been written down in terms of the Killing vectors. All these quantities have been defined in a completely covariant manner. We shall now proceed to establish covariant connections between the gyroscopic precession, *i.e.* the Frenet-Serret torsions τ_1 and τ_2, on the one hand

and the inertial forces on the other. First, we shall consider the simpler case of static spacetimes.

3.5.1 Static Spacetimes

In the present section, we establish the relation between the Frenet-Serret parameters τ_1 and τ_2 with the centrifugal force. As shown earlier, the Coriolis-Lense-Thirring force is identically zero in static spacetimes.

We have derived in equations (3.22) and (3.23), the Frenet-Serret torsions τ_1 and τ_2 for a stationary spacetime. As has been mentioned earlier, for a static spacetime $\xi^a\eta_a = 0$ and $\zeta^a = \xi^a$ in the above equations, as well as in the expressions for the inertial forces. With this specialization, the centrifugal force can be written from equation (3.41) as

$$Z_b = e^{-(\phi-\alpha)}\omega\kappa d_b \tag{3.43}$$

Substituting equation (3.43) in equations (3.22) and (3.23) we arrive at the relations

$$\tau_1^2 = \frac{\beta^2}{\omega^2}\left[a^b Z_b\right]^2 \tag{3.44}$$

and

$$\tau_2^2 = \frac{\beta^2}{\omega^2}\left[\frac{\varepsilon^{abcd}}{\sqrt{-g}}n_a\tau_b a_c Z_d\right]^2 \tag{3.45}$$

where

$$\beta = \frac{e^{(\phi-\alpha)}}{\kappa} \tag{3.46}$$

The equations above relate the gyroscopic precession directly to the centrifugal force. The two torsions τ_1 and τ_2, equivalent to the two components of the precession, are respectively proportional to the scalar and cross products of the acceleration and the centrifugal force. We shall discuss the consequences of these relations later on. In the next section, we derive similar relations in axially symmetric stationary spacetimes.

3.5.2 Stationary Spacetimes

In order to establish direct relations between the Frenet-Serret parameters and inertial forces in axially symmetric stationary spacetimes, we decompose all the parameters with respect to the irrotational congruence. From equation (3.14) we have

$$\begin{aligned} \mathcal{A} &= (\xi^a\xi_a) + 2\omega\,(\eta^a\xi_a) + \omega^2\,(\eta^a\eta_a) \\ \mathcal{B} &= (\eta^a\xi_a) + \omega\,(\eta^a\eta_a) \end{aligned}$$

We decompose the angular speed ω with reference to the fundamental angular speed of the irrotational congruence $\omega_0 = -\frac{(\xi^a\eta_a)}{(\eta^a\eta_a)}$,

$$\omega = \tilde{\omega} + \omega_0. \tag{3.47}$$

Then $\mathcal{A}$ and $\mathcal{B}$ simplifies to,

$$\begin{aligned} \mathcal{A} &= \zeta^a\zeta_a + \tilde{\omega}^2\,\eta^a\eta_a \\ \mathcal{B} &= \tilde{\omega}\,\eta^a\eta_a \end{aligned} \tag{3.48}$$

Similarly, their derivatives also can be written as,

$$\begin{aligned} \mathcal{A}_a &= (\zeta^b\zeta_b)_{,a} + 2\tilde{\omega}\,\mathcal{C}_a + \tilde{\omega}^2\,(\eta^b\eta_b)_{,a} \\ \mathcal{B}_a &= \mathcal{C}_a + \tilde{\omega}\,(\eta^b\eta_b)_{,a} \end{aligned} \tag{3.49}$$

where

$$\mathcal{C}_a \equiv (\xi^b\eta_b)_{,a} + \omega_0\,(\eta^b\eta_b)_{,a} \tag{3.50}$$

or equivalently,

$$\mathcal{C}_a = -(\xi^b\eta_b)\omega_{0,a} \tag{3.51}$$

From equations (3.14),(3.48) and (3.49) we can show that

$$\begin{aligned} d_a &= -e^{2\psi}\frac{e^{-(\phi+\alpha)}\tilde{\omega}}{2\kappa}\{(\zeta^p\zeta_p)\mathcal{C}_a + \tilde{\omega}[(\zeta^p\zeta_p)(\eta^q\eta_q)_{,a} - (\eta^p\eta_p)(\zeta^q\zeta_q)_{,a}] \\ &- \tilde{\omega}^2\,(\eta^p\eta_p)\mathcal{C}_a\} \end{aligned} \tag{3.52}$$

Further, it is is easy to see that $\mathcal{C}_a$ is directly proportional to the Coriolis force,

$$\mathcal{C}_a = - e^{-2\psi} \tilde{\omega}^{-1} C_a \tag{3.53}$$

where C_a is the Coriolis-Lense-Thirring force. Then equation (3.52) takes on the form

$$d_a = \frac{e^{(\phi-\alpha)}}{\tilde{\omega}\kappa} \left\{ Z_a - \frac{1}{2} \left[1 + \tilde{\omega}^2 e^{2(\alpha-\phi)} \right] C_a \right\} \tag{3.54}$$

Where Z_a is the centrifugal force.

Substituting this in equation(3.22) for τ_1^2 we get the relation,

$$\tau_1^2 = \frac{\beta^2}{\tilde{\omega}^2} \left[g^{ab} a_a \left(Z_b + \beta_1 C_b \right) \right]^2 \tag{3.55}$$

where

$$\begin{aligned} \beta &= \frac{e^{(\phi-\alpha)}}{\kappa} \\ \beta_1 &= -\frac{1}{2} \left[1 + \tilde{\omega}^2 e^{2(\alpha-\phi)} \right] \end{aligned} \tag{3.56}$$

Again, from equation (3.23) and (3.52), we obtain the expression

$$\tau_2^2 = \frac{\beta^2}{\tilde{\omega}^2} \left[\frac{\varepsilon^{abcd}}{\sqrt{-g}} n_a \tau_b a_c \left(Z_d + \beta_1 C_d \right) \right]^2 \tag{3.57}$$

These relations are more complicated than those we have derived in the static case. Nevertheless, they closely resemble the latter with the centrifugal force replaced by the combination of the centrifugal and Coriolis forces $(Z_a + \beta_1 C_a)$. The static case formulae are obtained from those of the stationary case by setting the Coriolis force to zero.

In this method we have directly expressed the Frenet-Serret parameters τ_1 and τ_2, which determine the gyroscopic precession, in terms of the inertial forces.

3.6 Gyroscopic Precession, Vorticity and Inertial Forces

In the last section, we have related the gyroscopic precession and inertial forces using the Frenet-Serret formalism. In other words, we have related the Frenet-Serret parameters τ_1 and τ_2 to the inertial forces. In this section we shall directly relate the gyroscopic precession and inertial forces yielding vector relations, whereas the relations derived in the last section were scalar. The vorticity is another important geometrical quantity of a trajectory which is embedded in a congruence. We have seen in the last chapter, that for a Killing congruence, the vorticity and the gyroscopic precession frequency are identical. By setting the parameter ω constant, we obtain direct relations between the vorticity and the inertial forces.

In the last chapter, we have seen that, for a quasi-Killing trajectory, the gyroscopic precession frequency can be written as,

$$\omega^a = \tilde{F}^{ab} u_b. \tag{3.58}$$

In order to relate the above equation and the inertial forces, we define the following quantities, following the notation by Geroch[33],

$$\underset{00}{\omega}^a \equiv \epsilon^{abcd} \xi_b \xi_{c;d} \tag{3.59}$$

$$\underset{01}{\omega}^a \equiv \frac{1}{2} \epsilon^{abcd} \left(\xi_b \eta_{c;d} + \eta_b \xi_{c;d} \right) \tag{3.60}$$

$$\underset{11}{\omega}^a \equiv \epsilon^{abcd} \eta_b \eta_{c;d}. \tag{3.61}$$

We also define the following scalar parameters

$$\begin{aligned} \lambda_{00} &= \xi^a \xi_a \\ \lambda_{01} &= \xi^a \eta_a \end{aligned} \tag{3.62}$$

$$\lambda_{11} = \eta^a \eta_a$$
$$\tau^2 = 2(\lambda_{01}^2 - \lambda_{00})\lambda_{11}$$

From equation (3.59) and using the properties of Killing vectors, the derivatives of the Killing vector can be expressed in terms of its vorticity, as follows,

$$\xi_{p;q} = \frac{\lambda_{00}^{-1}}{2}\epsilon_{pqrs}\xi^r \underset{00}{\omega}^s + \lambda_{00}^{-1}\xi_{[p}\lambda_{00,q]} \tag{3.63}$$

Similarly from equation (3.61) we get

$$\eta_{p;q} = \frac{\lambda_{11}^{-1}}{2}\epsilon_{pqrs}\eta^r \underset{11}{\omega}^s + \lambda_{11}^{-1}\eta_{[p}\lambda_{00,q]} \tag{3.64}$$

From equations (3.63) and (3.64), we express the vorticity $\underset{00}{\omega}^a$, $\underset{01}{\omega}^a$, and $\underset{11}{\omega}^a$ in terms of the scalar parameters defined in equation (3.63) as follows.

$$\underset{00}{\omega}^a = 2\tau^{-2}\epsilon^{aqbc}\xi_b\eta_c\left(\lambda_{00}\lambda_{01,q} - \lambda_{01}\lambda_{00,q}\right) \tag{3.65}$$

$$\underset{01}{\omega}^a = \tau^{-2}\epsilon^{aqbc}\xi_b\eta_c\left(\lambda_{11}\lambda_{00,q} - \lambda_{00}\lambda_{11,q}\right) \tag{3.66}$$

$$\underset{11}{\omega}^a = 2\tau^{-2}\epsilon^{aqbc}\xi_b\eta_c\left(\lambda_{11}\lambda_{01,q} - \lambda_{01}\lambda_{11,q}\right) \tag{3.67}$$

The gyroscopic precession frequency along a quasi-Killing trajectory is given by

$$\omega^a = \hat{F}^{ab}u_b \tag{3.68}$$

From equation (3.59, 3.60 and 3.61) we can express ω^a as,

$$\omega^a = \frac{e^{2\psi}}{2}\left[\underset{00}{\omega}^a + 2\omega\,\underset{01}{\omega}^a + \omega^2\,\underset{11}{\omega}^a\right]. \tag{3.69}$$

Decomposing ω with respect to the irrotational congruence,

$$\omega = \tilde{\omega} + \omega_0$$

where ω_0 is $-\xi^a\eta_a/\eta^b\eta_b$, we get

$$\omega^a = \frac{e^{2\psi}}{2}\left[\left(\underset{00}{\omega}^a + 2\omega_0\,\underset{01}{\omega}^a + \omega_0^2\,\underset{11}{\omega}^a\right) + 2\tilde{\omega}\left(\underset{01}{\omega}^a + \omega_0\,\underset{11}{\omega}^a\right) + \tilde{\omega}^2\,\underset{11}{\omega}^a\right] \tag{3.70}$$

or

$$\omega^a = \frac{e^{2\psi}}{\sqrt{2}}\tau^{-1}\epsilon^{ab}\left[\left(\frac{\tau^2}{2}+\tilde{\omega}^2\lambda_{11}^2\right)\left(\frac{\lambda_{01}}{\lambda 11}\right)_{,b}+\tilde{\omega}\left(\frac{\tau^2}{2\lambda_{11}}\right)^2\left(\frac{2\lambda_{11}^2}{\tau^2}\right)_{,b}\right]. \tag{3.71}$$

Using the fact that,

$$\left(\frac{\lambda_{01}}{\lambda_{11}}\right)_{,k} = \frac{e^{-2\psi}}{\tilde{\omega}\lambda_{11}}\, C_k, \tag{3.72}$$

$$\left(\frac{2\lambda_{11}^2}{\tau^2}\right)_{,k} = \frac{2e^{-2\psi}}{\tilde{\omega}^2}\left(\frac{2\lambda_{11}}{\tau^2}\right) Z_k, \tag{3.73}$$

we can express ω^a in terms of the inertial forces as follows,

$$\omega^a = \sqrt{2}\tau^{-1}\left(\frac{\tau^2}{2\lambda_{11}}\right)\frac{1}{\tilde{\omega}}\epsilon^{ab}\left[Z_b+\frac{1}{2}\left(1+\tilde{\omega}^2\frac{2\lambda_{11}^2}{\tau^2}\right)C_b\right]. \tag{3.74}$$

We have

$$\frac{\tau^2}{2\lambda_{11}} = e^{2\phi}, \quad e^{2\alpha} = \lambda_{11}$$

so that we get

$$\omega^a = \frac{e^{(\phi-\alpha)}}{\tilde{\omega}}\epsilon^{ab}\left[Z_b+\frac{1}{2}\left(1+\tilde{\omega}e^{2(\alpha-\phi)}\right)C_b\right] \tag{3.75}$$

or

$$\omega^a = \frac{\beta}{\tilde{\omega}}\epsilon_{abcd}\tau^b\left(Z^c+\beta_1 C^c\right)n^d \tag{3.76}$$

This is the direct relation between the gyroscopic precession and inertial forces, in contrast to the Frenet-Serret approach given in the preceding sections.

In the last two sections, we have established the relationship between inertial forces and gyroscopic precession for circular orbits in stationary axially symmetric spacetimes. One of the important consequences of these relations is to show the simultaneous reversal of the gyroscopic precession and the centrifugal force in static spacetimes, which we shall carry out in the next section. We shall also prove that, in general neither centrifugal force nor gyroscopic precession reversal occurs at the photon orbits in the stationary spacetimes.

3.7 Reversal of Gyroscopic Precession and Inertial Forces

In this section, we study the reversal of the gyroscopic precession and the centrifugal force. Abramowicz, Carter and Lasota[1] first proved that the centrifugal force reversal occurs at the photon orbits in static spacetimes. In this section, we shall show that the gyroscopic precession reversal also occurs at the photon orbits.

The condition for the reversal of the gyroscopic precession is given by

$$\omega^a_{FS} = \tau_1 e^a_{(3)} + \tau_2 e^a_{(1)} = 0 \tag{3.77}$$

Since $e^a_{(1)}$ and $e^a_{(3)}$ are linearly independent vector fields at each point, this condition is the same as requiring

$$\tau_1 = \tau_2 = 0 \tag{3.78}$$

By considering the actual structure of τ_2, it is easy to show that τ_2 becomes zero on a plane about which the metric components are reflection invariant. The equatorial plane in the black hole spacetime is an example of this.

We shall now examine the vanishing of the Frenet-Serret torsions in relation to the inertial forces.

3.7.1 Static Spacetimes

In what follows, we shall prove a theorem that relates the simultaneous reversal of the centrifugal force and the gyroscopic precession to the existence of a null circular orbit. First we shall prove that the gyroscopic precession reversal occurs if and only if the gyroscope is transported along a timelike trajectory whose spatial orbit coincides with a null geodesic.

We start from the condition for gyroscopic precession reversal, *i.e.* $\tau_1 = \tau_2 = 0$, and show that at the point where this occurs a null circular geodesic must exist. In the second part we take a circular null geodesic and show that the gyroscopic precession reversal occurs at such orbits.

Setting $\tau_2 = 0$ in equation (3.23) and noting that the only non vanishing components of n_a and τ_a are respectively, n_0 and τ_3, we arrive at the condition

$$\mathcal{A}_1\mathcal{B}_2 = \mathcal{A}_2\mathcal{B}_1 \tag{3.79}$$

Further setting $\tau_1 = 0$ in equation (3.22), we obtain

$$g^{11}\left(\frac{\mathcal{A}_1\mathcal{B}_1}{\mathcal{B}} - \frac{\mathcal{A}_1^2}{\mathcal{A}}\right) + g^{22}\left(\frac{\mathcal{A}_2\mathcal{B}_2}{\mathcal{B}} - \frac{\mathcal{A}_2^2}{\mathcal{A}}\right) = 0. \tag{3.80}$$

We shall now assume that the gyroscope is transported along a circular orbit which is not a geodesic, *i.e.* $\kappa \neq 0$. This we do in anticipation of the result that a null geodesic - not a timelike one, - exists with its spatial trajectory identical to that of this timelike orbit. Now $\kappa \neq 0$ implies $\mathcal{A}_1 \neq 0$ and $\mathcal{A}_2 \neq 0$ from equation (3.14). Then from equations (3.79) and (3.80) we arrive at

$$\mathcal{A}\mathcal{B}_1 - \mathcal{B}\mathcal{A}_1 = 0 \tag{3.81}$$

and

$$\mathcal{A}\mathcal{B}_2 - \mathcal{B}\mathcal{A}_2 = 0 \tag{3.82}$$

Combining the above two equations,

$$\mathcal{A}\mathcal{B}_a - \mathcal{B}\mathcal{A}_a = 0$$

Then, equation (3.14) reduces this condition to

$$(\xi^b\xi_b)(\eta^c\eta_c)_{,a} - (\xi^b\xi_b)_{,a}(\eta^c\eta_c) = 0 \tag{3.83}$$

With the help of this equation, we can show that, if a circular geodesic exists where precession reverses, then it has to be null, as follows.

The condition for circular geodesics is

$$(\xi^b \xi_b)_{,a} + \omega^2 (\eta^b \eta_b)_{,a} \;=\; 0 \tag{3.84}$$

This can be proved from the geodesic equation, assuming that the four velocity u^a is proportional to $\xi^a + \omega\eta^a$. Using condition (3.83), this reduces to

$$\frac{(\xi^b \xi_b)_{,a}}{(\xi^c \xi_c)} \left[(\xi^d \xi_d) \;+\; \omega^2 (\eta^d \eta_d)\right] = 0 \tag{3.85}$$

Since $\frac{(\xi^b \xi_b)_{,a}}{2(\xi^c \xi_c)}$ is the gravitational force, which is assumed to be nonzero, this is equivalent to

$$(\xi^d \xi_d) \;+\; \omega^2 (\eta^d \eta_d) \;=\; 0 \tag{3.86}$$

This means that the geodesic, if one exists, is null. Now we shall show that in fact a geodesic must exist at the point of precession reversal.

If a geodesic does not exist at the point of reversal, then

$$(\xi^b \xi_b)_{,a} \;+\; \omega^2 (\eta^b \eta_b)_{,a} \;\neq\; 0 \tag{3.87}$$

for all values of ω. However, equation (3.83) may be recast as

$$(\xi^b \xi_b)_{,a} \;-\; \left(\frac{\xi^c \xi_c}{\eta^c \eta_c}\right) (\eta^b \eta_b)_{,a} \;=\; 0. \tag{3.88}$$

This shows that the geodesic condition is satisfied for $\omega^2 \;=\; -\left(\frac{\xi^c \xi_c}{\eta^c \eta_c}\right)$. Therefore there does exist a geodesic and we have already shown that it has to be null. We shall now prove the converse, *i.e.* if a circular null geodesic exists, then τ_1 and τ_2 are zero at the null geodesic.

The condition for a circular null geodesic is given by equation (3.83). Dividing this equation by $(\xi^a \xi_a)(\eta^b \eta_b)$, we see that it reduces to $\left[\ln\left(\frac{\eta^b \eta_b}{\xi^c \xi_c}\right)\right]_{,k}$ which is proportional to Z_a from equation (3.41) and is equal to zero. Further, from the dependence of τ_1 and τ_2 on Z_a, from equations (3.44) and (3.45) we see that $\tau_1 \;=\; \tau_2 \;=\; 0$. We may note the fact that both the gyroscopic precession and the centrifugal force the reverse simultaneously, as is evident from equations (3.44) and (3.45). We have therefore proved the following theorem.

Theorem: In the case of circular orbits in static spacetimes reversal of gyroscopic precession and centrifugal force takes place at some point, if and only if a null geodesic exists at that point.

In the next section we study the reversal of the gyroscopic precession and the centrifugal force in stationary spacetimes.

3.7.2 Stationary Spacetimes

In section 3.5 we have derived expressions for τ_1 and τ_2, that embody gyroscopic precession, in terms of inertial forces, namely the centrifugal force Z_a and Coriolis-Lense-Thirring force C_a. These are complicated expressions and ω does not stand out as an overall multiplicative coefficient. Consequently, the reversal of gyroscopic precession is not related directly to that of these forces individually. As has been discussed in reference [58], these reversals occur at different places and also not at the null geodesic. Nevertheless, one can see from equation (3.55) and (3.57) that the gyroscopic precession reverses at a point where the combination of the centrifugal and the Coriolis forces given by $(Z_a + \beta_1 C_a)$, goes to zero.

We shall derive the angular velocity of a timelike orbit whose three dimensional trajectory coincides with a null geodesic in terms of inertial forces. Although there are no reversals at the null geodesic, this should give an idea of how these forces are structured along the null trajectory.

Conditions for the existence of circular null geodesic are

$$\mathcal{A} \equiv (\xi^a \xi_a) + 2\bar{\omega}\,(\eta^a \xi_a) + \bar{\omega}^2\,(\eta^a \eta_a) = 0 \tag{3.89}$$

and

$$\mathcal{A}_a \equiv (\xi^b \xi_b)_{,a} + 2\bar{\omega}(\eta^b \xi_b)_{,a} + \bar{\omega}^2(\eta^b \eta_b)_{,a} = 0 \tag{3.90}$$

The expression for $\mathcal{A}$ can also be written as

$$\mathcal{A} = \zeta^a \zeta_a + \bar{\bar{\omega}}^2 \eta^a \eta_a, \tag{3.91}$$

where

$$\bar{\omega} = \bar{\bar{\omega}} - \frac{\xi^b \eta_b}{\eta^c \eta_c} \tag{3.92}$$

Then $\mathcal{A} = 0$ implies

$$\bar{\bar{\omega}} = \pm\sqrt{-\frac{\zeta^a \zeta_a}{\eta^a \eta_a}} \tag{3.93}$$

Further, from equation (3.90)

$$\mathcal{A}_a = \frac{1}{\eta^p \eta_p}\left\{(\eta^q \eta_q)(\zeta^b \zeta_b)_{,a} - (\zeta^b \zeta_b)(\eta^q \eta_q)_{,a}\right\} \pm \sqrt{-\frac{\zeta^b \zeta_b}{\eta^q \eta_q}}\mathcal{C}_a = 0 \tag{3.94}$$

This has to be zero for a null geodesic. For a timelike curve with the same spatial orbit, but having angular velocity $\tilde{\omega}$ with respect to n^a,we have from equations (3.35) and (3.53)

$$Z_k = \frac{e^{2\psi}}{2}\,(\omega - \omega_0)^2\,\frac{1}{\zeta^a \zeta_a}\left[(\zeta^b \zeta_b)\,(\eta^c \eta_c)_{,k} - (\eta^b \eta_b)\,(\zeta^c \zeta_c)_{,k}\right] \tag{3.95}$$

$$C_a = -e^{-2\psi}\tilde{\omega}^{-1} C_a \tag{3.96}$$

Substituting in equation (3.94), we get

$$\frac{1}{\eta^p \eta_p}\left\{2e^{-2\psi}(\zeta^p \zeta_p)\tilde{\omega}^{-2} Z_a\right\} \mp \left\{2\sqrt{-\frac{\zeta^b \zeta_b}{\eta^q \eta_q}}e^{-2\psi}\tilde{\omega}^{-1} C_a\right\} = 0 \tag{3.97}$$

This reduces to the equation

$$\sqrt{-\frac{\zeta^p \zeta_p}{\eta^q \eta_q}}Z_a \mp \tilde{\omega} C_a = 0 \tag{3.98}$$

which gives $\tilde{\omega}$ in terms of centrifugal and Coriolis forces.

3.8 Gyroscopic Precession and Inertial Forces in Conformal Static Spacetimes

Some further insight into the gyroscopic precession and the inertial forces may be gained by considering them in a space conformal to the original one, as given in Abramowicz, Carter and Lasota [1]. In the case of the static metric, we carry out the conformal transformation

$$\hat{g}_{ab} = e^{-2\phi} g_{ab} \tag{3.99}$$

If we choose

$$e^{-2\phi} = g^{00} = \frac{1}{g_{00}} = \frac{1}{\hat{\phi}} \tag{3.100}$$

then, $\hat{g}_{00} = \hat{g}^{00} = 1$. The spatial part of metric $\hat{g}_{ab}$ corresponds to optical geometry defined in reference [1], for identifying inertial forces in such geometry. Purely in the conformal space, without referring to the original g_{ab}, we have

$$\hat{u}^a \hat{\nabla}_a \hat{u}^b = 0 \tag{3.101}$$

for a stationary observer with four velocity $\hat{u}^a = (1, 0, 0, 0)$, where $\hat{\nabla}_a$ is the covariant derivative with respect to the conformal metric $\hat{g_{ab}}$. The two four velocities u^a and $\hat{u}^a$ are related by $u^a = e^{-\phi}\hat{u}^a$. Equation (3.101) indicates that because of dilation, $\hat{u}^a$ follows a geodesic trajectory in the conformal metric. This is equivalent to the statement that the only force acting on a particle at rest in the original space is the gravitational force which is not felt in the conformal space. Since the gravitational force is independent of the velocity, no particle will experience it in the conformal space. In other words, the gravitational force is effectively removed to some extent by dilation given in equation (3.100). Consequently, if a particle is moving in a circular trajectory, then the only force acting on it is the centrifugal force.

If ξ^a is a Killing vector in the original space, then ξ^a is also a Killing vector in the conformal space if

$$\mathcal{L}_\xi \hat{\phi} = 0 \tag{3.102}$$

This is trivially true in coordinates adapted to the Killing vector ξ^a. Then the Killing vectors, in the original spacetime are also Killing vectors in the conformal spacetime. Therefore, $\hat{\xi^a} = (1,0,0,0)$ is the timelike Killing vector and $\hat{\eta^a} = (0,0,0,1)$ is the spacelike Killing vector which generates circular orbits in the conformal spacetime. The quasi-Killing trajectories

$$\hat{\chi}^a = \hat{\xi^a} + \omega \hat{\eta^a} \tag{3.103}$$

generate circular orbits and the only force acting on these particles is the centrifugal force. It is easy to prove that the expression for the centrifugal force is now

$$\hat{Z}_a = \hat{u}^b \hat{\nabla}_b \hat{u}_a \tag{3.104}$$

where

$$\hat{u}_a = e^{\hat{\psi}} \hat{\chi}_a \; and \; e^{-2\hat{\psi}} = \hat{\chi}_a \hat{\chi}^a \tag{3.105}$$

3.8.1 Gyroscopic Precession in the Conformal Space

The gyroscopic precession in the conformal spacetime can be computed exactly as before. The Frenet-Serret parameters for circular quasi-Killing trajectories can be written as

$$\hat{\kappa}^2 = -\frac{1}{4}\left(\frac{\hat{g}^{11}\hat{\mathcal{A}}_1^2 + \hat{g}^{22}\hat{\mathcal{A}}_2^2}{\hat{\mathcal{A}}^2}\right) \tag{3.106}$$

$$\hat{\tau}_1^2 = \left(\frac{\hat{\mathcal{B}}^2}{4\hat{\Delta}_3\left(\hat{g}^{11}\hat{\mathcal{A}}_1^2 + \hat{g}^{22}\hat{\mathcal{A}}_2^2\right)}\right) \cdot \left(\frac{\hat{g}^{11}\hat{\mathcal{A}}_1\hat{\mathcal{B}}_1 + \hat{g}^{22}\hat{\mathcal{A}}_2\hat{\mathcal{B}}_2}{\hat{\mathcal{B}}} - \frac{\hat{g}^{11}\hat{\mathcal{A}}_1^2 + \hat{g}^{22}\hat{\mathcal{A}}_2^2}{\hat{\mathcal{A}}}\right)^2 \tag{3.107}$$

$$\hat{\tau}_2^2 = \frac{\hat{g}^{11}\hat{g}^{22}\left(\hat{\mathcal{A}}_1\hat{\mathcal{B}}_2 - \hat{\mathcal{A}}_2\hat{\mathcal{B}}_1\right)^2}{4\hat{\Delta}_3\left(\hat{g}^{11}\hat{\mathcal{A}}_1^2 + \hat{g}^{22}\hat{\mathcal{A}}_2^2\right)} \tag{3.108}$$

where

$$\hat{\mathcal{A}} = \hat{\xi}^a\hat{\xi}_a + \omega^2\hat{\eta}^a\hat{\eta}_a = \frac{\mathcal{A}}{\hat{\phi}}$$

$$\hat{\mathcal{B}} = \omega\,\hat{\eta}^a\hat{\eta}_a = \frac{\mathcal{B}}{\hat{\phi}} \tag{3.109}$$

$$\hat{\Delta}_3 = (\hat{\xi}^a\hat{\xi}_a)(\hat{\eta}^b\hat{\eta}_b) \tag{3.110}$$

and

$$\hat{\mathcal{A}}_a = \omega^2(\hat{\eta}^b\hat{\eta}_b)_{,a} \quad ;a = 1,2$$

$$\hat{\mathcal{B}}_a = \omega(\hat{\eta}^b\hat{\eta}_b)_{,a} \quad ;a = 1,2 \tag{3.111}$$

One can then show that

$$\hat{\mathcal{A}}_a = \frac{\hat{\phi}\mathcal{A}_a - \mathcal{A}\hat{\phi}_{,a}}{\hat{\phi}^2}$$

$$\hat{\mathcal{B}}_a = \frac{\hat{\phi}\mathcal{B}_a - \mathcal{B}\hat{\phi}_{,a}}{\hat{\phi}^2} = \frac{\hat{\mathcal{A}}_a}{\omega} \tag{3.112}$$

With the help of the above equations, $\hat{\kappa}^2$ can be related to κ^2. After some simplification we have,

$$\hat{\kappa}^2 = \hat{\phi}\kappa^2 - \frac{1}{4\hat{\phi}}(g^{ab}\,\hat{\phi}_{,a}\,\hat{\phi}_{,b}) + \frac{1}{2\mathcal{A}}(g^{ab}\,\mathcal{A}_a\,\hat{\phi}_{,b}) \tag{3.113}$$

From the definition of $\hat{\kappa}$ and the expression for the centrifugal force as in (3.104), it is clear that the two are one and the same. This is because the contribution from the gravitational force has been removed and the acceleration that appears is due to the centrifugal force alone. We can relate $\hat{\tau}_1^2$ to $\hat{\kappa}^2$ by using the expression for $\hat{\tau}_1^2$ to obtain

$$\hat{\tau}_1{}^2 = -\frac{\hat{\kappa}^2}{\hat{\Delta}_3\omega^2} \tag{3.114}$$

The above equation is similar to equation(3.44) which relates τ_1 to the centrifugal force. It can also be shown that

$$\hat{\tau}_2^2 = 0 \tag{3.115}$$

everywhere in the conformal spacetime.

From equations (3.114) and (3.115) it is clear that the gyroscopic precession also reverses when $\tilde{\kappa} = 0$ and that in turn corresponds to the centrifugal force reversal. Also $\tilde{\kappa} = 0$ corresponds to the geodesic condition in the conformal space, which represents the null geodesics in the original space as given in reference [1].

To sum up, we have factored out the contribution due to the gravitational force by conformal transformation and have shown in a simple manner the simultaneous reversal of both the gyroscopic precession and the centrifugal force at the photon orbit.

3.9 Gravito-electric and Gravito-magnetic Fields

In the last chapter we have defined the gravito-electromagnetic fields in stationary spacetimes. In this section we shall define these fields in a covariant manner. Here we shall make use of the properties of a Killing vector, *i.e.*, that the derivative of a Killing vector $\xi_{a;b}$ is an antisymmetric tensor. We take the analogy between the Maxwell field tensor for electromagnetic field $\mathcal{F}_{ab}$ and F_{ab} for a Killing vector field. Using the above analogy one can define the gravito-electric and gravito-magnetic field with respect to any timelike vector field. In order to establish relations between the inertial forces and gravito-electromagnetic fields, we define these fields with respect to the global rest observers n^a. These fields with respect to observers following the integral curves of n^a, can be defined as follows.
Gravito-electric field:

$$E^a = F^{ab} n_b \tag{3.116}$$

Gravito-magnetic field:

$$H^a = \tilde{F}^{ab} n_b \tag{3.117}$$

where $\tilde{F}^{ab}$ is the dual of F^{ab},

$$\tilde{F}^{ab} = \frac{1}{2}(\sqrt{-g})^{-1}\varepsilon^{abcd}F_{cd} \tag{3.118}$$

In the above, as before, $F^{ab} = e^{\psi}(\xi_{a;b} + \omega\eta_{a;b})$. The equation of motion is

$$\dot{u}^a = F^{ab}u_b \tag{3.119}$$

Projecting onto the space orthogonal to n^a with $h_{ab} = g_{ab} - n_a n_b$ and decomposing u_a as given in (2.127), we get

$$\dot{u}_{\perp a} = \gamma\left[F_{ac}n^c + v(F_{ac}\tau^c - n_a F_{bc}n^b\tau^c)\right] \tag{3.120}$$

where γ is the normalization factor. This equation can be written in the form

$$\dot{u}_{\perp a} = \gamma\left[F_{ac}n^c + v\sqrt{-g}\varepsilon_{abcd}n^b\tau^c H^d\right], \tag{3.121}$$

or

$$\dot{u}_{\perp a} = \gamma[E + v \times H] \tag{3.122}$$

We can therefore define

Gravito-electric force:

$$f_{GEa} = \gamma F_{ac}n^c \tag{3.123}$$

Gravito-magnetic force:

$$f_{GHa} = \gamma v\sqrt{-g}\varepsilon_{abcd}n^b\tau^c H^d = \gamma v(F_{bc}\tau^c - n_a F_{bc}n^b\tau^c) \tag{3.124}$$

These define the gravito-electromagnetic fields. We have also split the force acting on a test particle in terms of gravito-electric and gravito-magnetic forces. In the next section we relate these forces to the inertial forces.

3.9.1 Relations Among Gravito-electric, Gravito-magnetic and Inertial Forces

3.9.1.1 Static Case

We have defined the gravito-electric field E_a by

$$\gamma E_a = \gamma F_{ac} n^c$$

If we substitute for $F_{ab} = e^{\psi}(\xi_{a;b} + \omega \eta_{a;b})$, we get

$$f_{GEa} = \gamma E_a = \gamma F_{ac} n^c = -e^{2(\psi+\phi)} G_a \tag{3.125}$$

So,

$$E_a = -e^{(\psi+\phi)} G_a \tag{3.126}$$

Here G_a is the gravitational force. Similarly, we have for the gravito-magnetic field

$$f_{GHa} = \gamma v (F_{ac} \tau^c - n_a n^b F_{bc} \tau^c)$$

The second term in this equation is identically zero because the Killing vector fields ξ^a and η^a commute and we get

$$\begin{aligned} f_{GHa} &= \gamma v \sqrt{-g} \varepsilon_{abcd} n^b \tau^c H^d \\ &= \gamma v F_{ac} \tau^c \\ &= \left[e^{2(\psi+\alpha)} \omega^2 G_a - Z_a \right] \end{aligned} \tag{3.127}$$

The above relation clearly shows the connection between the gravito-magnetic force on the one hand and the gravitational and centrifugal forces on the other.

3.9.1.2 Stationary Case

In the stationary case, n^a is given by equation (3.30). As before we decompose $\omega = \tilde{\omega} + \omega_0$, where ω_0 is given by (3.5). Then a straightforward computation gives the expression for the gravito-electric field.

$$E_a = -e^{(\psi+\phi)}G_a \ + \ e^{-(\psi+\phi)}C_a \tag{3.128}$$

and the gravito-electric force,

$$f_{GEa} = \gamma E_a = -e^{2(\psi+\phi)}G_a \ + \ C_a \tag{3.129}$$

This shows the relation of gravito-electric field or force to both the gravitational and centrifugal forces. In the stationary case also we have,

$$n_a n^b F_{bc}\tau^c \ \equiv \ 0 \tag{3.130}$$

Then it follows

$$\begin{aligned} f_{GHa} &\equiv \gamma v\sqrt{-g}\varepsilon_{abcd}n^d\tau^c H^d \\ &= \gamma v F_{ac}\tau^c \\ &= \left[\frac{C_a}{2} \ + \ e^{2(\psi+\alpha)}\tilde{\omega}^2 G_a \ - \ Z_a\right] \end{aligned} \tag{3.131}$$

Hence gravito-magnetic force is related to all the three inertial forces – gravitational, centrifugal and Coriolis.

3.9.2 Gravito-electric and Gravito-magnetic Fields with Respect to Comoving Frame

In the previous section, we have defined gravito-electric and gravito-magnetic fields with respect to the irrotational congruence. Similarly these fields can be defined with respect to the four velocity u^a of the particle as follows.

Gravito-electric field:

$$\tilde{E}^a \ = \ F^{ab}u_b \tag{3.132}$$

Gravito-magnetic field:

$$\tilde{H}^a = \tilde{F}^{ab} u_b \tag{3.133}$$

Where $\tilde{F}^{ab}$ is dual to F^{ab} as before. The equation of motion takes the form

$$a^a = \tilde{E}^a \tag{3.134}$$

The precession frequency can be written simply as

$$\omega^a = \tilde{H}^a \tag{3.135}$$

Following Honig, Schücking and Vishveshwara [42], the Frenet-Serret parameters κ, τ_1 and τ_2 can be expressed in terms of gravito-electric and gravito-magnetic fields.

$$\kappa = |\tilde{E}| \tag{3.136}$$

where

$$|\tilde{E}| = \sqrt{-\tilde{E}^a \tilde{E}_a} \tag{3.137}$$

$$\tau_1 = \frac{|\tilde{P}|}{|\tilde{E}|} \tag{3.138}$$

where

$$\tilde{P}^a = \varepsilon^{abcd} \tilde{E}_b \tilde{H}_a u_d = \tilde{E} \times \tilde{H} \tag{3.139}$$

$$|\tilde{P}| = \sqrt{-\tilde{P}^a \tilde{P}_a} \tag{3.140}$$

and

$$\tau_2 = -\frac{\tilde{H}^a \tilde{E}_a}{|\tilde{E}|} \tag{3.141}$$

the Frenet-Serret tetrad components can also be expressed in terms of $\tilde{E}^a$, $\tilde{H}^a$ and $\tilde{P}^a$,

$$\begin{aligned} e^a_{(1)} &= \frac{\tilde{E}^a}{|\tilde{E}|} \\ e^a_{(2)} &= \frac{\tilde{P}^a}{|\tilde{P}|} \\ e^a_{(3)} &= \frac{\varepsilon^{abcd} \tilde{E}_b \tilde{P}_c u_d}{\tilde{P}^r \tilde{E}_r} \end{aligned} \tag{3.142}$$

In reference [42], these expressions had been derived for charged particle motion in a constant electrcmagnetic field. We have now demonstrated the exact analogue in the case of gravito-electric and gravito-magnetic fields. The one-to-one correspondence is indeed remarkable.

3.10 Conclusions

The main purpose of the present chapter was to establish a covariant connection between the gyroscopic precession on the one hand and the analogies of inertial forces on the other. This has been accomplished in the case of axially symmetric stationary spacetimes for circular orbits. In the special case of static spacetimes the gyroscopic precession can be directly related to the centrifugal force. From this we have been able to prove that both precession and centrifugal force reverse at a photon orbit, provided the latter exists. In the case of stationary spacetimes, the corresponding relations are more complicated. The place of centrifugal force is now taken by a combination of centrifugal and Coriolis-Lense-Thirring forces. As a result, the gyroscopic precession and the centrifugal force do not reverse, in general, at the photon orbit. We have also studied some of the above aspects in the spacetime conformal to the original static spacetime. In this approach, part of the gravitational effect is factored out thereby achieving a certain degree of simplicity and transparency in displaying interrelations and the reversal phenomenon. Closely related to these considerations is the idea of gravito-electric and gravito-magnetic fields. We have covariantly defined these with respect to the globally hypersurface orthogonal vector field, that constitutes the general relativistic equivalent of a Newtonian rest frame. In this instance, these fields can be related to the inertial forces. When these fields are formulated with respect to the orbit under consideration, they lead to a striking similarity in the corresponding physical quantities that arise for a charge moving in an actual, constant electromagnetic field. We have thus established connections and correspondences among several interesting general relativistic phenomena. In the next chapter, we shall compute the gyroscopic precession and inertial forces in some of the black hole spacetimes.

Chapter 4

Application to Black Hole Spacetimes

4.1 Introduction

In the last two chapters we have presented the formalisms for the phenomena of gyroscopic precession and inertial forces. We have also established interrelations between them. In order to get further physical insight, we apply these formalisms to specific examples. These include static spacetimes - in which both gyroscopic precession and centrifugal force reversals occur at the null geodesics - and stationary spacetimes where this does not occur. The simultaneous reversal of gyroscopic precession and centrifugal force occurs in the Schwarzschild spacetime at $r = 3M$, where a circular null geodesic exists[1, 4, 57, 60]. We take the Ernst spacetime as a typical example for a static axially symmetric spacetime in order to illustrate the simultaneous reversal. In addition, the Schwarzschild spacetime and the Melvin universe can be treated as special cases of the Ernst spacetime. As has been shown in the last chapter neither centrifugal force nor gyroscopic precession reversal occurs at the circular null geodesics in stationary spacetimes. In this chapter we take the Kerr-Newman spacetime as an example of axially symmetric stationary spacetimes.

In this chapter we first study gyroscoic precession in static spacetimes using the Frenet-Serret formalism. In section 4.2.1 we give the general expressions for circular quasi-Killing trajectories. These results take a simple form on the equatorial plane. In section 4.2.1.2 we study the precession along a circular geodesic and show that in the limit $\kappa \rightarrow 0$, the Frenet-Serret parameter τ_1 is well defined. We may also note that the Frenet-Serret parameter $\tau_1^2 = \omega_{kepler}^{-2}$, where the ω_{kepler} is the Keplerian orbital frequency. In section 4.2.2 we study inertial forces in the Ernst spacetime. As we have seen earlier, the Euler force is zero for quasi-Killing trajectories and the Coriolis-Lense-Thirring force is zero in static spacetimes, so that we have only centrifugal force in addition to the gravitational force in static spacetimes. By setting the magnetic field parameter $B = 0$, we obtain the Schwarzschild solution as a special case of the Ernst spacetime. In section 4.2.3 we study gyroscopic precession and centrifugal force in the Schwarzschild spacetime. The Melvin universe can also be treated as a special case of the Ernst spacetime by setting the mass parameter $M = 0$. These are considered in section 4.2.4. One of the main motivations is to investigate the reversal of gyroscopic precession and centrifugal force in the Ernst spacetime. In section 4.2.5 we show that both gyroscopic precession and centrifugal force reversal occurs at the circular null geodesics in the Ernst spacetime as expected. Similar studies can be carried out in the case of the Schwarzschild spacetime and the Melvin universe as special cases.

In section 4.3 we take Kerr-Newman spacetime as an example of stationary axially symmetric spacetimes. One of the aims is to highlight the differences in the phenomenon of gyroscopic precession and inertial forces, as observed in a stationary spacetimei in constrast to a static one. The general expression for the Frenet-Serret parameters along circular orbit is given in section 4.3.1. These results are specialized to the case of equatorial plane and to circular geodesics in section 4.3.1.1 and 4.3.1.2. In the case of a stationary spacetime, we see that no direct relation between τ_1 for circular geodesics and Keplerian angular velocity exists as in the case of a static spacetime. In section 4.3.2 we compute inertial forces in the Kerr-Newman spacetime using the formalism given by Abramowicz, Nurowski and Wex[2]. In section 4.4 we study gyroscopic precession and inertial forces in the Kerr spacetime as a special case of the Kerr-Newman solution by setting the charge parameter $Q = 0$. Gyroscopic precession in the Kerr spacetime was studied by Iyer and Vishveshwara[43]. In section 4.4.1 we present the expression for gyroscopic precession as a special case of the Kerr-Newman solution. The expressions for inertial forces in the Kerr spacetime was given in section 4.4.2. Chakrabarti, Prasanna and Sai Iyer[45, 20] computed inertial forces in the Kerr spacetime using the formalism given in the reference[1], where the inertial forces are defined in the three-space orthogonal to the timelike Killing vector ξ^a. We utilize the covariant formalism given by Abramowicz, Nurowski and Wex, in which inertial forces are defined with respect to an irrotational congruence which is the generalization of a Newtonian global rest frame[35, 75]. By setting the angular momentum parameter $a = 0$ we obtain the Reissner-Nordstrom spacetime as a special case of the Kerr-Newman solution. In section 4.5 we study gyroscopic precession and inertial forces in the Reissner-Nordstrom spacetime. As the Reissner-Nordstrom spacetime is a static spacetime, we observer that the simultaneous reversal of gyroscopic precession and centrifugal force occurs at the circular photon orbits. In section 4.6 we study where the reversal of gyroscopic precession and centrifugal force in the Kerr-Newman spacetime occur. From the condition for reversal, one can clearly notice that the reversal of gyroscopic precession and centrifugal force occurs at different points in the spacetime.

4.2 Gyroscopic Precession and Inertial Forces in the Ernst Spacetime

In this section we compute the gyroscopic precession frequency and the inertial forces in the Ernst spacetime[30]. The Ernst spacetime is a typical example of static axially symmetric spacetime. The previous study by Prasanna indicated that the centrifugal force reversal occurs in the Ernst spacetime at the photon orbits. Here we show that gyroscopic precession reversal also occurs at the photon orbits. As has been shown in the last chapter the simultaneous reversal of gyroscopic precession and centrifugal force at the photon orbits is in fact a generic property of all static spacetimes. The Ernst spacetime allows two circular null geodesics and in this section we show that the reversal occurs at both the orbits.

The Ernst metric can be written in the form

$$ds^2 = \lambda^2\left(1-\frac{2M}{r}\right)dt^2 - \frac{\lambda^2}{\left(1-\frac{2M}{r}\right)}dr^2 - \lambda^2 r^2 d\theta^2 - \frac{r^2\sin^2\theta}{\lambda^2}d\phi^2 \tag{4.1}$$

with

$$\lambda = 1 + B^2 r^2 \sin^2\theta. \tag{4.2}$$

Here M and B are respectively the mass and the magnetic field in geometrized units. The Ernst spacetime represents a Schwarzschild black hole immersed in an axially symmetric magnetic field which becomes uniform asymptotically. When the magnetic field is zero($B = 0$), the solution reduces to the Schwarzschild spacetime as a special case. If the mass parameter $M = 0$, the Ernst solution becomes the Melvin universe, which is an empty universe with a constant magnetic field B. Next we compute the gyroscopic precession and inertial forces for a particle moving in a circular orbits around the black hole.

4.2.1 Gyroscopic Precession in the Ernst Spacetime

In this section we use the Frenet-Serret formalism as described in the second chapter in order to compute the gyroscopic precession. The Frenet-Serret parameters κ, τ_1 and τ_2 and the Frenet-Serret tetrad components are computed for the circular trajectories around the black hole. These circular orbits can be represented by the quasi-Killing congruence, with the four velocity,

$$u^a = e^{\psi}(\xi^a + \omega\eta^a) \tag{4.3}$$

here ω is a function of r and θ, and e^{ψ} is the normalizing factor as mentioned before.

The Frenet-Serret parameters for circular orbits can be written as follows,

$$\kappa^2 = \frac{\mathcal{K}_1}{\lambda^2 r^2 \mathcal{K}_2} \tag{4.4}$$

$$\tau_1^2 = \frac{\left(1-\frac{2M}{r}\right)\mathcal{K}_3}{\lambda^2\mathcal{K}_1\mathcal{K}_2}\omega^2\sin^2\theta \tag{4.5}$$

$$\tau_2^2 = \frac{M^2\mathcal{K}_4}{\lambda^4 r^2\mathcal{K}_1}\omega^2\cos^2\theta \tag{4.6}$$

where

$$\begin{aligned}
\mathcal{K}_1 &= \left(1-\frac{2M}{r}\right)r^2\left\{\frac{\lambda}{r^2}\left[M+B^2r^2(2r-3M)\sin^2\theta\right]\right.\\
&\quad\left. + \frac{\omega^2 r\sin^2\theta}{\lambda^2}\left[\frac{2B^2r^2\sin^2\theta}{\lambda}-1\right]\right\}^2\\
&+ r^4\cos^2\theta\sin^2\theta\left\{2B^2\lambda\left(1-\frac{2M}{r}\right)+\frac{\omega^2}{\lambda^2}\left[\frac{2B^2r^2\sin^2\theta}{\lambda}-1\right]\right\}^2\\
\mathcal{K}_2 &= \left[\lambda^2\left(1-\frac{2M}{r}\right)-\frac{\omega^2r^2}{\lambda^2}\sin^2\theta\right]^2\\
\mathcal{K}_3 &= \left[\left\{\frac{\lambda}{r^2}\left[M+B^2r^2\sin^2\theta(2r-3M)\right]\right.\right.\\
&+ \left.\frac{\omega^2 r\sin^2\theta}{\lambda^2}\left[\frac{2B^2r^2\sin^2\theta}{\lambda}-1\right]\right\}\\
&\cdot \left\{\left(1-\frac{2M}{r}\right)r\left[\frac{2B^2r^2\sin^2\theta}{\lambda}-1\right]\right.
\end{aligned} \tag{4.7}$$

$$
\begin{aligned}
&+ \frac{1}{\lambda}\left[M + B^2 r^2 \sin^2\theta(2r-3M)\right]\Big\} \\
&+ r^2\cos^2\theta\left\{2\left(1-\frac{2M}{r}\right)\lambda B^2 + \frac{\omega^2}{\lambda^2}\left[\frac{2B^2r^2\sin^2\theta}{\lambda}-1\right]\right\}\cdot \\
&\cdot \left\{\left[\frac{2B^2r^2\sin^2\theta}{\lambda}-1\right]+2B^2r^2\sin^2\theta\right\}\Big]^2 \\
\mathcal{K}_4 &= \left(2B^2r^2\sin^2\theta-1\right)^2\left(1+B^2r^2\sin^2\theta\right)^2
\end{aligned}
\tag{4.8}
$$

The components of the bases are given as

$$
\begin{aligned}
e^a_{(0)} &= \frac{1}{\sqrt{\mathcal{A}}}(1,0,0,\omega) \\
e^a_{(1)} &= -\frac{1}{2\kappa\mathcal{A}}(0, g^{11}\mathcal{A}_{(1)}, g^{22}\mathcal{A}_{(2)}, 0) \\
e^a_{(2)} &= \frac{1}{\sqrt{\mathcal{A}}\sqrt{-\Delta_3}}(\mathcal{B},0,0,-g_{00}) \\
e^a_{(3)} &= \frac{\sqrt{g^{11}g^{22}}}{2\kappa\mathcal{A}}(0,-\mathcal{A}_{(2)},\mathcal{A}_{(1)},0)
\end{aligned}
\tag{4.9}
$$

where

$$
\begin{aligned}
\mathcal{A} &= \lambda^2\left(1-\frac{2M}{r}\right)-\frac{\omega^2r^2}{\lambda^2}\sin^2\theta \\
\mathcal{A}_{(1)} &= \frac{2\lambda}{r^2}\left[M+B^2r^2\sin^2\theta(2r-3M)\right]+\frac{2\omega^2 r\sin^2\theta}{\lambda^2}\left[\frac{2B^2r^2\sin^2\theta}{\lambda}-1\right] \\
\mathcal{A}_{(2)} &= 2r^2\cos\theta\sin\theta\left\{2\left(1-\frac{2M}{r}\right)\lambda B^2+\frac{\omega^2}{\lambda^2}\left[\frac{2B^2r^2\sin^2\theta}{\lambda}-1\right]\right\} \\
\mathcal{B} &= -\frac{\omega r^2\sin^2\theta}{\lambda^2}
\end{aligned}
\tag{4.10}
$$

These results are general. However, considerable simplification occurs in the above expressions when we specialize to orbits in the equatorial plane and, further, to circular geodesics.

4.2.1.1 Equatorial Plane

The black hole spacetimes have reflection symmetry about the equatorial plane, which is represented by $\theta = \frac{\pi}{2}$. By setting $\theta = \pi/2$ in equations (4.4, 4.5 and 4.6) we obtain,

$$\begin{aligned}
\kappa^2 &= \frac{\left(1-\frac{2M}{r}\right)\left\{\frac{\lambda}{r^2}\left[M+B^2r^2(2r-3M)\right]+\frac{\omega^2 r}{\lambda^2}\left[\frac{2B^2r^2}{\lambda}-1\right]\right\}^2}{\lambda^2\left[\lambda^2\left(1-\frac{2M}{r}\right)-\frac{\omega^2r^2}{\lambda^2}\right]^2} \\
\tau_1^2 &= \frac{\left\{\left(1-\frac{2M}{r}\right)\left[\frac{2B^2r^2}{\lambda}-1\right]r+\frac{1}{\lambda}\left[M+B^2r^2(2r-3M)\right]\right\}^2}{\lambda^2r^2\left[\lambda^2\left(1-\frac{2M}{r}\right)-\frac{\omega^2r^2}{\lambda^2}\right]^2}\omega^2 \qquad (4.11) \\
\tau_2^2 &= 0
\end{aligned}$$

$$\begin{aligned}
e^a_{(0)} &= \frac{1}{\sqrt{\lambda^2\left(1-\frac{2M}{r}\right)-\frac{\omega^2r^2}{\lambda^2}}}(1,0,0,\omega) \\
e^a_{(1)} &= \sqrt{\left(1-\frac{2M}{r}\right)\frac{1}{\lambda^2}}(0,1,0,0) \\
e^a_{(2)} &= \frac{1}{r\sqrt{\left(1-\frac{2M}{r}\right)\left[\lambda^2\left(1-\frac{2M}{r}\right)-\frac{\omega^2r^2}{\lambda^2}\right]}}\left(-\frac{\omega r^2}{\lambda^2},0,0,-\lambda^2\left(1-\frac{2M}{r}\right)\right) \qquad (4.12) \\
e^a_{(3)} &= \frac{1}{\lambda r}(0,0,1,0)
\end{aligned}$$

One observes that $\tau_2 = 0$ on the equatorial plane. This is in fact true for any plane about which the spacetime has as reflection symmetry. Since $\tau_2 = 0$, the gyroscope precesses about $e_{(3)}$, *i.e.* about an axis orthogonal to the orbital plane at a rate given by τ_1.

4.2.1.2 Circular Geodesic Motion

In chapter 2 we have shown that the Frenet-Serret parameters κ represents the magnitude of the four acceleration of the particle. One can clearly see the fact that the Frenet-Serret equations are not well defined for geodesics as a single curve. However, one can define the geodesics as a member of a congruence for which the Frenet-Serret parameters are smooth functions on the spacetime. In such a case, geodesics are defined as a limiting case $\kappa \to 0$; also, the corresponding Frenet-Serret parameters τ_1 and τ_2 are well defined as it was shown in reference [43]. Here we study the circular geodesics in the Ernst spacetime. We consider the circular geodesics as a member of a quasi-Killing congruence with suitable ω. In such a case, circular geodesic motion can be obtained as the limiting case when κ vanishes. The Keplerian orbital frequency in this case is given by

$$\omega^2 = \frac{\lambda^4}{(1 - B^2r^2)\, r^3} \left[M + B^2r^2(2r - 3M)\right] \tag{4.13}$$

As a liming case, since the ratio $\frac{\mathcal{A}_{(2)}}{\kappa}$ goes to zero as $\kappa \to 0$ indicates that $\tau_2 = 0$ for geodesics and the ratio $\mathcal{A}_{(1)}\ /\ \kappa$ is well defined in this limit, one can determine τ_1 and it is given by

$$\tau_1^2 = \frac{(1 - B^2r^2)}{\lambda^4 r^3} \left[M + B^2r^2(2r - 3M)\right] \tag{4.14}$$

As has been discussed in reference [43], we can now compute the total angle of gyroscopic precession $\Delta\phi$ with respect to a fiducial direction fixed in space, when the gyroscope is transported along the orbit in one full revolution. This angle is

$$\begin{aligned} \Delta\phi &= \mp\tau_1 \frac{2\pi}{\omega}\sqrt{\mathcal{A}} \ + 2\pi \\ &= \mp 2\pi \left[\left(\frac{1 - B^2r^2}{\lambda^3}\right) \sqrt{\left(1 - \frac{2M}{r}\right) - \frac{[M + B^2r^2(2r - 3M)]}{r(1 - B^2r^2)}} - 1 \right] \end{aligned} \tag{4.15}$$

In the next section we compute the inertial forces for circular trajectories in the Ernst spacetime.

4.2.2 Inertial Forces in the Ernst Spacetime

Prasanna[60] first computed the centrifugal force in the Ernst spacetime in order to demonstrate the reversal of centrifugal force at photon orbits in the case of the

Schwarzschild spacetime. Here we give the general expressions for the centrifugal force and the gravitational forces for circular quasi-Killing trajectories in the Ernst spacetime. For the sake of brevity, we shall leave out the intermediate steps and give the final results. The gravitational force is given by

$$G_k = -\frac{1}{\lambda^2}\left(1-\frac{2M}{r}\right)^{-1}(0, g_1, g_2, 0) \tag{4.16}$$

where

$$\begin{aligned} g_1 &= \frac{\lambda}{r^2}\left[M + B^2r^2\sin^2\theta(2r-3M)\right] \\ g_2 &= 2\left(1-\frac{2M}{r}\right)\lambda B^2r^2\cos\theta\sin\theta \end{aligned} \tag{4.17}$$

Similarly the centrifugal force is given by

$$Z_k = \frac{\omega^2}{\mathcal{A}\lambda^2\left(1-\frac{2M}{r}\right)}(0, z_1, z_2, 0) \tag{4.18}$$

where

$$\begin{aligned} z_1 &= -\frac{\sin^2\theta}{\lambda}\left[(r-3M) - B^2r^2\sin^2\theta(3r-5M)\right] \\ z_2 &= \left(1-\frac{2M}{r}\right)\frac{r^2}{\lambda^2}\sin\theta\cos\theta\left(3B^2r^2\sin^2\theta - 1\right) \end{aligned} \tag{4.19}$$

On the equatorial plane ($\theta = \pi/2$) these expressions reduce to

$$\begin{aligned} g_1 &= \frac{\lambda}{r^2}\left[M + B^2r^2(2r-3M)\right] \\ g_2 &= 0 \end{aligned} \tag{4.20}$$

so that

$$G_k = -\left(1-\frac{2M}{r}\right)^{-1}\frac{1}{\lambda r^2}\left[M + B^2r^2(2r-3M)\right](0,1,0,0)\,. \tag{4.21}$$

Similarly,

$$\begin{aligned} z_1 &= -\frac{1}{\lambda}\left[(r-3M) - B^2r^2(3r-5M)\right] \\ z_2 &= 0 \end{aligned} \tag{4.22}$$

so that

$$Z_k = -\frac{\left[(r-3M) - B^2r^2(3r-5M)\right]\omega^2}{\mathcal{A}\lambda^3\left(1-\frac{2M}{r}\right)}(0,1,0,0) \qquad (4.23)$$

In the following sections we obtain the gyroscopic precession and the inertial forces in the Schwarzschild spacetime and in the Melvin universe as special cases of the Ernst spacetime.

4.2.3 Gyroscopic Precession and Inertial Forces in the Schwarzschild Spacetime

In this section we compute gyroscopic precession and inertial forces in the Schwarzschild spacetime as a special case of the Ernst spacetime, by setting parameter $B = 0$. The metric for the Schwarzschild spacetime can be written as,

$$ds^2 = \left(1-\frac{2M}{r}\right)dt^2 - \frac{1}{\left(1-\frac{2M}{r}\right)}dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2. \qquad (4.24)$$

Where M is the mass parameter.

The Frenet-Serret parameters κ, τ_1 and τ_2 for circular orbits can be deduced form equations (4.4, 4.5 and 4.6), by setting $B = 0$ or $\lambda = 1$.

$$\kappa^2 = \frac{\mathcal{K}_1}{r^2\mathcal{K}_2} \qquad (4.25)$$

$$\tau_1^2 = \frac{\left(1-\frac{2M}{r}\right)\mathcal{K}_3}{\mathcal{K}_1\mathcal{K}_2}\omega^2\sin^2\theta \qquad (4.26)$$

$$\tau_2^2 = \frac{M^2}{r^2\mathcal{K}_1}\omega^2\cos^2\theta \qquad (4.27)$$

where

$$\begin{aligned}\mathcal{K}_1 &= \left(1-\frac{2M}{r}\right)r^2\left[\frac{M}{r^2} - \omega^2 r\sin^2\theta\right]^2 \\ &+ \omega^4 r^4\cos^2\theta\sin^2\theta\end{aligned} \qquad (4.28)$$

$$\mathcal{K}_2 = \left[\left(1-\frac{2M}{r}\right)-\omega^2r^2\sin^2\theta\right]^2$$
$$\mathcal{K}_3 = \left[\left(\frac{M}{r^2}-\omega^2r\sin^2\theta\right)(3M-r)+r^2\omega^2\cos^2\theta\right]^2 \quad (4.29)$$

In the equatorial plane, the above equation can be simplified and written as,

$$\kappa^2 = \frac{r^2\left(1-\frac{M}{r}\right)\left(\frac{M}{r^3}-\omega^2\right)}{\left(1-\frac{2M}{r}-r^2\omega^2\right)^2} \quad (4.30)$$
$$\tau_1^2 = \omega^2\frac{\left(1-\frac{3M}{r}\right)^2}{\left(1-\frac{2M}{r}-r^2\omega^2\right)^2} \quad (4.31)$$
$$\tau_2^2 = 0. \quad (4.32)$$

The expressions for inertial forces in Schwarzschild spacetime can also be obtained as a special case of the Ernst spacetime and are given by,

gravitational force

$$G_k = -\left(1-\frac{2M}{r}\right)^{-1}(0,\frac{1}{r^2},0,0) \quad (4.33)$$

and the centrifugal force is given by

$$Z_k = \frac{\omega^2}{\mathcal{A}\left(1-\frac{2M}{r}\right)}(0,z_1,z_2,0) \quad (4.34)$$

where,

$$z_1 = -\sin^2\theta(r-3M) \quad (4.35)$$
$$z_2 = -\left(1-\frac{2M}{r}\right)r^2\sin\theta\ \cos\theta \quad (4.36)$$
$$\mathcal{A} = \left(1-\frac{2M}{r}-\omega^2r^2\sin^2\theta\right) \quad (4.37)$$

On the equatorial plane ($\theta=\pi/2$) the centrifugal force reduces to

$$Z_k = -\frac{(r-3M)\omega^2}{\left(1-\frac{2M}{r}-\omega^2r^2\right)\left(1-\frac{2M}{r}\right)}(0,1,0,0) \quad (4.38)$$

From the above equation it is clear that both gyroscopic precession and centrifugal force reversal occurs at $r=3M$. A more detailed study of reversal of gyroscopic precession and centrifugal force is given in section 4.2.5.

4.2.4 Gyroscopic Precession and Inertial Forces in the Melvin Universe

The Melvin universe may be treated as a special case of the Ernst spacetime by setting $M = 0$ and the spacetime metric is given by

$$ds^2 = \lambda^2 \left[dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{4.39}$$

with,

$$\lambda = 1 + B^2r^2\sin^2\theta \tag{4.40}$$

As mentioned earlier, the Melvin universe represents a flat spacetime with constant magnetic field.

All the relevant quantities can be read off from the formulae already given for the Ernst spacetime by setting the mass parameter $M = 0$. We note that, because of the inherent cylindrical symmetry, one finds $\tau_2 = 0$ for all values of the angle θ.

Specializing to the equatorial plane $\theta = \pi/2$, we have

$$\begin{aligned}\kappa^2 &= \frac{\left\{2\lambda B^2 r + \frac{\omega^2 r}{\lambda^2}\left(\frac{2B^2r^2}{\lambda} - 1\right)\right\}^2}{\lambda^2\left[\lambda^2 - \frac{\omega^2r^2}{\lambda^2}\right]^2} \\ \tau_1^2 &= \frac{\omega^2\left(3B^2r^2 - 1\right)^2}{\left[\lambda^4 - \omega^2r^2\right]^2} \\ \tau_2^2 &= 0\end{aligned} \tag{4.41}$$

Further, for geodesic motion with $\kappa = 0$ we get

$$\omega^2 = \frac{2B^2\lambda^4}{(1 - B^2r^2)} \tag{4.42}$$

and

$$\tau_1^2 = \frac{2B^2}{\lambda^4}\left(1 - B^2r^2\right) \tag{4.43}$$

The gyroscopic precession for a full orbital revolution turns out to be

$$\Delta\phi = \mp 2\pi\left[\frac{1}{\lambda^3}\sqrt{(1 - B^2r^2)(1 - 3B^2r^2)} - 1\right] \tag{4.44}$$

Gravitational and centrifugal forces in the Melvin Universe are found by setting, as before, $M = 0$. So,

$$\begin{aligned} G_k &= -\frac{2}{\lambda}B^2 r \sin\theta \,(0, 1, r\cos\theta, 0) \\ Z_k &= \frac{\omega^2 \sin\theta}{\lambda\left[\lambda^4 - \omega^2 r^2 \sin\theta\right]} \\ &\quad \left(0, \left[r - 3B^2 r^3 \sin^2\theta\right], r^2\cos\theta\left[2B^2 r^2 \sin^2\theta - 1\right], 0\right) \end{aligned} \tag{4.45}$$

On the equatorial plane, these reduce to

$$\begin{aligned} G_k &= -\frac{2B^2 r}{\lambda}(0, 1, 0, 0) \\ Z_k &= -\frac{\omega^2\left(r - 3B^2 r^3\right)}{\lambda\left(\lambda^4 - \omega^2 r^2\right)}(0, 1, 0, 0) \end{aligned} \tag{4.46}$$

In the next section we study the reversal of centrifugal force and gyroscopic precession in the Ernst spacetime.

4.2.5 Reversal of Centrifugal Force and Gyroscopic Precession in the Ernst Spacetime

One of the interesting results that emerged from the generalization of inertial forces to the general theory of relativity is the reversal of centrifugal force. In the last chapter we have proved the theorem on the simultaneous reversal of gyroscopic precession and centrifugal force in static spacetimes. In this section we shall demonstrate this interesting result in the case of the Ernst spacetime.

4.2.5.1 Centrifugal Force

In order to study the reversal we write the centrifugal force as follows,

$$Z_k = C_F \tilde{Z}_k, \tag{4.47}$$

where $\tilde{Z}_k$ is the unit vector in the conformal space with the metric $\tilde{h}_{ik}$. On the equatorial plane, to which earlier calculations by other authors have been confined to, this reduces to

$$\tilde{Z} = -\left(1 - \frac{2M}{r}\right)^{-1} (0, 1, 0, 0) \tag{4.48}$$

and

$$C_F = \frac{\frac{r}{\lambda^3}\left\{\left(1 - \frac{3M}{r}\right) - B^2 r^2 \left(3 - \frac{5M}{r}\right)\right\}\omega^2}{\left[\left(1 - \frac{2M}{r}\right)\lambda^2 - \frac{\omega^2 r^2}{\lambda^2}\right]} \tag{4.49}$$

This agrees with the expression derived by Prasanna in reference [60] making use of a formalism developed earlier than the one presented in [2] which we have followed. Reversal of centrifugal force in the Ernst spacetime has been discussed in detail in reference [60]. As in the case of the Schwarzschild spacetime, this reversal occurs where there is a circular photon orbit. In the Ernst spacetime, depending upon the value of BM, there can be one, two or no circular photon orbits. Accordingly, centrifugal force can also reverse at these circular null geodesics. The condition for the existence of such a null geodesic is given by

$$3(BM)^2 R^3 - 5(BM)^2 R^2 - R + 3 = 0, \tag{4.50}$$

where $R \equiv r/M$. It can be shown that this equation gives the location of centrifugal reversal as well.

By setting the parameter $B = 0$ we get the reversal condition for the Schwarzschild spacetime. The magnitude of centrifugal force takes the form,

$$C_F = \frac{(r - 3M)\omega^2}{\left[\left(1 - \frac{2m}{r}\right) - \omega^2 r^2\right]}. \tag{4.51}$$

Also, the condition for circular null geodesics is given by,

$$r = 3M \tag{4.52}$$

The reversal in the Melvin universe can also be studied by setting $M = 0$. We find that centrifugal force reverses in the Melvin Universe at $r = \frac{1}{\sqrt{3}B}$.

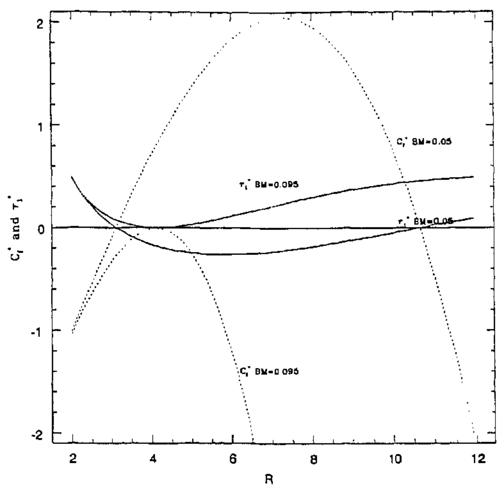

Figure 4.1: Plot of τ_1^* and C_f^* as functions of R for $BM = 0.05$ and 0.095 in the Ernst spacetime.

4.2.5.2 Gyroscopic Precession

Gyroscopic precession along the equatorial orbits in the Ernst spacetime is given by τ_1 of equation (4.23). The orbit at which the precession reverses sign can be located by equating τ_1 to zero. With some algebra it can be shown that this yields exactly the condition (4.50). Gyroscopic precession therefore reverses at the circular photon orbits as in the case of the centrifugal force. We note that this reversal is independent of the value of ω. Figure 4.1 shows plots of functions τ_1^* and C_F^* which are equivalent respectively to τ_1 and C_F with the ω dependence factored out. Thus,

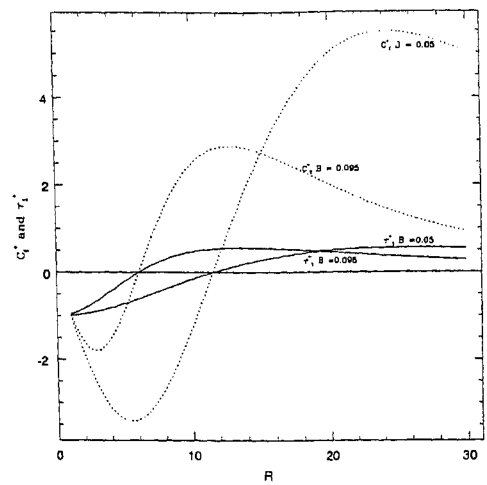

Figure 4.2: Plot of τ_1^* and C_f^* as functions of R for $B = 0.05$ and 0.095 In the Melvin universe.

$$
\begin{aligned}
\tau_1^* &= \alpha\, \tau_1 \\
&= \frac{1}{\lambda R}\left\{\left(1-\frac{2}{R}\right)\left(\frac{2B^2M^2R^2}{\lambda}-1\right)R-\frac{1}{\lambda}\left[1+B^2M^2R^2(2R-3)\right]\right\} \\
C_F^* &= (\omega M)^{-1}\alpha\, C_F \qquad (4.53) \\
&= \frac{R}{\lambda^3}\left\{\left(1-\frac{3}{R}\right)-B^2M^2R^2\left(3-\frac{5}{R}\right)\right\}
\end{aligned}
$$

where,

$$\alpha = \left(\omega\lambda^2\right)^{-1}\left[\lambda^4\left(1-\frac{2M}{r}\right)-\omega^2r^2\right] \qquad (4.54)$$

Again, in the special case of Melvin Universe ($M = 0$) the above formulae reduce to,

$$\begin{aligned} \tau_1^* &= \beta\tau_1 \\ &= (3B^2r^2 - 1) \\ C_F^* &= \omega^{-1}\beta C_F \\ &= \frac{r}{\lambda}(3B^2r^2 - 1) \end{aligned} \tag{4.55}$$

where,

$$\beta = \omega^{-1}\left(\lambda^4 - \omega^2 r^2\right) \tag{4.56}$$

Gyroscopic precession, as in the case of centrifugal force, reverses at $r = \frac{1}{\sqrt{3}B}$. Figure 4.2 shows examples of this phenomenon for some values of B.

So far in this section we have emphasized the simultaneous reversal of gyroscopic precession and centrifugal force in static spacetimes. As we have seen in the last chapter, the description becomes more complicated in the case of stationary spacetimes. Neither centrifugal force nor gyroscopic precession reversal occurs at the circular null geodesics. Also, there are two circular null geodesics corresponding to co-rotating and counter-rotating orbits. In the next section we study the phenomena of gyroscopic precession and inertial forces in stationary spacetimes, by taking specific examples such as the Kerr-Newman spacetime.

4.3 Gyroscopic Precession and Inertial Forces in the Kerr-Newman Spacetime

In the case of a stationary spacetime, the inherent rotation of the spacetime manifests as inertial frame dragging and plays an important role in the phenomena of gyroscopic precession and the general relativistic analogues of inertial forces. As has been shown in the last chapter, due to the effect of frame dragging a gyroscope transported along the stationary observers undergoes precession. Similarly, Coriolis-Lense-Thirring force is non-zero in a stationary spacetime, which gives rise to interesting results. In the last chapter we have seen that, in the case of stationary spacetimes, the gyroscopic precession is related to a combination of centrifugal and Coriolis-Lense-Thirring force. This leads to the fact that the simultaneous reversal of gyroscopic precession and centrifugal force does not occur in stationary spacetimes. In this section we study the gyroscopic precession and inertial forces for circular quasi-Killing trajectories in the Kerr-Newman spacetime as an example of stationary axially symmetric spacetimes.

The Kerr-Newman metric represents a charged Kerr solution. When the charge Q, is set to zero, we obtain the Kerr spacetime as a special case. The Reissner-Nordstrom solution is a special case of the Kerr-Newman solution when the angular momentum parameter a is zero.

The spacetime metric for the Kerr-Newman solution can be written in the form,

$$\begin{aligned} ds^2 &= \left(1-\frac{\mu}{\Sigma}\right)dt^2 + \frac{\mu a}{\Sigma}\sin^2\theta dt d\phi \\ &- \left(a^2+r^2+\frac{\mu a^2}{\Sigma}\sin^2\theta\right)\sin^2\theta d\phi^2 - \frac{\Sigma}{\Delta}dr^2 - \Sigma d\theta^2 \end{aligned} \tag{4.57}$$

with M and Q being mass and charge respectively, and

$$\begin{aligned} \mu &= 2Mr - Q^2 \\ \Delta &= r^2 - 2Mr + a^2 + Q^2 \\ \Sigma &= r^2 + a^2\cos^2\theta \end{aligned} \tag{4.58}$$

The black hole or the null surface in the case of Kerr-Newman solution is represented by the equation,

$$r^2 - 2Mr + a^2 + Q^2 = 0. \tag{4.59}$$

on which the irrotational congruence defined by the equation,

$$\zeta^a = \xi^a - \frac{\xi^b \eta_b}{\eta^c \eta_c} \eta^a,$$

becomes null. The stationary limit is defined by the surface on which the timelike Killing vector ξ^a becomes null, which is given by the condition,

$$r^2 - 2Mr + a^2 \cos^2\theta + Q^2 = 0. \tag{4.60}$$

Next we compute the gyroscopic precession and inertial forces in the Kerr-Newman spacetime.

4.3.1 Gyroscopic Precession

By making use of the formulae given in the last chapter we can compute the Frenet-Serret quantities for a given circular orbit of fixed but arbitrary values of r, θ and ω. Thus

$$\begin{aligned} \kappa^2 &= \frac{\mathcal{K}_1}{\Sigma \mathcal{K}_2} \\ \tau_1^2 &= \frac{\Delta \mathcal{K}_3}{\Sigma \mathcal{K}_1 \mathcal{K}_2} \sin^2\theta \\ \tau_2^2 &= \frac{\mathcal{K}_4}{\Sigma^5 \mathcal{K}_1} \cos^2\theta \end{aligned} \tag{4.61}$$

where

$$\begin{aligned} \mathcal{K}_1 &= \Delta \left[\frac{\lambda}{\Sigma^2} \left(1 - \omega a \sin^2\theta\right)^2 - \omega^2 r \sin^2\theta \right]^2 \\ &+ \sin^2\theta \cos^2\theta \left[\omega^2 \Delta + \frac{\mu}{\Sigma^2} \left\{ \left(r^2 + a^2\right)\omega - a \right\}^2 \right]^2 \end{aligned} \tag{4.62}$$

$$\mathcal{K}_2 = \left[1 - \omega^2 \sin^2\theta \left(r^2 + a^2\right) - \frac{\mu}{\Sigma} \left(1 - \omega a \sin^2\theta\right)^2 \right]^2 \tag{4.63}$$

$$\begin{aligned} \mathcal{K}_3 = &\left[\left\{ \frac{\lambda}{\Sigma^2} \left(1 - \omega a \sin^2\theta\right)^2 - \omega^2 r \sin^2\theta \right\} \cdot \right. \\ &\left\{ \omega r - \frac{\mu \omega r}{\Sigma} \left(1 - \omega a \sin^2\theta\right) - \frac{\lambda}{\Sigma^2} \left(1 - \omega a \sin^2\theta\right) \left[\left(r^2 + a^2\right)\omega - a \right] \right\} \end{aligned}$$

$$
\begin{aligned}
&+ \quad \cos^2\theta \left\{\omega^2\Delta + \frac{\mu}{\Sigma^2}\left[\left(r^2+a^2\right)\omega - a\right]^2\right\} \\
&\cdot \quad \left\{\frac{\mu a}{\Sigma^2}\left(1-\omega a \sin^2\theta\right)^2 - \omega\right\}\Bigg]^2 \qquad (4.64) \\
\mathcal{K}_4 &= \Big[\omega\mu a r \sin^2\theta \left\{\left(a^2+r^2\right)\omega - a\right\} - \lambda\omega\left(1-\omega a \sin^2\theta\right)\left(r^2+a^2\right) \\
&+ \quad \frac{\mu\lambda a}{\Sigma}\left(1-\omega a \sin^2\theta\right)^2\Big]^2 \qquad (4.65)
\end{aligned}
$$

and

$$\epsilon = r^2 - a^2\cos^2\theta, \qquad \lambda = M\epsilon - Q^2 r. \qquad (4.66)$$

The components of the bases are given by,

$$
\begin{aligned}
e^a_{(0)} &= \frac{1}{\sqrt{\mathcal{A}}}(1,0,0,\omega) \\
e^a_{(1)} &= -\frac{1}{2\kappa\mathcal{A}}(0, g^{11}\mathcal{A}_{(1)}, g^{22}\mathcal{A}_{(2)}, 0) \\
e^a_{(2)} &= \frac{1}{\sqrt{\mathcal{A}}\sqrt{-\Delta_3}}(\mathcal{B},0,0,-\mathcal{C}) \qquad (4.67) \\
e^a_{(3)} &= \frac{\sqrt{g^{11}g^{22}}}{2\kappa\mathcal{A}}(0, -\mathcal{A}_{(2)}, \mathcal{A}_{(1)}, 0)
\end{aligned}
$$

where $\mathcal{A}_{(1)}, \mathcal{A}_{(2)}$, $\mathcal{B}$ and $\mathcal{C}$ given as

$$
\begin{aligned}
\mathcal{A} &= 1 - \omega^2\sin^2\theta\left(r^2+a^2\right) - \frac{\mu}{\Sigma}\left(1-\omega a\sin^2\theta\right)^2 \\
\mathcal{A}_{(1)} &= 2\frac{\lambda}{\Sigma^2}\left(1-\omega a \sin^2\theta\right)^2 - 2\omega^2 r\sin^2\theta \\
\mathcal{A}_{(2)} &= -2\cos\theta\sin\theta\left[\omega^2\Delta + \frac{\mu}{\Sigma^2}\left\{\left(r^2+a^2\right)\omega - a\right\}^2\right] \qquad (4.68) \\
\mathcal{B} &= \frac{\mu a\sin^2\theta}{\Sigma}\left(1-\omega a\sin^2\theta\right) - \omega\sin^2\theta\left(r^2+a^2\right) \\
\mathcal{C} &= 1 - \frac{\mu}{\Sigma}\left(1-\omega a\sin^2\theta\right)
\end{aligned}
$$

The above formulae simplify when we specialize to trajectories lying in the equatorial plane, especially, to geodesic orbits.

4.3.1.1 Equatorial Plane

We take $\theta = \pi/2$ so that the Frenet-Serret scalar simplifies to,

$$\kappa^2 = \frac{\Delta}{r^2}\frac{\left\{\frac{1}{r^3}(Mr-Q^2)(1-\omega a)^2-\omega^2 r\right\}^2}{\left[1-\omega^2(r^2+a^2)-\left(\frac{2Mr-Q^2}{r^2}\right)(1-\omega a)^2\right]^2} \quad (4.69)$$

$$\tau_1^2 = \frac{\left\{\omega r-\left(\frac{2Mr-Q^2}{r}\right)(1-\omega a)\omega-\left(\frac{Mr-Q^2}{r^3}\right)(1-\omega a)\left\{(r^2+a^2)\omega-a\right\}\right\}^2}{r^2\left\{1-\omega^2(r^2+a^2)-\left(\frac{2Mr-Q^2}{r^2}\right)(1-\omega a)^2\right\}^2} \quad (4.70)$$

$$\tau_2^2 = 0 \quad (4.71)$$

The base components can be written as,

$$e^a_{(0)} = \frac{1}{\sqrt{\left[1-\omega^2(r^2+a^2)-\left(\frac{2Mr-Q^2}{r^2}\right)(1-\omega a)^2\right]}}(1,0,0,\omega)$$

$$e^a_{(1)} = \sqrt{\frac{\Delta}{r^2}}(0,1,0,0)$$

$$e^a_{(2)} = \frac{\left(\left[\left(\frac{2Mr-Q^2}{r^2}\right)(1-\omega a)a-\omega(r^2+a^2)\right],0,0-\left[1-\left(\frac{2Mr-Q^2}{r^2}\right)(1-\omega a)\right]\right)}{\sqrt{\left\{\Delta\left[1-\omega^2(a^2+r^2)-\left(\frac{2Mr-Q^2}{r^2}\right)(1-\omega a)^2\right]\right\}}}$$

$$e^a_{(3)} = \frac{1}{r}(0,0,1,0) \quad (4.72)$$

Since $\tau_2 = 0$, the gyroscope precesses about the vector $e_{(3)}$, that is, about the axis orthogonal to the orbital plane, at a rate given by τ_1.

4.3.1.2 Circular Geodesics

The special case of circular geodesic motion results in the limit of vanishing κ. The geodetic orbital frequency ω is given by,

$$\omega^{-1} = a \pm \sqrt{\frac{r^4}{Mr-Q^2}} \quad (4.73)$$

The precession rate τ_1, which is finite in this limit, is,

$$\tau_1^2 = \frac{Mr-Q^2}{r^4} \quad (4.74)$$

As has been outlined in reference[43] the total angle of gyroscopic precession $\Delta\phi$ can be computed relative to a fiducial direction fixed in space when the gyroscope is transported around the orbit in one full circle. This angle is given by

$$\begin{aligned}
\Delta\phi &= \mp r_1 \frac{2\pi}{\omega}\sqrt{\mathcal{A}} \;\; + 2\pi \\
&= \mp 2\pi \left[\left\{ 1 \pm 2a\sqrt{\frac{Mr - Q^2}{r^4}} - \left(3Mr - 2Q^2\right)\frac{1}{r^2} \right\}^{\frac{1}{2}} - 1 \right]
\end{aligned} \tag{4.75}$$

4.3.2 Inertial Forces in the Kerr-Newman Spacetime

The formalism developed in reference [2] and summarized in the previous subsection can be applied to the Kerr-Newman spacetime in a straightforward manner. Results pertaining to the Kerr and the Reissner-Nordstrom spacetimes may be deduced by setting $Q = 0$ and $a = 0$ respectively. The forces in the Kerr-Newman spacetime for circular orbits with fixed but arbitrary values of r, θ and ω are as follows.

Gravitational force

$$G_k = -\frac{1}{\Delta}\left(0, g_1, g_2, 0\right) \tag{4.76}$$

where,

$$\begin{aligned}
g_1 &= \left[(r - M) - \frac{\Delta\left\{r - \frac{\lambda}{\Sigma^2}a^2\sin^2\theta\right\}}{\left\{(r^2 + a^2) + \frac{\mu}{\Sigma}a^2\sin^2\theta\right\}} \right] \\
g_2 &= -\frac{\mu\Delta a^2\left(r^2 + a^2\right)\sin\theta\cos\theta}{\Sigma\left\{\left(r^2 + a^2\right)\Sigma + \mu a\sin^2\theta\right\}}
\end{aligned} \tag{4.77}$$

Coriolis-Lense-Thirring force

$$C_k = -\frac{\mathcal{W}}{\mathcal{A}\,\mathcal{G}_3}\left(0, c_1, c_2, 0\right) \tag{4.78}$$

where,

$$\mathcal{W} = \omega - \frac{\mu a}{\left[\Sigma\left(r^2 + a^2\right) + \mu a^2\sin^2\theta\right]}$$

$$\begin{aligned}
\mathcal{G}_3 &= \left(r^2+a^2\right)+\frac{\mu}{\Sigma}a^2\sin^2\theta \qquad (4.79)\\
c_1 &= 2\left[\frac{\lambda}{\Sigma^2}\left(r^2+a^2\right)+\frac{\mu r}{\Sigma}\right]a\sin^2\theta\\
c_2 &= -2\frac{\Delta\mu}{\Sigma^2}a^3\sin^3\theta\cos\theta
\end{aligned}$$

Centrifugal force

$$Z_k=\frac{\mathcal{W}^2}{\mathcal{A}}\left(0,z_1,z_2,0\right) \qquad (4.80)$$

where,

$$\begin{aligned}
z_1 &= \frac{\sin^2\theta}{\Delta}\left[(r-M)\left\{\left(r^2+a^2\right)+\frac{\mu}{\Sigma}a^2\sin^2\theta\right\}-2\Delta\left\{r-\frac{\lambda}{\Sigma^2}a^2\sin^2\theta\right\}\right]\\
z_2 &= -\sin\theta\cos\theta\left\{\left(r^2+a^2\right)+\frac{\mu}{\Sigma^2}a^2\sin^2\theta\left[2\left(r^2+a^2\right)+\Sigma\right]\right\} \qquad (4.81)
\end{aligned}$$

As usual, on the equatorial plane ($\theta=\pi/2$) these expressions simplify to

$$G_k = \frac{(r-M)\left\{(r^2+a^2)r^2+(2Mr-Q^2)a^2\right\}-\frac{\Delta}{r}\left\{r^4-(Mr-Q^2)a^2\right\}}{\Delta\left\{(r^2+a^2)r^2+(2Mr-Q^2)a^2\right\}}(0,-1,0,0) \qquad (4.82)$$

$$C_k = \frac{2a\mathcal{W}}{\mathcal{A}\,\mathcal{G}_3}\left\{\left(\frac{Mr-Q^2}{r^3}\right)a^2+\left(\frac{3Mr-Q^2}{r}\right)\right\}(0,-1,0,0) \qquad (4.83)$$

$$Z_k = \frac{\mathcal{W}^2}{\mathcal{A}}\left(0,z_1,0,0\right) \qquad (4.84)$$

where,

$$z_1=\frac{1}{\Delta r^3}\left[(r-M)r\left\{\left(r^2+a^2\right)r^2+\left(2Mr-Q^2\right)a^2\right\}-2\Delta\left\{r^4-\left(Mr-Q^2\right)a^2\right\}\right] \qquad (4.85)$$

As mentioned earlier, by setting the charge parameter Q to zero in the Kerr-Newman solution, one obtains the Kerr solution. In the next section we obtain the gyroscopic precession and inertial forces in the Kerr spacetime by setting the parameter $Q=0$ in the above results.

4.4 Gyroscopic Precession and Inertial Forces in the Kerr Spacetime

The Kerr metric is one of the most important solutions to the Einstein equations from the astrophysical point of view[47]. It represents a vacuum solution with a rotating black hole. The spacetime metric for the Kerr solution in Boyer-Lindquist form can be written as[18],

$$\begin{aligned} ds^2 &= \left(1-\frac{2Mr}{\Sigma}\right)dt^2 + \frac{2Mra}{\Sigma}\sin^2\theta dt d\phi \\ &- \left(a^2+r^2+\frac{2Mra^2}{\Sigma}\sin^2\theta\right)\sin^2\theta d\phi^2 - \frac{\Sigma}{\Delta}dr^2 - \Sigma d\theta^2 \end{aligned} \tag{4.86}$$

with M and a being mass and angular momentum parameter respectively, and

$$\begin{aligned} \Delta &= r^2 - 2Mr + a^2 \\ \Sigma &= r^2 + a^2\cos^2\theta \end{aligned} \tag{4.87}$$

The gyroscopic precession in the Kerr spacetime was studied in detail by Iyer and Vishveshwara[43]. Here we obtain these result as a special case of the Kerr-Newman spacetime by setting the parameter $Q = 0$. We also study the inertial forces in the Kerr spacetime using the formalism given in reference[2].

4.4.1 Gyroscopic Precession

By setting the parameter $Q = 0$ one gets the values of κ, τ_1 and τ_2 for the Kerr solution from the Kerr-Newman case which can be written as,

$$\begin{aligned} \kappa^2 &= \frac{\mathcal{K}_1}{\Sigma\mathcal{K}_2} \\ \tau_1^2 &= \frac{\Delta\mathcal{K}_3}{\Sigma\mathcal{K}_1\mathcal{K}_2}\sin^2\theta \\ \tau_2^2 &= \frac{\mathcal{K}_4}{\Sigma^5\mathcal{K}_1}\cos^2\theta \end{aligned} \tag{4.88}$$

where

$$
\begin{aligned}
\mathcal{K}_1 &= \Delta\left[\frac{\lambda}{\Sigma^2}\left(1-\omega a\sin^2\theta\right)^2-\omega^2 r\sin^2\theta\right]^2 \\
&+ \sin^2\theta\cos^2\theta\left[\omega^2\Delta+\frac{2Mr}{\Sigma^2}\left\{\left(r^2+a^2\right)\omega-a\right\}^2\right]^2 &(4.89)\\
\mathcal{K}_2 &= \left[1-\omega^2\sin^2\theta\left(r^2+a^2\right)-\frac{2Mr}{\Sigma}\left(1-\omega a\sin^2\theta\right)^2\right]^2 &(4.90)\\
\mathcal{K}_3 &= \left[\left\{\frac{\lambda}{\Sigma^2}\left(1-\omega a\sin^2\theta\right)^2-\omega^2 r\sin^2\theta\right\}\cdot\right.\\
&\cdot \left\{\omega r-\frac{2M\omega r^2}{\Sigma}\left(1-\omega a\sin^2\theta\right)-\frac{\lambda}{\Sigma^2}\left(1-\omega a\sin^2\theta\right)\left[\left(r^2+a^2\right)\omega-a\right]\right\}\\
&+ \cos^2\theta\left\{\omega^2\Delta+\frac{2Mr}{\Sigma^2}\left[\left(r^2+a^2\right)\omega-a\right]^2\right\}\\
&\cdot \left.\left\{\frac{2Mra}{\Sigma^2}\left(1-\omega a\sin^2\theta\right)^2-\omega\right\}\right]^2 &(4.91)\\
\mathcal{K}_4 &= \left[2M\omega a r^2\sin^2\theta\left\{\left(a^2+r^2\right)\omega-a\right\}-\lambda\omega\left(1-\omega a\sin^2\theta\right)\left(r^2+a^2\right)\right.\\
&+ \left.\frac{2Mr\lambda a}{\Sigma}\left(1-\omega a\sin^2\theta\right)^2\right]^2
\end{aligned}
$$

and

$$\epsilon=r^2-a^2\cos^2\theta,\qquad \lambda=M\epsilon. \tag{4.92}$$

The components of the bases are given by,

$$
\begin{aligned}
e^a_{(0)} &= \frac{1}{\sqrt{\mathcal{A}}}(1,0,0,\omega)\\
e^a_{(1)} &= -\frac{1}{2\kappa\mathcal{A}}(0,g^{11}\mathcal{A}_{(1)},g^{22}\mathcal{A}_{(2)},0)\\
e^a_{(2)} &= \frac{1}{\sqrt{\mathcal{A}}\sqrt{-\Delta_3}}(\mathcal{B},0,0,-\mathcal{C}) &(4.93)\\
e^a_{(3)} &= \frac{\sqrt{g^{11}g^{22}}}{2\kappa\mathcal{A}}(0,-\mathcal{A}_{(2)},\mathcal{A}_{(1)},0)
\end{aligned}
$$

where $\mathcal{A}_{(1)},\mathcal{A}_{(2)}$, $\mathcal{B}$ and $\mathcal{C}$ given as

$$\mathcal{A} = 1-\omega^2\sin^2\theta\left(r^2+a^2\right)-\frac{2Mr}{\Sigma}\left(1-\omega a\sin^2\theta\right)^2$$

$$\begin{aligned}
\mathcal{A}_{(1)} &= 2\frac{\lambda}{\Sigma^2}\left(1-\omega a\sin^2\theta\right)^2 - 2\omega^2 r\sin^2\theta \\
\mathcal{A}_{(2)} &= -2\cos\theta\sin\theta\left[\omega^2\Delta + \frac{2Mr}{\Sigma^2}\left\{\left(r^2+a^2\right)\omega - a\right\}^2\right] \\
\mathcal{B} &= \frac{2Mar\sin^2\theta}{\Sigma}\left(1-\omega a\sin^2\theta\right) - \omega\sin^2\theta\left(r^2+a^2\right) \\
\mathcal{C} &= 1-\frac{2Mr}{\Sigma}\left(1-\omega a\sin^2\theta\right)
\end{aligned} \tag{4.94}$$

As usual, the above formulae assume simpler forms when we specialize to trajectories lying in the equatorial plane, especially, to geodesic orbits.

4.4.1.1 Equatorial Plane

We take $\theta = \pi/2$ so that

$$\kappa^2 = \frac{\Delta M^2}{r^6}\frac{\left\{(a\omega-1)^2 - \frac{r^3\omega^2}{M}\right\}^2}{\left\{1-\omega^2\left(r^2+a^2\right) - \frac{2M}{r}\left(1-\omega a\right)^2\right\}^2} \tag{4.95}$$

$$\tau_1^2 = \frac{\left\{\omega r - 2M\left(1-\omega a\right)\omega - \left(\frac{M}{r^2}\right)\left(1-\omega a\right)\left\{\left(r^2+a^2\right)\omega - a\right\}\right\}^2}{r^2\left\{1-\omega^2\left(r^2+a^2\right) - \left(\frac{2M}{r}\right)\left(1-\omega a\right)^2\right\}^2} \tag{4.96}$$

$$\begin{aligned}
\tau_2^2 &= 0 \\
e^a_{(0)} &= \frac{1}{\sqrt{\left[1-\omega^2\left(r^2+a^2\right) - \left(\frac{2M}{r}\right)\left(1-\omega a\right)^2\right]}}\left(1,0,0,\omega\right) \\
e^a_{(1)} &= \sqrt{\frac{\Delta}{r^2}}\left(0,1,0,0\right) \\
e^a_{(2)} &= \frac{\left(\left[\left(\frac{2M}{r}\right)\left(1-\omega a\right)a - \omega\left(r^2+a^2\right)\right],0,0-\left[1-\left(\frac{2M}{r}\right)\left(1-\omega a\right)\right]\right)}{\sqrt{\left\{\Delta\left[1-\omega^2\left(a^2+r^2\right) - \left(\frac{2M}{r}\right)\left(1-\omega a\right)^2\right]\right\}}} \\
e^a_{(3)} &= \frac{1}{r}\left(0,0,1,0\right)
\end{aligned} \tag{4.97}$$

As before, since $\tau_2 = 0$, the gyroscope precesses about the vector $e_{(3)}$, that is, about the axis orthogonal to the orbital plane, at a rate given by τ_1.

4.4.1.2 Circular Geodesics

The special case of circular geodesic motion results in the limit of vanishing κ. The geodetic orbital frequency ω is given by

$$\omega^{-1} = a \pm \sqrt{\frac{r^3}{M}} \tag{4.98}$$

The precession rate τ_1, which is finite in this limit, is

$$\tau_1^2 = \frac{M}{r^3} \tag{4.99}$$

As has been outlined in reference[43], the total angle of gyroscopic precession $\Delta\phi$ can be computed relative to a fiducial direction fixed in space, when the gyroscope is transported around the orbit in one full circle. This angle is given by

$$\begin{aligned} \Delta\phi &= \mp\tau_1 \frac{2\pi}{\omega}\sqrt{\mathcal{A}} \ + 2\pi \\ &= \mp 2\pi \left[\left(1 - \frac{3M}{r} \pm 2a\sqrt{\frac{M}{r^3}}\right)^{\frac{1}{2}} - 1\right] \end{aligned} \tag{4.100}$$

4.4.2 Inertial Forces in the Kerr Spacetime

Forces in the Kerr spacetime can be read off from the formulae given for the Kerr-Newman metric by setting $Q = 0$. For the sake of simplicity we shall consider only the equatorial orbits ($\theta = \pi/2$). Chakrabarti, Prasanna and Sai Iyer have discussed centrifugal force in the Kerr spacetime [45, 20]. They have made use of an earlier formalism developed by Abramowicz, Carter and Lasota [1], which considers the forces in the quotient space orthogonal to the timelike Killing vector ξ^a. On the other hand the formalism of reference[2], which we are employing, defines the quotient space orthogonal to the irrotational vector field n^i. The advantage of the latter formalism is that n^i is timelike all the way down to the event horizon and the n^i congruence, being globally hypersurface orthogonal, defines observers who measure and synchronize the global time t. They are the general relativistic analogues of the Newtonian rest observers [35]. Thus in the present formalism the forces in the Kerr spacetime for equatorial orbits are given by

$$
\begin{aligned}
G_k &= \frac{(r-M)\left\{(r^2+a^2)\,r+2Ma^2\right\}-\frac{\Delta}{r}\left\{r^3-Ma^2\right\}}{\Delta\left\{(r^2+a^2)\,r+2Ma^2\right\}}(0,-1,0,0) && (4.101)\\
C_k &= \frac{2a\mathcal{W}}{\mathcal{A}\,\mathcal{G}_3}\left\{\frac{M}{r^2}a^2+3M\right\}(0,-1,0,0) && (4.102)\\
Z_k &= \frac{\mathcal{W}^2}{\mathcal{A}}(0,z_1,0,0) && (4.103)
\end{aligned}
$$

where

$$
\begin{aligned}
\Delta &\equiv r^2+a^2-2Mr\\
\mathcal{A} &= 1-\omega^2(r^2+a^2)-\frac{2M}{r}(1-\omega a)^2\\
\mathcal{G}_3 &= (r^2+a^2)+\frac{2M}{r}a^2 && (4.104)\\
\mathcal{W} &= \omega-\frac{2Ma}{(r^2+a^2)r+2Ma^2}\\
z_1 &= \frac{1}{\Delta r^2}\left[(r-M)\left\{(r^2+a^2)r^2+2Mra^2\right\}-2\Delta\left\{r^3-Ma^2\right\}\right]
\end{aligned}
$$

Another important special case of the Kerr-Newman solution is the Reissner-Nordstrom spacetime. In the next section we present the expression for gyroscopic precession and inertial forces in the Reissner-Nordstrom spacetime as a special case of the Kerr-Newman spacetime.

4.5 Gyroscopic Precession and Inertial Forces in the Reissner-Nordstrom Spacetime

The Reissner-Nordstrom solution is a spherically symmetric static solution with a charge Q. It reduces to the Schwarzschild spacetime when the charge is zero. It is a special case of the Kerr-Newman solution, where $a \to 0$. The Reissner-Nordstrom metric can be written as,

$$ds^2 = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 - \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 - r^2 \left(dr^2 + \sin^2\theta d\phi^2\right) \quad (4.105)$$

Here, we study the gyroscopic precession and the inertial forces in the Reissner-Nordstrom spacetime.

4.5.1 Gyroscopic Precession in the Reissner-Nordstrom Spacetime

Gyroscopic precession in the Reissner-Nordstrom spacetime can be deduced directly by setting $a = 0$ in the result derived in the case of the Kerr-Newman metric. For the sake of brevity we shall confine ourselves to the equatorial plane and specialize to geodesic circular orbits. In the equatorial plane, $\theta = \pi/2$, we have

$$\begin{aligned} \kappa^2 &= \frac{r^2\left[\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)\left\{\frac{1}{r^3}\left(M - \frac{Q^2}{r}\right) - \omega^2\right\}^2\right]}{\left[1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \omega^2 r^2\right]^2} \\ \tau_1^2 &= \frac{\left(1 - \frac{3M}{r} + \frac{2Q^2}{r^2}\right)^2}{\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \omega^2 r^2\right)^2}\omega^2 \\ \tau_2^2 &= 0 \end{aligned} \quad (4.106)$$

Further setting $\kappa = 0$ for geodesic motion we get the Keplerian frequency

$$\omega = \sqrt{\left(M - \frac{Q^2}{r}\right)\frac{1}{r^3}} \quad (4.107)$$

In this limit the gyroscopic precession rate about the axis perpendicular to the orbital plane simplifies to

$$\tau_1 = \omega \quad (4.108)$$

the Keplerian orbital frequency. The gyroscopic precession angle for a complete orbital motion is then

$$\Delta\phi = -2\pi\left[\left(1 - \frac{3M}{r} + \frac{2Q^2}{r^2}\right)^{\frac{1}{2}} - 1\right] \tag{4.109}$$

4.5.2 Inertial Forces in the Reissner-Nordstrom Spacetime

When the Kerr-Newman angular momentum parameter a is made zero, we get the results for the Reissner-Nordstrom spacetime. In a static spacetime the Coriolis force reduces identically to zero so that we are left with only the gravitational and the centrifugal forces. Once again, we confine ourselves to the equatorial orbits and write down these forces:

$$G_k = \frac{(r-m)r - \Delta}{\Delta r}(0, -1, 0, 0) \tag{4.110}$$

$$Z_k = \frac{\omega^2}{1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \omega^2 r^2}\left(0, \frac{r}{\Delta}\left[-r^2 + 3Mr + 2Q^2\right], 0, 0\right) \tag{4.111}$$

In the next section we study the reversal of gyroscopic precession and centrifugal force in the Kerr-Newman spacetimes.

4.6 Reversal of Gyroscopic Precession and Centrifugal Force in the Kerr-Newman Spacetime

As has been shown in the last chapter, in stationary spacetimes the simultaneous reversal of gyroscopic precession and centrifugal force does not takes place. This is because of the fact that the gyroscopic precession depends on both centrifugal force and Coriolis-Lense-Thirring force. In this section we study these phenomena in the Kerr-Newman spacetime as a typical example of stationary axially symmetric spacetimes.

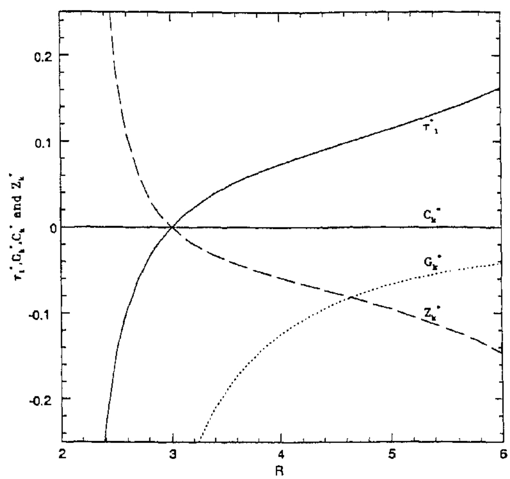

Figure 4.3: Plot of τ_1, C_k, G_k and Z_k as functions of R in the Reissner-Nordstrom spacetime.

In the case of the equatorial orbits, gyroscopic precession reverses when τ_1 changes its sign. This orbit can be located by the condition $\tau_1 = 0$. From equation (4.70) for τ_1^2 we get then

$$\omega r^4 - (2Mr - Q^2)(1 - \omega a)\omega r^2 - (Mr - Q^2)(1 - \omega a)\left\{(r^2 + a^2)\omega - a\right\} = 0 \quad (4.112)$$

The roots of this equation depend on the value of ω. This is so in the case of Kerr spacetime as well when $Q = 0$. In the case of Reissner-Nordstrom metric ($a = 0$) the root is independent of ω. From equation(4.84) we may locate the orbit for which the centrifugal force reverses by the condition $C_k = 0$. This leads to an equation which

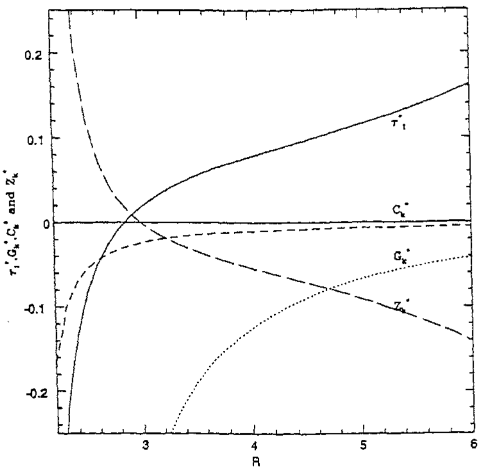

Figure 4.4: Plot of τ_1, C_k, G_k and Z_k as functions of R in the Kerr spacetime for the angular momentum parameter $a = 0.1$ and $\omega = 0.1$.

is independent of ω:

$$(r - M)r\left\{(r^2 + a^2)r^2 + (2Mr - Q^2)a\right\} - 2\Delta\left\{r^4 - (Mr - Q^2)a^2\right\} = 0 \qquad (4.113)$$

Thus, whereas the centrifugal force reversal is independent of ω, reversal of gyroscopic precession is not. The orbits where these two reversals occur do not coincide in general. Nevertheless, in the case of Reissner-Nordstrom metric both gyroscopic precession and centrifugal force reverse at the same location, namely at the circular photon orbit, as they should in an axisymmetric static spacetime[57]. This orbit

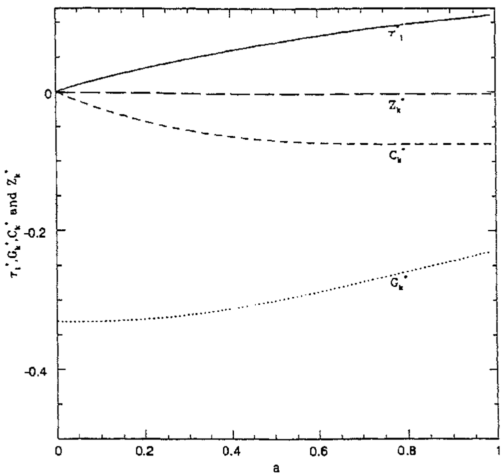

Figure 4.5: Variation of τ_1, C_k, G_k and Z_k as functions of angular momentum parameter a in the Kerr-Newman spacetime with $R = 3M$, $Q = 0.1$ and $\omega = 0.1$.

occurs at

$$r = \frac{3M \pm \sqrt{9M^2 - 8Q^2}}{2} \tag{4.114}$$

The above considerations are reflected in the plots of τ_1 and C_k in the figures(4.3-4.6).

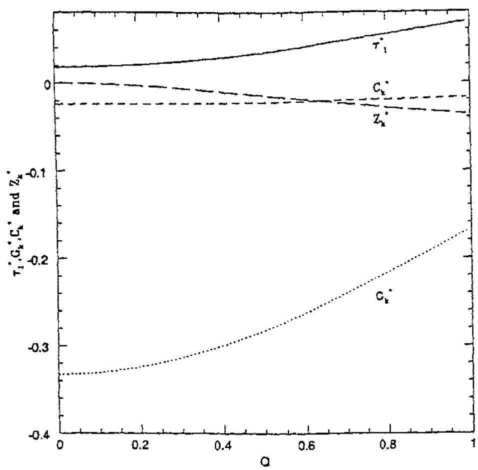

Figure 4.6: Variation of τ_1, C_k, G_k and Z_k as functions of angular charge Q in the Kerr-Newman spacetime with $R = 3M$, $a = 0.1$ and $\omega = 0.1$.

4.7 Conclusions

Based on the detailed derivations of reference [57] we have discussed gyroscopic precession in the Ernst spacetime for circular orbits. We have also considered the general relativistic equivalent of centrifugal force as defined in reference [2]. These results are quite general in the sense that orbits need not be confined to the equatorial plane and the angular speed of the orbiting particle is arbitrary. By setting the mass parameter of the Ernst spacetime equal to zero, the Melvin universe can also be treated as a special case. Substantial simplification occurs if the orbits are taken to be in the equatorial plane and even more so if they are geodesics. Centrifugal force in the Ernst spacetime has been studied in reference [60] utilizing an earlier formalism. Our results agree with those presented in [60]. The main interest of [60] was the centrifugal force reversal at the photon orbits. In the context of the Schwarzschild spacetime, it had been argued on qualitative grounds that gyroscopic precession must also reverse at the photon orbits. Quantitative calculations have borne out this conclusion. Once again, in the Ernst spacetime we show explicitly that gyroscopic precession, just like centrifugal force, reverses at the photon orbits. We may note that, since such null geodesics are confined to the equatorial plane, reversal cannot occur elsewhere. This fact can easily be ascertained from the formulae we have derived for the orbits that are off the equatorial plane. Depending on the magnitude of the product of the magnetic field and the mass parameter, there can be one, two or no photon orbits in the Ernst spacetime. Gyroscopic precession and centrifugal force have been plotted in some typical cases. The Melvin universe, a special case of the Ernst spacetime, admits a single photon orbit the location of which is inversely proportional to the magnetic field. Reversal of the two effects have been considered in this case as well.

We have derived detailed formulae describing gyroscopic precession along the circular orbits in the Kerr-Newman spacetime. The Kerr and the Reissner-Nordstrom metrics have been treated as special cases by setting Q and a equal to zero respectively. These formulae simplify significantly for orbits in the equatorial plane, especially when they are geodesics. We have also considered the general relativistic equivalents of inertial forces as defined in reference[2]. They are with reference to observers following a timelike globally hypersurface orthogonal congruence. As has been pointed out in reference [35], they form the general relativistic analogues of Newtonian rest observers. Further, as has been discussed in reference [43], the connecting vector of this congruence undergoes Fermi-Walker transport and hence is locked on to a gyroscope. Consequently, it is natural to study the precession of gyroscopes as seen by a Frenet-Serret frame using such a congruence. The dependence of gyroscopic precession and the inertial forces on a and Q has also been studied. As has been discussed in reference[57], in static spacetimes, *e.g.* Reissner-Nordstrom, only gravitational and centrifugal forces exist. Both gyroscopic precession and centrifugal force reverse at the photon orbits simultaneously. In the case of stationary metrics, *e.g.* Kerr-Newman, the situation is more complicated. Coriolis force is also present now. Whereas in the case of these two inertial forces, ω stands as an overall multiplicative factor, it enters into the precession formulae in a more complicated way. Therefore, the reversal of centrifugal force is independent of ω but it is not so in the case of gyroscopic precession.

Chapter 5

Inertial Forces and Einstein's Equations

5.1 Introduction

In the last chapter we have established direct relations between the phenomena of gyroscopic precession and inertial forces. It has also been shown that, both gyroscopic precession and centrifugal force reverse at circular null geodesics in static spacetimes. For example, this occurs at $R = 3M$ in the case of the Schwarzschild spacetime. Therefore, these phenomena are important in the case of black hole solutions and compact objects whose radii lie within the circular null geodesics. Iyer, Vishveshwara and Dhurandhar[44] in fact showed that such ultra compact configurations are possible and are stable within the framework of general theory of relativity. The centrifugal force reversal might influence the equilibrium configurations of such ultra compact objects. Equilibrium configurations of relativistic fluids is of considerable importance in astrophysics, since they represent compact objects such as neutron stars. The rotation induces interesting as well as intriguing effects on equilibrium configurations of relativistic fluids. One such interesting effect in the case of slowly rotating configurations was first observed by Chandrashekar and Miller[21, 55]. Their studies showed that the ellipticity of a slowly rotating configuration increases to a maximum value and then decreases with the decreasing radius, whereas in Newtonian gravity the ellipticity is a monotonic function of the radius of the star. Abramowicz and Miller[12] suggested that this is due to the reversal of the general relativistic analogue of centrifugal force. Using the centrifugally corrected Newtonian equations they reproduced the phenomenon of reversal of ellipticity behavior of relativistic Maclaurin ellipsoids. Gupta, Iyer and Prasanna[36, 37] investigated the behavior of ellipticity and centrifugal reversal for slowly rotating perfect fluids with various equations of state. They used the formalism developed by Hartle and Thorne[39, 40], where rotation is treated to the first order. In order to understand the influence of centrifugal force and its reversal on such compact objects, it would be advantageous to have a general treatment, in which we express Einstein's equations for an axially symmetric stationary system in terms of inertial forces. In this chapter we establish direct covariant relations between inertial forces and Einstein's equations with a perfect fluid as the source term. This formalism might be useful in understating the existing axially symmetric stationary solutions in terms of relativistic analogues of inertial forces. Also, in particular, one might be able to study the equilibrium configuration of slowly as well as rapidly rotating ultra compact objects.

In oder to establish direct relations between inertial forces and Einstein's equations, we use the Geroch formalism. It is well known that for a spacetime admitting Killing vectors the Einstein equations can be simplified by projecting the field equations on the lower dimensional manifold defined by the space of trajectories along the Killing vector fields. We use the field equations in the Geroch formalism[32, 33]. If the spacetime admits one Killing vector, the field equations can be projected on to a three-dimensional manifold. The details of the formalism for one Killing vector case(general stationary spacetimes) are given in *appendix A*. In this formalism the vacuum field equations are expressed in terms of the norm and the twist(vorticity) of the Killing vector field. In section 5.2 we describe the Geroch formalism for the case of two Killing vectors[33]. In this case the Einstein equations are simplified on the two-manifold S, defined by the infinetisimal two-surfaces which are everywhere orthogonal to the two-surface formed by the commuting Killing vectors. In this case the field equations are further simplified in terms of scalar products of the Killing vector fields ξ^a and η^a. In section 5.2.1 we specialize this formalism to the case of axially symmetric stationary spacetimes. In this case, the elements of S are surface forming and the two-manifold S can be represented by one of such surfaces. The Geroch formalism has been extended for gravitational fields with matter field by Hansen and Wincour[38]. Their formalism is outlined in *appendix B*. In section 5.3.1, we specialize this formalism to the case where the source is described by a perfect fluid. We establish direct relations between inertial forces and Einstein's equations in section 5.4. In order to establish these relations, we first express the Einstein equations in terms of scalar potentials which define the inertial forces. We also show that inertial forces are also vector fields on the two-manifold S on which the field equations are defined. In section 5.4.2 we express the field equations for source free spacetimes in term of inertial forces. The field equations with a perfect fluid source in terms of inertial forces are given in section 5.4.3. Finally we end the chapter with a few concluding remarks.

5.2 Einstein's Equations with Two Killing Vectors

For a spacetime admitting Killing vectors, the field equations can be simplified in terms of the norm and the twist of the Killing vectors. The formalism given by

Geroch for spacetimes admitting one Killing vector is described in *appendix A*. The formalism given in the last section can be further simplified for spacetimes with two Killing vectors. In this section we briefly describe the Geroch formalism with two Killing vectors. First we outline the general formalism and then specialize to the case of axially symmetric stationary spacetimes where the convection is assumed to be zero.

Let $\mathcal{M}$ be a four-dimensional manifold with metric g_{ab} satisfying the source-free Einstein's equation represented by,

$$R_{ab} = 0.$$

Let the metric g_{ab}, admit a pair of Killing vectors ξ^a and η^a, which commute:

$$\xi^a \nabla_b \eta^a - \eta^b \nabla_b \xi^a = 0, \tag{5.1}$$

where ∇_b is the covariant derivative on spacetime manifold $\mathcal{M}$. In this chapter we use the $(-, +, +, +)$ signature for the metric tensor. Since the Killing vectors commute, one can construct the canonical coordinate system in which the metric depends on only two independent variables. If one constructs a two-manifold $\mathcal{S}$, with independent variable, then the metric and the Killing vector can completely describe the manifold $\mathcal{S}$. As in the case of one Killing vector, here also one can express the field equations on the two-manifold $\mathcal{S}$. The formalism was given by Geroch[33]. We briefly summarize the general formalism as given in reference [33] and apply it to axially symmetric stationary vacuum fields.

Since the Killing vector fields, ξ^a and η^a are commuting, one can construct two-surfaces spanned by these Killing vector fields. The two-manifold $\mathcal{S}$ is generated by the infinitesimal two-surface which is orthogonal to the surface formed by the Killing vector fields ξ^a and η^a at each point in the spacetime. As in the case of one Killing vector field here also we define the tensor field $\tilde{T}^{a\cdots c}{}_{b\cdots d}$ on $\mathcal{S}$ which has *one-to-one* correspondence with the tensor field $T^{a\cdots c}_{b\cdots d}$ on $\mathcal{M}$. A tensor field $T^{a\cdots c}_{b\cdots d}$ on $\mathcal{M}$ can be related to a tensor field $\tilde{T}^{a\cdots c}_{b\cdots d}$ on $\mathcal{S}$ which satisfies the conditions

$$\begin{aligned}
\xi_a T^{a\cdots c}{}_{b\cdots d} &= 0, \quad \cdots \quad \xi^d T^{a\cdots c}{}_{b\cdots d} = 0, \\
\eta_a T^{a\cdots c}{}_{b\cdots d} &= 0, \quad \cdots \quad \eta^d T^{a\cdots c}{}_{b\cdots d} = 0, \\
\pounds_\xi T^{a\cdots c}{}_{b\cdots d} &= 0 \\
\pounds_\eta T^{a\cdots c}{}_{b\cdots d} &= 0
\end{aligned} \tag{5.2}$$

Also, $\tilde{\mu}$ is a scalar field on $\mathcal{S}$ which has *one-to-one* correspondence with a scalar μ on $\mathcal{M}$ if,

$$\pounds_\eta \mu = \pounds_\xi \mu = 0, \tag{5.3}$$

We write the inner product of Killing vectors as,

$$\underset{00}{\lambda} = \xi^a \xi_a, \tag{5.4}$$

$$\underset{01}{\lambda} = \xi^a \eta_a \tag{5.5}$$

$$\underset{11}{\lambda} = \eta^a \eta_a \tag{5.6}$$

It is easy to show that the scalars $\underset{00}{\lambda}$, $\underset{01}{\lambda}$ and $\underset{11}{\lambda}$ are functions on $\mathcal{S}$. If we assume that the surface spanned by the Killing vectors ξ^a and η^a is timelike, then we have the scalar

$$\tau^2 \equiv 2\left[\underset{01}{\lambda}^2 - \underset{00}{\lambda}\,\underset{11}{\lambda}\right] > 0. \tag{5.7}$$

The metric on the two-manifold $\mathcal{S}$ can be expressed in terms of the Killing vector fields ξ^a and η^a, and their scalar products as follows,

$$h_{ab} = g_{ab} + 2\tau^{-2}\,\underset{11}{\lambda}\,\xi_a\xi_b + 2\tau^{-2}\,\underset{00}{\lambda}\,\eta_a\eta_b - 4\tau^{-2}\,\underset{01}{\lambda}\,\xi_{(a}\eta_{b)} \tag{5.8}$$

and the antisymmetric tensor in this case is given by,

$$\epsilon_{ab} = 2^{\frac{1}{2}}\tau^{-1}\epsilon_{abcd}\xi^c\eta^d. \tag{5.9}$$

The derivative operator DE_a on $\mathcal{S}$ can be defined using the two-metric and the covariant derivative ∇_a on $\mathcal{M}$ as,

$$D_p T^{a\cdots c}{}_{b\cdots d} \;=\; h^q_p h^a_m \cdots h^c_n g^r_b \cdots h^s_d \nabla_q T^{m\cdots n}{}_{r\cdots s} \tag{5.10}$$

The derivative operator satisfies all the conditions for the derivative operator listed in *Appendix A*. In particular one can see that

$$D_p\, h_{ab} \;=\; 0 \tag{5.11}$$

The source free field equations are expressed in terms of the twist and norms of the Killing vectors. The twist fields for the Killing vectors can be defined as,

$$\underset{00}{\omega}{}^a \;=\; \epsilon^{abcd}\xi_b\nabla_c\xi_d \tag{5.12}$$

$$\underset{01}{\omega}{}^a \;=\; \frac{1}{2}\epsilon^{abcd}\left(\xi_b\nabla_c\eta_d \;+\; \eta_b\nabla_c\xi_d\right) \tag{5.13}$$

$$\underset{11}{\omega}{}^a \;=\; \epsilon^{abcd}\eta_b\nabla_c\eta_d \tag{5.14}$$

In a general spacetime with two Killing vector fields, the twists defined above need not satisfy the conditions (5.2). This indicates the fact that $\underset{00}{\omega}$, $\underset{01}{\omega}$ and $\underset{11}{\omega}$ are in general not vector fields in the two-manifold $\mathcal{S}$. By projecting these twists using the operator (5.8), one can obtain the corresponding vector fields on $\mathcal{S}$. We define the projections of twists on to $\mathcal{S}$ as,

$$\underset{00}{\nu}{}^a \;=\; h^a_b\,\underset{00}{\omega}{}^b \tag{5.15}$$

$$\underset{01}{\nu}{}^a \;=\; h^a_b\,\underset{01}{\omega}{}^b \tag{5.16}$$

$$\underset{11}{\nu}{}^a \;=\; h^a_b\,\underset{11}{\omega}{}^b \tag{5.17}$$

In addition. we also define two constants C_0 and C_1 as,

$$C_0 \;=\; \epsilon^{abcd}\xi_a\eta_b\nabla_c\xi_d \tag{5.18}$$

$$C_1 \;=\; \epsilon^{abcd}\xi_a\eta_b\nabla_c\eta_d \tag{5.19}$$

For spacetimes which satisfy the source free field equations, one can show that C_0 and C_1 are constant. These constants represents convective flows. One can show that the vanishing of both constants C_0 and C_1 is a necessary and sufficient condition that the infinitesimal two-surface orthogonal to the two-surface formed by the Killing vectors are surface forming[35], *i.e.* the Killing vector fields satisfy orthogonal transitivity.

The vector fields $\underset{00}{\nu}$, $\underset{01}{\nu}$ and $\underset{11}{\nu}$ are defined on the two-dimensional manifold $\mathcal{S}$. Their derivatives with respect to D_a can we simplified to,

$$D^{[a}\,\underset{00}{\nu}^{b]} = 2^{-\frac{1}{2}}\tau^{-1}\left[C_0^2\,\underset{01}{\lambda} - C_0C_1\,\underset{00}{\lambda}\right]\epsilon^{ab} \tag{5.20}$$

$$D^{[a}\,\underset{01}{\nu}^{b]} = \frac{1}{2}2^{-\frac{1}{2}}\tau^{-1}\left[C_0^2\,\underset{11}{\lambda} - C_1^2\,\underset{00}{\lambda}\right]\epsilon^{ab} \tag{5.21}$$

$$D^{[a}\,\underset{11}{\nu}^{b]} = 2^{-\frac{1}{2}}\tau^{-1}\left[C_0C_1\,\underset{11}{\lambda} - C_1^2\,\underset{01}{\lambda}\right]\epsilon^{ab} \tag{5.22}$$

One can also show that the divergence of the twists are zero in the case of source free field equations($R_{ab} = 0$),*i.e*,

$$D^a\,\underset{00}{\nu_a} = D^a\,\underset{01}{\nu_a} = D^a\,\underset{11}{\nu_a} = 0. \tag{5.23}$$

The derivatives of the Killing vectors now can be expressed in terms of their twists and norms as follows,

$$\nabla_b\xi_b = \frac{1}{2}\,\underset{00}{\lambda}^{-1}\,\epsilon_{abcd}\xi^c\,\underset{00}{\omega}^d + \underset{00}{\lambda}^{-1}\,\xi_{[b}D_{a]}\,\underset{00}{\lambda} \tag{5.24}$$

$$\nabla_b\eta_b = \frac{1}{2}\,\underset{11}{\lambda}^{-1}\,\epsilon_{abcd}\eta^c\,\underset{11}{\omega}^d + \underset{11}{\lambda}^{-1}\,\eta_{[b}D_{a]}\,\underset{11}{\lambda} \tag{5.25}$$

Here we use the fact that the Killing vector ξ^a and η^a are commuting, hence we have the identity,

$$\frac{1}{2}D_a\,\underset{01}{\lambda} = \xi^b\nabla_a\eta_b = \eta^b\nabla_a\xi_b. \tag{5.26}$$

Using the equations (5.24) and (5.25) one can write the vector fields $\underset{00}{\nu^a}$, $\underset{01}{\nu^a}$ and $\underset{11}{\nu^a}$ in terms of the scalar products of the Killing vectors as,

$$\begin{aligned}
\underset{00}{\nu^a} &= 2^{\frac{1}{2}}\tau^{-1}\epsilon^{ab}\left[-\underset{01}{\lambda}\, D_b \underset{00}{\lambda} + \underset{00}{\lambda}\, D_b \underset{01}{\lambda}\right], \\
\underset{01}{\nu^a} &= \frac{1}{2}\, 2^{\frac{1}{2}}\tau^{-1}\epsilon^{ab}\left[-\underset{11}{\lambda}\, D_b \underset{00}{\lambda} + \underset{00}{\lambda}\, D_b \underset{11}{\lambda}\right], \qquad (5.27) \\
\underset{11}{\nu^a} &= 2^{\frac{1}{2}}\tau^{-1}\epsilon^{ab}\left[-\underset{11}{\lambda}\, D_b \underset{01}{\lambda} + \underset{01}{\lambda}\, D_b \underset{11}{\lambda}\right],
\end{aligned}$$

Taking the derivatives of the ν's from equation (5.27) and multiplying with ϵ_{ab} we get,

$$\epsilon_{pa}\, D^p \underset{00}{\nu^a} = 2^{\frac{1}{2}}\; \underset{01}{\lambda}\, D^b\left[\tau^{-1} D_b \underset{00}{\lambda}\right] - \underset{00}{\lambda}\, D^b\left[\tau^{-1} D_b \underset{01}{\lambda}\right] \qquad (5.28)$$

and using the equation (5.22), we obtain the identity,

$$\underset{01}{\lambda}\, D^a\left[\tau^{-1}\, D_a \underset{00}{\lambda}\right] - \underset{00}{\lambda}\, D^a\left[\tau^{-1}\, D_a \underset{01}{\lambda}\right] = \tau^{-1}\left[C_0^2 \underset{01}{\lambda} - C_0 C_1 \underset{00}{\lambda}\right].$$

Similar results can be proved for $\underset{01}{\nu^a}$ and $\underset{11}{\nu^a}$. Now we have,

$$\underset{01}{\lambda}\, D^a\left[\tau^{-1}\, D_a \underset{00}{\lambda}\right] - \underset{00}{\lambda}\, D^a\left[\tau^{-1}\, D_a \underset{01}{\lambda}\right] = \tau^{-1}\left[C_0^2 \underset{01}{\lambda} - C_0 C_1 \underset{00}{\lambda}\right] \qquad (5.29)$$

$$\underset{11}{\lambda}\, D^a\left[\tau^{-1}\, D_a \underset{00}{\lambda}\right] - \underset{00}{\lambda}\, D^a\left[\tau^{-1}\, D_a \underset{11}{\lambda}\right] = \tau^{-1}\left[C_0^2 \underset{11}{\lambda} - C_1^2 \underset{00}{\lambda}\right] \qquad (5.30)$$

$$\underset{01}{\lambda}\, D^a\left[\tau^{-1}\, D_a \underset{11}{\lambda}\right] - \underset{11}{\lambda}\, D^a\left[\tau^{-1}\, D_a \underset{01}{\lambda}\right] = \tau^{-1}\left[C_1^2 \underset{01}{\lambda} - C_0 C_1 \underset{11}{\lambda}\right] \qquad (5.31)$$

Clearly the above equations are linearly dependent and one cannot solve them to obtain expressions for $D^a\left[\tau^{-1}\, D_a \underset{00}{\lambda}\right]$, $D^a\left[\tau^{-1}\, D_a \underset{00}{\lambda}\right]$ and $D^a\left[\tau^{-1}\, D_a \underset{11}{\lambda}\right]$. To obtain the field equations we compute the divergence of the scalar products as follows,

$$\begin{aligned}
D^a D_a \underset{00}{\lambda} &= h^{ab}\, \nabla_a \left(h_b^m \nabla_m \underset{00}{\lambda}\right) \\
&= 2h^{ab}\, \nabla_a \left(\xi^m \nabla_b \xi_m\right) \\
&= 2h^{ab}\xi^m\, \nabla_a \nabla_b \xi_m + 2h^{ab}\left(\nabla_a \xi^m\right)\left(\nabla_b \xi_m\right) \qquad (5.32)
\end{aligned}$$

from the fact that

$$\nabla_a \nabla_b \xi_c \;=\; R_{abcd}\, \xi^d$$

and $R_{ab} \;=\; 0$, we simplify the equation (5.32) to,

$$D^a D_b \underset{00}{\lambda} \;=\; 2\tau^{-2} \underset{00}{\lambda} \left[D^a \underset{00}{\lambda} D_a \underset{11}{\lambda} - D^a \underset{01}{\lambda} D_a \underset{01}{\lambda}\right] + \tau^{-1} D_a\tau\, D^a \underset{00}{\lambda} + 2\tau^{-2} C_0^2 \quad (5.33)$$

Similarly one can write for $\underset{01}{\lambda}$ and $\underset{11}{\lambda}$,

$$\begin{aligned} D^a D_b \underset{01}{\lambda} \;&=\; 2\tau^{-2} \underset{01}{\lambda} \left[D^a \underset{00}{\lambda} D_a \underset{11}{\lambda} - D^a \underset{01}{\lambda} D_a \underset{01}{\lambda}\right] \qquad (5.34) \\ &+\; \tau^{-1} D_a\tau\, D^a \underset{01}{\lambda} + 2\tau^{-2} C_0 C_1 \end{aligned}$$

$$\begin{aligned} D^a D_b \underset{11}{\lambda} \;&=\; 2\tau^{-2} \underset{11}{\lambda} \left[D^a \underset{00}{\lambda} D_a \underset{11}{\lambda} - D^a \underset{01}{\lambda} D_a \underset{01}{\lambda}\right] \qquad (5.35) \\ &+\; \tau^{-1} D_a\tau\, D^a \underset{11}{\lambda} + 2\tau^{-2} C_1^2 \end{aligned}$$

The equations (5.33), (5.35) and (5.36) can be simplified to,

$$\begin{aligned} D^a[\tau^{-1} D_a \underset{00}{\lambda}] \;&=\; 2\tau^{-3} \underset{00}{\lambda} \left[D^a \underset{00}{\lambda} D_a \underset{11}{\lambda} - D^a \underset{01}{\lambda} D_a \underset{01}{\lambda}\right] \\ &+\; 2\tau^{-3} C_0^2 \qquad (5.36) \end{aligned}$$

$$\begin{aligned} D^a[\tau^{-1} D_a \underset{01}{\lambda}] \;&=\; 2\tau^{-3} \underset{01}{\lambda} \left[D^a \underset{00}{\lambda} D_a \underset{11}{\lambda} - D^a \underset{01}{\lambda} D_a \underset{01}{\lambda}\right] \\ &+\; 2\tau^{-3} C_0 C_1 \qquad (5.37) \end{aligned}$$

$$\begin{aligned} D^a[\tau^{-1} D_a \underset{11}{\lambda}] \;&=\; 2\tau^{-3} \underset{11}{\lambda} \left[D^a \underset{00}{\lambda} D_a \underset{11}{\lambda} - D^a \underset{01}{\lambda} D_a \underset{01}{\lambda}\right] \\ &+\; 2\tau^{-3} C_1^2 \qquad (5.38) \end{aligned}$$

Using the derivative operator one can define the Riemann tensor on the two-dimensional space $\mathcal{S}$. If k_a is an arbitrary vector field on $\mathcal{S}$ then,

$$\begin{aligned} \frac{1}{2}\mathcal{R}_{abcd} k^d \;&=\; D_{[a} D_{b]} k_c \\ &=\; h_{[a}{}^m h_{b]}{}^n h_c^p \nabla_m \left[h_n{}^r h_p{}^s \nabla_r k_s\right] \qquad (5.39) \end{aligned}$$

Expanding the above equation and using the fact that k_a is arbitrary we get,

$$\begin{aligned}
\frac{1}{2}\mathcal{R}_{abcd}k^d &= h_{[a}{}^m h_{b]}{}^n h_c{}^p k^q \Big[\frac{1}{2}R_{mnpq} + \tau^{-2} \underset{11}{\lambda} \nabla_m\xi_n \nabla_q\xi_p + \tau^{-2} \underset{00}{\lambda} \nabla_m\eta_n \nabla_q\eta_p \\
&- \tau^{-2} \underset{01}{\lambda} \nabla_m\xi_n \nabla_q\eta_p - \tau^{-2} \underset{01}{\lambda} \nabla_m\eta_n \nabla_q\xi_p \\
&- \tau^{-2} \underset{11}{\lambda} \nabla_m\xi_p \nabla_n\xi_q - \tau^{-2} \underset{00}{\lambda} \nabla_m\eta_p \nabla_n\eta_q \\
&+ \tau^{-2} \underset{01}{\lambda} \nabla_m\xi_p \nabla_n\eta_q + \tau^{-2} \underset{01}{\lambda} \nabla_m\eta_p \nabla_n\xi_q
\end{aligned} \tag{5.40}$$

Contracting the equation (5.40) and taking the field equation ($R_{ab} = 0$) we can write the two-dimensional scalar curvature in terms of λ's as,

$$\begin{aligned}
\mathcal{R} &= \tau^{-2}\left[D^a \underset{00}{\lambda} D_a \underset{11}{\lambda} - D^a \underset{01}{\lambda} D_a \underset{01}{\lambda}\right] \\
&+ 6\tau^{-4}\left[2C_0C_1 \underset{01}{\lambda} - C_0^2 \underset{11}{\lambda} - C_1^2 \underset{00}{\lambda}\right]
\end{aligned} \tag{5.41}$$

Equations (5.36), (5.37), (5.38) and (5.41) are equivalent to the Einstein field equations with two Killing vectors. From this two-dimensional formalism one can retrieve the four-dimensional equations[32, 33]. This formalism simplifies considerably when applied to stationary axially symmetric spacetimes with orthogonal transitivity, which we describe next.

5.2.1 Axially Symmetric Stationary Spacetimes

In this section we specialize the formalism for the Einstein equations to an axially symmetric stationary spacetime with orthogonal transitivity. As has been mentioned in chapter 3, if the orthogonal transitivity conditions are satisfied, the infinitesimal two-surfaces orthogonal to the two-surface formed by the Killing vectors ξ^a and η^a are also surface forming. These two families of two surfaces are everywhere orthogonal to each other in the spacetime. In this case the two-manifold S can be represented by one of such surfaces which are orthogonal to the surfaces formed by the commuting Killing vectors.

The necessary and sufficient conditions for the orthogonal transitivity are given by,

$$C_0 = \epsilon^{abcd}\xi_a\eta_b \nabla_c\xi_d = 0 \tag{5.42}$$

and

$$C_1 = \epsilon^{abcd}\xi_a\eta_b \nabla_c\eta_d = 0 \tag{5.43}$$

With the above conditions Einstein's field equations are given by equations (5.36), (5.37) and (5.38) can be simplified to,

$$D^a[\tau^{-1} D_a \underset{00}{\lambda}] = 2\tau^{-3} \underset{00}{\lambda} \left[D^a \underset{00}{\lambda} D_a \underset{11}{\lambda} - D^a \underset{01}{\lambda} D_a \underset{01}{\lambda}\right] \tag{5.44}$$

$$D^a[\tau^{-1} D_a \underset{01}{\lambda}] = 2\tau^{-3} \underset{01}{\lambda} \left[D^a \underset{00}{\lambda} D_a \underset{11}{\lambda} - D^a \underset{01}{\lambda} D_a \underset{01}{\lambda}\right] \tag{5.45}$$

$$D^a[\tau^{-1} D_a \underset{11}{\lambda}] = 2\tau^{-3} \underset{11}{\lambda} \left[D^a \underset{00}{\lambda} D_a \underset{11}{\lambda} - D^a \underset{01}{\lambda} D_a \underset{01}{\lambda}\right] \tag{5.46}$$

and the 2-dimensional scalar curvature (5.41) can be expressed as,

$$\mathcal{R} = \tau^{-2} \left[D^a \underset{00}{\lambda} D_a \underset{11}{\lambda} - D^a \underset{01}{\lambda} D_a \underset{01}{\lambda}\right] \tag{5.47}$$

We use this formalism to obtain direct relations between inertial forces and the Einstein equations. In a more realistic model one would like to establish the relation between the Einstein equations with source terms to inertial forces. In the next section we outline the field equations with the source term.

5.3 The Einstein Equations with Source

The source free formalism developed by Geroch [32, 33] for spacetimes admitting two Killing vector fields has been generalized to the equations with sources by Hansen and Wincour[38]. The details of the formalism is given in *appendix B*. Below we specialize to the case where the source is described by a perfect fluid.

5.3.0.1 Perfect Fluid

The energy momentum tensor for a perfect fluid source can be written as

$$T_{ab} = (\mu + p)u_a u_a + g_{ab}p \tag{5.48}$$

where u^a is the four velocity of the fluid. We decompose the four velocity u^a as

$$\underset{0}{u} = u_m \xi^m \tag{5.49}$$

$$\underset{1}{u} = u_m \eta^m \tag{5.50}$$

The field equations given in equations (B.22), (B.23) and (B.24) in the *appendix B* are simplified to the following form,

$$\begin{aligned} D^m\left[\tau^{-1} D_m \underset{00}{\lambda}\right] &= 2\tau^{-3} \underset{00}{\lambda} \left[D^m \underset{00}{\lambda} D_m \underset{11}{\lambda} - D^m \underset{01}{\lambda} D_m \underset{01}{\lambda}\right] \\ &+ 2\tau^{-3} C_0^2 - 16\pi\tau^{-1} \left[(\mu+p) \underset{0}{u} \underset{0}{u} + \frac{1}{2}(\mu - p) \underset{00}{\lambda}\right] \end{aligned} \tag{5.51}$$

$$\begin{aligned} D^m\left[\tau^{-1} D_m \underset{01}{\lambda}\right] &= 2\tau^{-3} \underset{01}{\lambda} \left[D^m \underset{00}{\lambda} D_m \underset{11}{\lambda} - D^m \underset{01}{\lambda} D_m \underset{01}{\lambda}\right] \\ &+ 2\tau^{-3} C_0 C_1 - 16\pi\tau^{-1} \left[(\mu+p) \underset{0}{u} \underset{1}{u} + \frac{1}{2}(\mu - p) \underset{01}{\lambda}\right] \end{aligned} \tag{5.52}$$

$$\begin{aligned} D^m\left[\tau^{-1} D_m \underset{11}{\lambda}\right] &= 2\tau^{-3} \underset{11}{\lambda} \left[D^m \underset{00}{\lambda} D_m \underset{11}{\lambda} - D^m \underset{01}{\lambda} D_m \underset{01}{\lambda}\right] \\ &+ 2\tau^{-3} C_1^2 - 16\pi\tau^{-1} \left[(\mu+p) \underset{1}{u} \underset{1}{u} + \frac{1}{2}(\mu - p) \underset{11}{\lambda}\right] \end{aligned} \tag{5.53}$$

The C_0 and C_1 are scalar functions on the two-space S, in the case of the non-vacuum spacetimes. As mentioned before these functions represent the convective circulation of the matter in the spacetime. They satisfy the equations,

$$D_a C_0 = -8\sqrt{2}\,\pi(\mu + p)\,\tau\ \underset{0}{u}\,\epsilon_a^m v_m \tag{5.54}$$

$$D_a C_1 = -8\sqrt{2}\,\pi(\mu + p)\,\tau\ \underset{1}{u}\,\epsilon_a^m v_m. \tag{5.55}$$

The two-dimensional Ricci scalar is given by the equation,

$$\begin{aligned} \mathcal{R} &= \tau^{-2}\left[D^m \underset{00}{\lambda} D_m \underset{11}{\lambda} - D^m \underset{01}{\lambda} D_m \underset{01}{\lambda}\right] \\ &- 6\tau^{-4}\left[\underset{11}{\lambda} C_0^2 + \underset{00}{\lambda} C_1^2 - 2\underset{01}{\lambda} C_0 C_1\right] \\ &+ 8\pi\,(\mu+p). \end{aligned} \tag{5.56}$$

We also have the hydrostatic support in the direction perpendicular and parallel to the flow of convective circulation as,

$$\begin{aligned} \tau^{-1}(\mu+p)\left[\underset{11}{\lambda} D_a(\underset{0}{u}\,\underset{0}{u}) + -2\underset{01}{\lambda} D_a(\underset{0}{u}\,\underset{1}{u}) + \underset{00}{\lambda} D_a(\underset{1}{u}\,\underset{01}{u})\right] &= -\tau(h_a^m + v_a v^m)D_m p \; + \\ (\mu+p)\epsilon_a{}^m v_m\left(\tau\epsilon^{np}D_n v_p - 2\sqrt{2}\tau^{-2}\left[\underset{11}{\lambda} C_0^2 - 2\underset{01}{\lambda} C_0 C_1 + \underset{00}{\lambda} C_1 C_1\right]\right) \end{aligned} \tag{5.57}$$

$$(\mu+p)v^m D_m \underset{0}{u} = -\underset{0}{u} v^m D_m p, \tag{5.58}$$

$$(\mu+p)v^m D_m \underset{1}{u} = -\underset{1}{u} v^m D_m p \tag{5.59}$$

The conservation of convective flux can be expressed as,

$$D^m(\tau\mu v_m) = -pD^m(\tau v_m) \tag{5.60}$$

We shall now specialize to the case of a fluid with vanishing convective circulation, *i.e.*, $v^a = 0$. The four-velocity for such systems can be written as,

$$u^a = e^{\psi}(\xi^a + \Omega\eta^a) \tag{5.61}$$

where, e^{ψ} is normalizing factor. We assume Ω is a scalar function on $\mathcal{S}$, *i.e.*

$$\pounds_\xi\, \Omega = \pounds_\eta\, \Omega = 0 \tag{5.62}$$

The field equations can be written as,

$$\begin{aligned} D^m\left[\tau^{-1}D_m \underset{00}{\lambda}\right] &= 2\tau^{-3}\underset{00}{\lambda}\left[D^m \underset{00}{\lambda} D_m \underset{11}{\lambda} - D^m \underset{01}{\lambda} D_m \underset{01}{\lambda}\right] \\ &+ 2\tau^{-3}C_0^2 + 16\pi\tau^{-1}\left[(\mu+3p)\underset{00}{\lambda} - (\mu+p)\tau^2 e^{2\psi}\Omega^2\right] \end{aligned} \tag{5.63}$$

$$D^m\left[\tau^{-1}D_m \underset{01}{\lambda}\right] = 2\tau^{-3}\underset{01}{\lambda}\left[D^m \underset{00}{\lambda} D_m \underset{11}{\lambda} - D^m \underset{01}{\lambda} D_m \underset{01}{\lambda}\right] + 2\tau^{-3}C_0C_1 + 8\pi\tau^{-1}\left[(\mu+3p)\underset{01}{\lambda} + (\mu+p)\tau^2 e^{2\psi}\Omega\right] \quad (5.64)$$

$$D^m\left[\tau^{-1}D_m \underset{11}{\lambda}\right] = 2\tau^{-3}\underset{11}{\lambda}\left[D^m \underset{00}{\lambda} D_m \underset{11}{\lambda} - D^m \underset{01}{\lambda} D_m \underset{01}{\lambda}\right] + 2\tau^{-3}C_1^2 + 8\pi\tau^{-1}\left[(\mu+3p)\underset{11}{\lambda} - (\mu+p)\tau^2 e^{2\psi}\right]. \quad (5.65)$$

Since the convective flows are zero, we have,

$$D_aC_0 = D_aC_1 = 0 \quad (5.66)$$

The two-dimensional Ricci scalar $\mathcal{R}$ on $\mathcal{S}$ can we written as,

$$\mathcal{R} = \tau^{-2}\left[D^m \underset{00}{\lambda} D_m \underset{11}{\lambda} - D^m \underset{01}{\lambda} D_m \underset{01}{\lambda}\right] \quad 6\tau^{-4}\left[\underset{11}{\lambda}C_0^2 + \underset{00}{\lambda}C_1^2 - 2\underset{01}{\lambda}C_0C_1\right] + 8\pi(\mu+p) \quad (5.67)$$

$$(\mu+p)\left[D_a\psi - e^{2\psi}\left(\underset{11}{\lambda}\Omega + \underset{01}{\lambda}\right)D_a\Omega\right] = D_ap \quad (5.68)$$

$$\begin{aligned} D_aD_b\tau &+ \frac{1}{2}\tau^{-1}\left[D_a \underset{11}{\lambda} D_b \underset{00}{\lambda} + D_a \underset{00}{\lambda} D_b \underset{11}{\lambda} - 2D_a \underset{01}{\lambda} D_b \underset{01}{\lambda}\right] \\ &- \frac{1}{2}\tau^{-1} h_{ab}\left[D^m \underset{11}{\lambda} D_m \underset{00}{\lambda} - D^m \underset{01}{\lambda} D_m \underset{01}{\lambda}\right] \\ &+ \tau^{-3} h_{ab}\left(\underset{11}{\lambda}C_0^2 + \underset{00}{\lambda}C_1^2 - \underset{01}{\lambda}C_0C_1\right) + 8\pi p\tau\, h_{ab} = 0 \end{aligned} \quad (5.69)$$

Equations (5.70) to (5.69) represents the Einstein equations with the source term.

Now we shall apply this formalism to the case of an axially symmetric stationary spacetime. In axially symmetric stationary spacetimes the Killing vectors ξ^a and η^a are linearly independent. The scalar functions C_0 and C_1 can be shown to be zero for a non-convective fluid source[19, 38]. The field equations can now be written as.

$$D^m\left[\tau^{-1}D_m \underset{00}{\lambda}\right] = 2\tau^{-3}\underset{00}{\lambda}\left[D^m \underset{00}{\lambda} D_m \underset{11}{\lambda} - D^m \underset{01}{\lambda} D_m \underset{01}{\lambda}\right] + 8\pi\tau^{-1}\left[(\mu+3p)\underset{00}{\lambda} - (\mu+p)\tau^2 e^{2\psi}\Omega^2\right] \quad (5.70)$$

$$D^m\left[\tau^{-1}D_m \underset{01}{\lambda}\right] = 2\tau^{-3}\underset{01}{\lambda}\left[D^m \underset{00}{\lambda} D_m \underset{11}{\lambda} - D^m \underset{01}{\lambda} D_m \underset{01}{\lambda}\right] + 8\pi\tau^{-1}\left[(\mu+3p)\underset{01}{\lambda} + (\mu+p)\tau^2 e^{2\psi}\Omega\right] \tag{5.71}$$

$$D^m\left[\tau^{-1}D_m \underset{11}{\lambda}\right] = 2\tau^{-3}\underset{11}{\lambda}\left[D^m \underset{00}{\lambda} D_m \underset{11}{\lambda} - D^m \underset{01}{\lambda} D_m \underset{01}{\lambda}\right] + 8\pi\tau^{-1}\left[(\mu+3p)\underset{11}{\lambda} - (\mu+p)\tau^2 e^{2\psi}\right]. \tag{5.72}$$

$$\mathcal{R} = \tau^{-2}\left[D^m \underset{00}{\lambda} D_m \underset{11}{\lambda} - D^m \underset{01}{\lambda} D_m \underset{01}{\lambda}\right] + 8\pi(\mu+p) \tag{5.73}$$

$$(\mu+p)\left[D_a\psi - e^{2\psi}\left(\underset{11}{\lambda}\Omega + \underset{01}{\lambda}\right)D_a\Omega\right] = D_a p \tag{5.74}$$

These form the field equations in an axially symmetric stationary spacetime with a perfect fluid as source. In the next section we express these field equations in terms of new scalar functions. In this new form Einstein's field equations can be directly related to the inertial forces in a covariant manner.

5.4 The Einstein Equations and Inertial Forces in Axially Symmetric Stationary Spacetimes

In the preceding sections we have given the formalisms in which Einstein's equations are in terms of norms and twists of the Killing vectors. On the other hand, in chapter 3 we have established the direct relation between vorticity and inertial forces in axially symmetric stationary spacetimes. In this section we shall establish a direct relation between inertial forces and Einstein's equations. As has been shown in chapter 3, inertial forces are defined with respect to a global rest frame. In the case of axially symmetric stationary spacetimes with orthogonal transitivity, the global rest frames are uniquely determined by the irrotational congruence[35]. Also, in the case of axially symmetric stationary spacetimes, the Einstein equations are represented on one of the surfaces in the family of surfaces $\mathcal{S}$, which are orthogonal to the two-surface formed by the commuting Killing vectors ξ^a and η^a. We represent this surface as our two-manifold $\mathcal{S}$. Also, one can show that for a fiducial test particles following quasi-Killing trajectories, inertial forces are tensor fields on two-manifold $\mathcal{S}$. We recast the Geroch formalism in terms of a set of new scalar functions, whose gradients are proportional to the inertial forces. The Einstein field equations in this new form can be directly related to the inertial forces.

The two-surfaces spanned by the Killing vectors ξ^a and η^a are the same as the two surfaces spanned by the irrotational vector field ζ^a defined by the equation (3.4), and the Killing vector η^a[35]. The two-metric on the surface $\mathcal{S}$ which is orthogonal to the surface formed by the Killing vectors is given by equation (5.8),

$$h_{ab} = g_{ab} + 2\tau^{-2} \underset{11}{\lambda}\, \xi_a\xi_b + 2\tau^{-2} \underset{00}{\lambda}\, \eta_a\eta_b - 4\tau^{-2} \underset{01}{\lambda}\, \xi_{(a}\eta_{b)} \tag{5.75}$$

and can be written in terms of the vector field ζ^a and η^a as,

$$h_{ab} = g_{ab} + n_a n_b - \tau_a\tau_b. \tag{5.76}$$

In the above

$$\begin{aligned} n^a &= e^{-\phi}\zeta^a \\ &= e^{-\phi}\left(\xi^a - \frac{\underset{01}{\lambda}}{\underset{11}{\lambda}}\eta^a\right) \end{aligned} \tag{5.77}$$

$$\tau^a = e^{-\alpha}\eta^a \tag{5.78}$$

where,

$$e^{2\phi} = \frac{\tau^2}{2\underset{11}{\lambda}} \tag{5.79}$$

$$e^{2\alpha} = \underset{11}{\lambda} \tag{5.80}$$

The vector fields ζ^a and η^a are both linearly independent and orthogonal to each other whereas the Killing vectors ξ^a and η^a are only linearly independent to each other. However, it is important to notice the fact that the vector field ζ^a is not a Killing vector field, it is only a quasi-Killing vector field as has been defined earlier.

We formulate the Geroch formalism for axially symmetric stationary spacetime in terms of the new scalar functions, which are defined below.

$$\underset{00}{f} = \frac{\underset{01}{\lambda}^2 - \underset{00}{\lambda}\,\underset{11}{\lambda}}{\underset{11}{\lambda}} = \frac{\tau^2}{2\,\underset{11}{\lambda}} \tag{5.81}$$

$$\underset{01}{f} = \frac{\underset{01}{\lambda}}{\underset{11}{\lambda}} a = -\omega_0 \tag{5.82}$$

$$\underset{11}{f} = \frac{\underset{11}{\lambda}^2}{\underset{01}{\lambda}^2 - \underset{00}{\lambda}\,\underset{11}{\lambda}} = \frac{2\,\underset{11}{\lambda}^2}{\tau^2}. \tag{5.83}$$

The functions $\underset{00}{\lambda}$, $\underset{01}{\lambda}$ and $\underset{11}{\lambda}$ also can be expressed in terms of the functions $\underset{00}{f}$, $\underset{01}{f}$, and $\underset{11}{f}$ uniquely. This allow one to rewrite the field equations in term of $\underset{00}{f}$, $\underset{01}{f}$, and $\underset{11}{f}$. In order to do so we write the equations (5.44), (5.45) and (5.46) as,

$$\begin{aligned} D^m D_m \underset{00}{\lambda} &= 2\tau^{-2}\,\underset{00}{\lambda}\left[D^m \underset{00}{\lambda}\, D_m \underset{11}{\lambda} - D^m \underset{01}{\lambda}\, D_m \underset{01}{\lambda}\right] \\ &+ \tau^{-1} D^m \tau\, D_m \underset{00}{\lambda} \end{aligned} \tag{5.84}$$

$$\begin{aligned} D^m D_m \underset{01}{\lambda} &= 2\tau^{-2}\,\underset{01}{\lambda}\left[D^m \underset{00}{\lambda}\, D_m \underset{11}{\lambda} - D^m \underset{01}{\lambda}\, D_m \underset{01}{\lambda}\right] \\ &+ \tau^{-1} D^m \tau\, D_m \underset{01}{\lambda} \end{aligned} \tag{5.85}$$

$$\begin{aligned} D^m D_m \underset{11}{\lambda} &= 2\tau^{-2}\,\underset{11}{\lambda}\left[D^m \underset{00}{\lambda}\, D_m \underset{11}{\lambda} - D^m \underset{01}{\lambda}\, D_m \underset{01}{\lambda}\right] \\ &+ \tau^{-1} D^m \tau\, D_m \underset{11}{\lambda} \end{aligned} \tag{5.86}$$

Using the above equations one can express the field equation in terms of $\underset{00}{f}$, $\underset{01}{f}$, and $\underset{11}{f}$ as follows,

$$D^m D_m \underset{00}{f} = \underset{00}{f}\,\underset{11}{f}(D^m \underset{01}{f})(D_m \underset{01}{f}) - \frac{1}{2\,\underset{11}{f}}(D^m \underset{00}{f})(D_m \underset{11}{f}) \tag{5.87}$$

$$D^m D_m \underset{01}{f} = -\frac{3}{2\,\underset{11}{f}}(D^m \underset{11}{f})(D_m \underset{01}{f}) - \frac{1}{\underset{00}{f}}(D^m \underset{00}{f})(D_m \underset{01}{f}) \tag{5.88}$$

$$D^m D_m \underset{11}{f} = -2 \underset{11}{f}^2 (D^m \underset{01}{f})(D_m \underset{01}{f}) + \frac{1}{2} \underset{11}{f} (D^m \underset{11}{f})(D_m \underset{11}{f}) - \frac{1}{\underset{00}{f}} (D^m \underset{00}{f})(D_m \underset{11}{f}). \quad (5.89)$$

The two-dimensional Ricci scalar given in the equation (5.47) takes the form,

$$\mathcal{R} = -\frac{1}{2 \underset{00}{f}} \left[\frac{1}{\underset{00}{f}} (D^m \underset{00}{f})(D_m \underset{00}{f}) + \frac{1}{\underset{11}{f}} (D^m \underset{00}{f})(D_m \underset{11}{f} + \underset{00}{f} \underset{11}{f} (D^m \underset{01}{f})(D_m \underset{01}{f}) \right] \quad (5.90)$$

The Einstein equations in the above form can be written in terms of inertial forces, since, gradients of the functions $\underset{00}{f}$, $\underset{01}{f}$, and $\underset{11}{f}$ are proportional to inertial forces acting on the test particle.

5.4.1 Inertial Forces on the Two Surface $\mathcal{S}$

In this section we define the inertial forces for a test particle following a quasi-Killing trajectory. If the spacetime is a source free solution to the Einstein equations then we assume that the four velocity corresponds to a fictitious test particle following a quasi-Killing trajectory. If the spacetime is described by a perfect fluid, then the four velocity u^a corresponds to the velocity of a fluid element. The four velocity of such a system can be written as,

$$u^a = e^{\psi} (\xi^a + \Omega \eta^a) \quad (5.91)$$

As we defined in chapter 3, we compute the inertial forces using the formalism given by Abramowicz, Nurowski and Wex[2]. The various inertial forces acting on the test particle can be written as follows.

Gravitational force:

$$G_k = -\frac{1}{2 \underset{00}{f}} (D_k \underset{00}{f}). \quad (5.92)$$

Centrifugal force:

$$Z_k = \frac{e^{\psi}}{2}\tilde{\Omega}^2 \underset{00}{f}\,(D_k \underset{11}{f}) \tag{5.93}$$

where

$$\tilde{\Omega} = \Omega + \underset{01}{f} \tag{5.94}$$

Coriolis-Lense-Thirring force:

$$C_k = e^{\psi} \underset{00}{f}\,\underset{11}{f}\tilde{\Omega}\,(D_k \underset{01}{f}) \tag{5.95}$$

It is easy to show that the scalars $\underset{00}{f}$, $\underset{01}{f}$, and $\underset{11}{f}$ are functions on the two-manifold $\mathcal{S}$. The inertial forces G_k, Z_k and C_k are also vector fields on $\mathcal{S}$, since they are proportional to the gradients of the functions $\underset{00}{f}$, $\underset{01}{f}$, and $\underset{11}{f}$. One also verifies that the inertial forces satisfy the conditions (5.2).

In the next section we express the vacuum field equations given in the last section in terms of the inertial forces.

5.4.2 Vacuum Field Equations in Terms of Inertial Forces

In this section we directly relate inertial forces acting on a fictitious test particle whose four-velocity is along a quasi-Killing trajectory in a source free axially symmetric stationary spacetime. The functions $\underset{00}{f}$, $\underset{01}{f}$, and $\underset{11}{f}$ are like potentials of the gravitational, Coriolis-Lense-Thirring and centrifugal forces respectively. The Coriolis-Lense-Thirring and centrifugal forces are proportional to the gradients of the potential functions. Using these potential functions one can write the field equations directly in terms of inertial forces.

Using the equation (5.87) and the expressions for inertial forces one obtain the divergence of the gravitational force as

$$D^m G_m = 2G^m G_m - \frac{e^{-2\psi}\tilde{\Omega}^{-2}}{\underset{00}{f}\,\underset{11}{f}}\,G^m Z_m - \frac{1}{2}\frac{e^{-4\psi}\tilde{\Omega}^{-2}}{\underset{00}{f}^2\,\underset{11}{f}}\,C^m C_m \tag{5.96}$$

Similarly form (5.88) and (5.90) we get,

$$D^m Z_m \;=\; Z^m \left[2D_m\psi \;+\; \frac{2}{\tilde{\Omega}}\, D_m\tilde{\Omega} \;-\; e^{-2\psi}\tilde{\Omega}^{-2}\frac{\underset{11}{f}}{\underset{00}{f}}\, Z_m \right] - \frac{e^{-2\psi}}{\underset{00}{f}} C^m C_m \quad (5.97)$$

$$D^m C_m \;=\; C^m \left[2D_m\psi \;-\; \frac{e^{-2\psi}\tilde{\Omega}^{-2}}{\underset{00}{f}\,\underset{11}{f}}\, Z_m \;+\; \frac{1}{\tilde{\Omega}}\, D_m\tilde{\Omega} \right] \quad (5.98)$$

$$\mathcal{R} \;=\; -\frac{1}{2\underset{00}{f}} \left[4\underset{00}{f}\, G^m G_m \;-\; 4\frac{e^{-2\psi}}{\underset{11}{f}}\, \tilde{\Omega}^{-2}\, Z^m G_m + \frac{e^{-4\psi}}{\underset{11}{f}\,\underset{00}{f}}\, \tilde{\Omega}^{-2}\, C^m C_m \right]. \quad (5.99)$$

The above equations represent the field equations for a source free axially symmetric stationary spacetime in terms of inertial forces.

5.4.2.1 Static Spacetime

The general results given for the stationary spacetimes can be specialized to static spacetimes by setting $\xi^a\eta_a = \underset{01}{\lambda} = 0$. As has been shown in chapter 3 the Coriolis-Lense-Thirring force C_k is identically zero in static spacetimes. The field equations take the form,

$$D^m G_m \;=\; 2G^m G_m - \frac{e^{-2\psi}\tilde{\Omega}^{-2}}{\underset{00}{f}\,\underset{11}{f}}\, G^m Z_m \quad (5.100)$$

$$D^m Z_m \;=\; Z^m \left[2D_m\psi \;+\; \frac{2}{\tilde{\Omega}}\, D_m\tilde{\Omega} \;-\; e^{-2\psi}\tilde{\Omega}^{-2}\frac{\underset{11}{f}}{\underset{00}{f}}\, Z_m \right] \quad (5.101)$$

$$\mathcal{R} \;=\; -\frac{1}{2\underset{00}{f}} \left[4\underset{00}{f}\, G^m G_m \;-\; 4\frac{e^{-2\psi}}{\underset{11}{f}}\, \tilde{\Omega}^{-2}\, Z^m G_m \right] \quad (5.102)$$

In the next section we derive these relations for field equations with a perfect fluid as source.

5.4.3 Field Equations With Source in Terms of Inertial Forces

In this section we establish direct relations between inertial forces and field equations with a perfect fluid source. The field equations are given by

$$R_{ab} - \frac{1}{2}Rg_{ab} = 8\pi T_{ab}. \tag{5.103}$$

The energy momentum tensor T_{ab} is described by a perfect fluid,

$$T^{ab} = (\mu + p)u^a u^b + pg^{ab} \tag{5.104}$$

We assume that the four-velocity of a fluid element is along the quasi-Killing trajectory and can be written as,

$$u^a = e^{\psi}(\xi^a + \Omega\eta^a). \tag{5.105}$$

Where ω satisfies the condition,

$$\pounds_{\xi}\, \Omega = \pounds_{\eta}\, \Omega = 0, \tag{5.106}$$

i.e. Ω is constant along each orbits.

In this case also we rewrite the field equations given by Hansen and Wincour[38] in terms of potential functions $\underset{00}{f}$, $\underset{01}{f}$, and $\underset{11}{f}$. First we write the field equations given in equations (B.22), (B.23) and (B.24) as

$$\begin{aligned} D^m D_m \underset{00}{\lambda} &= 2\tau^{-2} \underset{00}{\lambda} \left[D^m \underset{00}{\lambda} D_m \underset{11}{\lambda} - D^m \underset{01}{\lambda} D_m \underset{01}{\lambda}\right] \\ &+ \tau^{-1} D^m \tau\, D_m \underset{00}{\lambda} - 2R_{mn}\xi^m\xi^n \end{aligned} \tag{5.107}$$

$$\begin{aligned} D^m D_m \underset{01}{\lambda} &= 2\tau^{-2} \underset{01}{\lambda} \left[D^m \underset{00}{\lambda} D_m \underset{11}{\lambda} - D^m \underset{01}{\lambda} D_m \underset{01}{\lambda}\right] \\ &+ \tau^{-1} D^m \tau\, D_m \underset{01}{\lambda} - 2R_{mn}\xi^m\eta^n \end{aligned} \tag{5.108}$$

$$\begin{aligned} D^m D_m \underset{11}{\lambda} &= 2\tau^{-2} \underset{11}{\lambda} \left[D^m \underset{00}{\lambda} D_m \underset{11}{\lambda} - D^m \underset{01}{\lambda} D_m \underset{01}{\lambda}\right] \\ &+ \tau^{-1} D^m \tau\, D_m \underset{11}{\lambda} - 2R_{mn}\eta^m\eta^n \end{aligned} \tag{5.109}$$

Now the field equations in terms of $\underset{00}{f}$, $\underset{01}{f}$, and $\underset{11}{f}$ can be written as,

$$\begin{aligned} D^m D_m \underset{00}{f} &= \underset{00}{f}\,\underset{11}{f}(D^m \underset{01}{f})(D_m \underset{01}{f}) - \frac{1}{2\underset{11}{f}}(D^m \underset{00}{f})(D_m \underset{11}{f}) \\ &+ 2\underset{00}{f}\, R_{mn} n^m n^n \end{aligned} \tag{5.110}$$

$$\begin{aligned} D^m D_m \underset{01}{f} &= -\frac{3}{2\underset{11}{f}}(D^m \underset{11}{f})(D_m \underset{01}{f}) - \frac{1}{\underset{00}{f}}(D^m \underset{00}{f})(D_m \underset{01}{f}) \\ &- \frac{2}{\underset{00}{f}\,\underset{11}{f}} R_{mn}\zeta^m \eta^n \end{aligned} \tag{5.111}$$

$$\begin{aligned} D^m D_m \underset{11}{f} &= -2\underset{11}{f}^2(D^m \underset{01}{f})(D_m \underset{01}{f}) + \frac{1}{2}\underset{11}{f}(D^m \underset{11}{f})(D_m \underset{11}{f}) \\ &- \frac{1}{\underset{00}{f}}(D^m \underset{00}{f})(D_m \underset{11}{f}) + 2\underset{11}{f}\,[R_{mn} n^m n^n + R_{mn}\tau^m \tau^n]. \end{aligned} \tag{5.112}$$

Also the two-dimensional Ricci scalar on S can be written as

$$\begin{aligned} \mathcal{R} &= -\frac{1}{2\underset{00}{f}}\left[\frac{1}{\underset{00}{f}}(D^m \underset{00}{f})(D_m \underset{00}{f}) + \frac{1}{\underset{11}{f}}(D^m \underset{00}{f})(D_m \underset{11}{f} \right. \\ &\left. + \underset{00}{f}\,\underset{11}{f}(D^m \underset{01}{f})(D_m \underset{01}{f})\right] + h^{mn} R_{mn} + R_{mn} n^m n^n - R_{mn}\tau^n \tau^m \end{aligned} \tag{5.113}$$

As in the case of vacuum field equations these equations can also be directly written in terms of forces instead of potential functions. The field equations in terms of inertial forces take the following form,

$$\begin{aligned} D^m G_m &= 2G^m G_m - \frac{e^{-2\psi}\tilde{\Omega}^{-2}}{\underset{00}{f}\,\underset{11}{f}} G^m Z_m - \frac{1}{2}\frac{e^{-4\psi}\tilde{\Omega}^{-2}}{\underset{00}{f}^2\,\underset{11}{f}} C^m C_m \\ &- R_{mn} n^m n^n \end{aligned} \tag{5.114}$$

$$\begin{aligned} D^m Z_m &= Z^m \left[2D_m\psi + \frac{2}{\tilde{\Omega}} D_m \tilde{\Omega} - e^{-2\psi}\tilde{\Omega}^{-2}\frac{\underset{11}{f}}{\underset{00}{f}} Z_m\right] \\ &- \frac{e^{-2\psi}}{\underset{00}{f}} C^m C_m + e^{2\psi}\tilde{\Omega}^2\, \underset{00}{f}\,\underset{11}{f}\,(R_{mn} n^n n^m + R_{mn}\tau^m \tau^n) \end{aligned} \tag{5.115}$$

$$D^m C_m = C^m \left[2D_m\psi - \frac{e^{-2\psi}\tilde{\Omega}^{-2}}{\underset{00}{f}\,\underset{11}{f}} Z_m + \frac{1}{\tilde{\Omega}} D_m\tilde{\Omega} \right] - 2e^{2\psi+\phi+\alpha}\tilde{\Omega} R_{mn} n^m \tau^n \quad (5.116)$$

$$\mathcal{R} = -\frac{1}{2\underset{00}{f}} \left[4\underset{00}{f}\, G^m G_m - 4\frac{e^{-2\psi}}{\underset{11}{f}} \tilde{\Omega}^{-2} Z^m G_m + \frac{e^{-4\psi}}{\underset{11}{f}\,\underset{00}{f}} \tilde{\Omega}^{-2} C^m C_m \right] + h^{mn} R_{mn} + R_{mn} n^m n^n - R_{mn} \tau^m \tau^n \quad (5.117)$$

In order to decompose the energy-momentum tensor with respect to the vector fields n^a and τ^a, we split the four-velocity u^a as follows,

$$u^a \equiv e^{\psi} (\xi^a + \Omega \eta^a) = \gamma (n^a + v\tau^a) . \quad (5.118)$$

where,

$$\gamma = e^{\psi+\phi}, \quad (5.119)$$

$$v = e^{-\phi+\alpha}\tilde{\Omega} \quad (5.120)$$

From the above and using the energy-momentum tensor given in the equation (5.104) we get,

$$R_{ab} n^a n^b = 8\pi \left[\gamma^2(\mu + p) + \frac{1}{2}(p - \mu) \right] \quad (5.121)$$

$$R_{ab} n^a \tau^b = -8\pi\gamma^2 v(\mu + p) \quad (5.122)$$

$$R_{ab} \tau^a \tau^b = 8\pi \left[\gamma^2 v^2(\mu + p) - \frac{1}{2}(p - \mu) \right] \quad (5.123)$$

Using the above equations the field equations given in (5.114) to (5.117) can be written as,

$$D^m G_m = 2G^m G_m - \frac{e^{-2\psi}\tilde{\Omega}^{-2}}{\underset{00}{f}\,\underset{11}{f}} G^m Z_m - \frac{1}{2}\frac{e^{-4\psi}\tilde{\Omega}^{-2}}{\underset{00}{f^2}\,\underset{11}{f}} C^m C_m - 8\pi \left[\gamma^2(\mu + p) + \frac{1}{2}(p - \mu) \right] \quad (5.124)$$

$$
\begin{aligned}
D^m Z_m &= Z^m \left[2D_m\psi + \frac{2}{\tilde{\Omega}} D_m\tilde{\Omega} - e^{-2\psi}\tilde{\Omega}^{-2} \frac{\underset{11}{f}}{\underset{00}{f}} Z_m \right] \\
&- \frac{e^{-2\psi}}{\underset{00}{f}} C^m C_m + 8\pi e^{2\psi}\tilde{\Omega}^2 \underset{00}{f}\,\underset{11}{f} \left[\gamma^2 v^2 (\mu + p) - \frac{1}{2}(p - \mu) \right]
\end{aligned}
\tag{5.125}
$$

$$
\begin{aligned}
D^m C_m &= C^m \left[2D_m\psi - \frac{e^{-2\psi}\tilde{\Omega}^{-2}}{\underset{00}{f}\,\underset{11}{f}} Z_m + \frac{1}{\tilde{\Omega}} D_m\tilde{\Omega} \right] \\
&+ 16\pi e^{2(\psi+\alpha)}\tilde{\Omega}^2\gamma^2(\mu + p)
\end{aligned}
\tag{5.126}
$$

$$
\begin{aligned}
\mathcal{R} &= -\frac{1}{2\underset{00}{f}} \left[4\underset{00}{f}\, G^m G_m - 4\frac{e^{-2\psi}}{\underset{11}{f}} \tilde{\Omega}^{-2} Z^m G_m + \frac{e^{-4\psi}}{\underset{11}{f}\,\underset{00}{f}} \tilde{\Omega}^{-2} C^m C_m \right] \\
&+ 8\pi(\mu + p)
\end{aligned}
\tag{5.127}
$$

In addition we also have,

$$
(\mu + p)\left[D_a\psi - e^{2(\psi+\alpha)}\tilde{\Omega} D_a\Omega \right] = D_a p \tag{5.128}
$$

Equations (5.124) to (5.128) represent the field equations for a stationary axially symmetric spacetime in terms of the inertial forces.

Next we relate the gravito-electromagnetic fields to the Einstein field equations in stationary spacetime.

5.5 Gravito-electromagnetic Fields and the Einstein Field Equations

In this section we shall express the Einstein field equations in terms of gravito-electromagnetic fields.

Let ξ^a be a timelike Killing vector in a stationary spacetime. The four velocity of an observer following along the Killing vector is given by

$$
u^a = e^{\psi}\xi^a \tag{5.129}
$$

As described in third chapter we define the gravito-electric and gravito-magnetic fields with respect to the comoving frame of the observer moving with four velocity u^a. We have, gravito-electric filed,

$$E_a = F_{ab}u^a \tag{5.130}$$

and the gravito-magnetic filed

$$B_a = \hat{F}_{ab}u^b, \tag{5.131}$$

where $\hat{F}_{ab}$ is dual of F_{ab}.

Using the properties of Killing vector fields we have,

$$\xi_{a;b} = \frac{1}{2}\lambda^{-1}\epsilon_{abcd}\omega^c\xi^d + \lambda^{-1}\xi_{[a}D_{b]}\lambda. \tag{5.132}$$

Here λ is the norm and ω is the twist of the the Killing vector ξ^a, which are given by the equations (A.15) and (A.16)i respectively.

$$\lambda = \xi^m\xi_m \tag{5.133}$$

$$\omega_a = \epsilon_{abcd}\xi^b\xi^{d;c} \tag{5.134}$$

Using the above definition we have

$$\omega^a = \lambda B^a \tag{5.135}$$

and

$$E_a = \frac{1}{2\lambda}D_a\lambda \tag{5.136}$$

The Einstein filed equations with one Killing vector can be reduce to following set of equations using the Geroch[32] formalism. the details are given in *appendix A*.

$$D^a\omega_a = \frac{3}{2}\lambda^{-1}\omega_m D^m\lambda \tag{5.137}$$

$$D^aD_a\lambda = \frac{1}{2}\lambda^{-1}D^m\lambda\, D_m\lambda - \lambda^{-1}\omega^m\omega_m - \kappa\lambda(\mu - 3p) \tag{5.138}$$

$$D_{[a}\omega_{b]} \;=\; 0 \tag{5.139}$$

$$\begin{aligned} \mathcal{R}_{ab} \;&=\; \frac{1}{2}\lambda^{-2}\left[\omega_a\omega_b - h_{ab}\omega_m\omega^m\right] + \frac{1}{2}\lambda^{-1}D_aD_b\lambda \\ &-\; \frac{1}{4}\lambda^{-2}D_a\lambda\; D_b\lambda \;+\; \frac{\kappa}{2}(\mu - p)h_{ab} \end{aligned} \tag{5.140}$$

In the above, source term is assumed to be a perfect fluid with energy momentum tensor

$$T_{ab} \;=\; (\mu + p)u_au_b \;+\; pg_{ab} \tag{5.141}$$

where u^a is four velocity along the Killing vector ξ^a as given in the equation (5.129).

Using the definition of gravito-electric and gravito-magnetic fields, we obtain the Einstein field equation as,

$$D^aB_a \;=\; E_aB^a \tag{5.142}$$

$$D_{[a}B_{b]} \;=\; -2E_{[a}B_{b]} \tag{5.143}$$

$$D^aE_a \;=\; -E^aE_a - \frac{1}{2}B^aB_a - \frac{1}{2}\kappa\,(\mu - 3p) \tag{5.144}$$

$$\begin{aligned} \mathcal{R}_{ab} \;&=\; \frac{1}{2}\left[B_aB_b - h_{ab}B_mB^m\right] \\ &=\; +D_aE_b - 2E_aE_b + \frac{\kappa}{2}\,(\mu - p)\,h_{ab} \end{aligned} \tag{5.145}$$

These equations represents the Einstein field equations in terms of gravito-electromagn fields.

In the case of source free field equations we have,

$$\mu \;=\; p \;=0 \tag{5.146}$$

The equations (5.142)- (5.150) reduces to the form,

$$D^aB_a \;=\; E_aB^a \tag{5.147}$$

$$D_{[a}B_{b]} = -2E_{[a}B_{b]} \tag{5.148}$$

$$D^aE_a = -E^aE_a - \frac{1}{2}B^aB_a \tag{5.149}$$

$$\mathcal{R}_{ab} = \frac{1}{2}\left[B_aB_b - h_{ab}B_mB^m\right] + D_aE_b - 2E_aE_b \tag{5.150}$$

These presents the source free Einstein field equations with one Killing vector field, in terms of gravito-electric and gravito-magnetic fields. In the case of two Killing vector fields such as in the case of axially symmetric stationary spacetimes, we have seen that field equations can be written only in terms of the scalar products of the Killing vector fields. The twists $\underset{00}{\omega}$, $\underset{01}{\omega}$ and $\underset{11}{\omega}$ do not represent independent equations. Because of this fact one can not write the Einstein field equations completely in terms of gravito-electric and gravito-magnetic fields as defined by the co-moving frame.

5.6 Conclusions

In this chapter we have directly connected the Einstein field equations to the inertial forces. The inertial force concept was first developed in order to get better insight into the motion of test particles in the general theory of relativity. When the formalism is applied to the trajectories along the directions of spacetime symmetries, one can also use inertial forces to understand the spacetime structure. This goal has been achieved in this chapter by directly expressing Einstein's equations in terms of inertial forces. This formalism may be useful in understanding the geometry and the physical significance of axially symmetric stationary spacetimes. Also, as mentioned earlier, several studies have been carried out relating the centrifugal force reversal to the equilibrium configurations of ultra compact objects. The present chapter does not deal with this problem directly. However, we hope that more insight into this problem can be gained using the formalism developed in this chapter and the discussions extended to include fast rotations.

Chapter 6

Concluding Remarks

In the present thesis we have examined some of the rotational effects in the general theory of relativity. These effects include the phenomenon of gyroscopic precession, the general relativistic analogues of inertial forces and gravito-electromagnetic fields. The phenomenon of gyroscopic precession is an effective tool to probe the rotational effects and can be used as a test for general relativistic rotational effects. Because of this reason one would like to relate any rotational effect to the phenomenon of gyroscopic precession. This is one of the main themes of the present thesis.

In our present study we have related the general relativistic analogues of inertial forces and gravito-electromagnetic fields to the precession frequency. Also we have established the relation between the inertial forces and the gravito-electromagnetic fields. We have carried out our investigation in general axially symmetric stationary spacetimes so that the formalisms are applicable to the case of black hole solutions as well as to compact objects. We have infact applied these formalisms to several black hole solutions in order to demonstrate some of the interesting effects.

For investigating gyroscopic precession we have used the Frenet-Serret formalism developed by Iyer and Vishveshwara[43]. In the second chapter we have shown that the precession frequency of a gyroscope transported along an arbitrary trajectory can be related to two of the Frenet-Serret parameters namely, τ_1 and τ_2. Interesting results emerge when the trajectories are along a timelike Killing vector field in spacetimes. Along a Killing trajectory not only is the precession frequency expressible in terms of the Frenet-Serret parameters, in addition one can show that all the Frenet-Serret scalars are conserved along the trajectory. Furthermore, all the basis vector fields of the Frenet-Serret frame satisfy Lorentz like equation of motion[43, 35].

The covariant formalism given by Abramowicz, Nurowski and Wex forms the basis for our study of the general relativistic inertial forces[2]. We apply this formalism to a particle moving along a quasi-Killing trajectory in axially symmetric stationary spacetimes. In this case we show that the forces are proportional to gradients of scalar potentials. We have established relation between inertial forces and gyroscopic precession using two approaches. In the first approach we have related the Frenet-Serret parameters τ_1 and τ_2 which represent the gyroscopic precession in terms of inertial forces. In a second approach we directly relate the gyroscopic precession frequency to the inertial forces yielding vector relations. Using these expressions we show that in static spacetimes, for circular trajectories, the simultaneous reversal of gyroscopic precession and centrifugal force reversal occurs only at circular null geodesics. The reversal of centrifugal force at the photon orbits was first shown by Abramowicz Carter and Lasota[1]. We have shown the general properties of simultaneous reversal of gyroscopic precession and centrifugal force at the circular photon orbits in static spacetimes. By applying this formalism in the Ernst spacetime we have explicitly demonstrated this phenomenon. In the case of the Ernst spacetime, there exists two circular null geodesics and simultaneous reversal of gyroscopic precession and centrifugal force occurs at both the photon orbits. We obtain similar results for the Schwarzschild spacetime and the Melvin universe as special cases of the Ernst spacetime by setting the parameter B and M zero respectively.

Contrary to the case of static spacetimes. in the case of stationary axially symmetric spacetimes, we have shown that neither centrifugal force nor gyroscopic precession reversal occurs at the circular null geodesics. In general, centrifugal force and gyroscopic precession reversals occur at different points in the spacetime. Using the Kerr-Newman spacetime as an example for axially symmetric stationary spacetime we study the reversal of gyroscopic precession and centrifugal force. We also investigated forces as functions of angular momentum parameter and charge parameter in the Kerr-Newman spacetime. Kerr solution is treated as a special case of the Kerr-Newman spacetime by setting the charge parameter Q to be zero. By setting angular momentum parameter a as zero in the Kerr-Newman spacetime one obtains the Reissner-Nordstrom solution which is a static spacetime. As expected, we observe that in the Reissner-Nordstrom spacetime both gyroscopic precession and centrifugal force reversals occur at the circular photon orbits.

We have used a different approach for treating the gravito-electromagnetic fields in contrast to conventional weak field approximation[23]. We use the properties of Killing vector fields in order to define the gravito-electromagnetic fields. The advantage of our approach is that one can define the gravito-electric and gravito-magnetic fields with respect to any given observer. By defining the gravito-electric and gravito-magnetic fields with respect to the global rest frame, we relate them to inertial forces. If one defines the gravito-electric and gravito-magnetic fields with respect to the co-moving frame of the particle, we show that one can obtain simple relations between gravito-electromagnetic fields and the Frenet-Serret parameters.

In the fifth chapter we have established the direct relation between the inertial forces and the Einstein field equations. As we have shown that the inertial forces are proportional to gradients of scalar potentials, we express the Einstein equations in terms of these potentials using the Geroch formalism[32, 33]. In the case of field equations with a perfect fluid source terms we use the formalism given by Hansen and Winicour[38]. The concept of inertial force was first developed to study the dynamics of the test particles in a given spacetime. Since we have established relations between inertial forces and the Einstein field equations, this formalism can be used for physical interpretations of exact solution in terms of inertial forces. This can be expected to be useful in studying the equilibrium configurations of relativistic rotating ultra compact objects.

Appendix A

Einsteins's Equations in Stationary Spacetimes

In this appendix we introduce the Geroch formalism which simplifies the Einstein equations for a source free stationary spacetime[32]. In this formalism the Einstein equations are represented on the three-space orthogonal to the Killing vector ξ^a. The field equations are completely expressed in terms of the magnitude and the vorticity of the Killing vector field ξ^a. Though the formalism is applicable with one arbitrary Killing vector, we assume that the spacetime is stationary, *i.e.*, the Killing vector field ξ^a is timelike at each point. The detailed derivation of the formalism is given in reference[32]. We briefly summarize the formalism for a stationary spacetime.

Let $\mathcal{M}$ be a stationary spacetime with metric g_{ab} and timelike Killing vector field ξ^a. We construct spacetime foliations at each point, which are orthogonal to the timelike Killing vector field ξ^a and are represented by $\mathcal{S}$. The projection operator for the quotient space can be defined as follows,

$$h_{ab} = g_{ab} - (\xi^m \xi_m)^{-1} \xi_a \xi_b \tag{A.1}$$

$$h^{ab} = g^{ab} - (\xi^m \xi_m)^{-1} \xi^a \xi^b \tag{A.2}$$

$$h_a^b = \delta_a^b - (\xi^m \xi_m)^{-1} \xi_a \xi^b \tag{A.3}$$

If the Killing vector ξ^a were hypersurface orthogonal, then it is possible to represent $\mathcal{S}$ as one of the hypersurfaces in $\mathcal{M}$ which is everywhere orthogonal to ξ^a. In the non-hypersurface orthogonal case, however, there is no natural way of introducing such a surface in $\mathcal{M}$.

The tensor field h_{ab} as given in equation (A.1), defines the metric on the three-quotient space orthogonal to the Killing vector ξ^a. The geometry on $\mathcal{S}$ is induced by the spacetime $\mathcal{M}$ with induced metric h_{ab}. Any tensor field on $\mathcal{M}$ projected on to the quotient space using the operator h_{ab} is a tensor field on $\mathcal{S}$. But we consider certain tensor fields $\tilde{T}_{a\cdots c}^{\ b\cdots d}$ on $\mathcal{S}$ which have a *one-to-one* correspondence with the tensor fields $T_{a\cdots c}^{\ b\cdots d}$ on $\mathcal{M}$. In order to have such a *one-to-one* correspondence the tensor field $T_{a\cdots c}^{\ b\cdots d}$ on $\mathcal{M}$ must obey the following conditions[32],

$$\xi^a T_{a\cdots c}^{\ b\cdots d} = 0 \quad \cdots \xi_d T_{a\cdots c}^{\ b\cdots d} = 0 \tag{A.4}$$

and

$$\pounds_\xi \ T_{a\cdots c}^{\ b\cdots d} = 0. \tag{A.5}$$

A scalar field $\tilde{\mu}$ on the quotient space $\mathcal{S}$ represents a scalar field μ on $\mathcal{M}$ if it satisfies the equation,

$$\pounds_\xi \ \mu = 0. \tag{A.6}$$

The above results are proved in reference[32]. The conditions (A.4), (A.5) and (A.6) define the scalar and tensor fields on $\mathcal{S}$. The antisymmetric permutation tensor ϵ^{abcd} on $\mathcal{S}$ can be defined as

$$\epsilon_{abc} = (\xi^m \xi_m)^{-\frac{1}{2}} \epsilon^{abcd} \xi_d \tag{A.7}$$

with

$$\epsilon_{abc} \epsilon^{abc} = 6. \tag{A.8}$$

Since the tensor fields satisfying the conditions (A.4) and (A.5) have *one-to-one* correspondence with the tensor field on $\mathcal{S}$, we drop the tilde: we shall represent the tensor fields on $\mathcal{S}$ merely as fields satisfying the condition (A.4) and (A.5). The covariant derivative D_a on the quotient space $\mathcal{S}$ can be defined as follows,

$$D_e T_{a\cdots c}^{\ b\cdots d} = h_e^p h_a^m \cdots h_c^n h_r^b \cdots h_s^a \nabla_p T_{m\cdots n}^{\ r\cdots s} \tag{A.9}$$

where ∇_p is the covariant derivative on $\mathcal{M}$. One can clearly see that the derivative operator D_a satisfies the conditions (A.4) and (A.5). In addition, it also satisfies the following conditions[32].

1. The derivative operator satisfies the Leibnitz rule, *i.e.*,

$$D_a(T_b \times B_c) = T_b \times D_a B_c + D_a T_b \times B_c, \tag{A.10}$$

   where $\times$ is the outer product operator.

2. The contraction of the derivative of any tensor field on $\mathcal{S}$ equals the derivative of its contraction.

3. If μ is any scalar field on $\mathcal{S}$, the $D_a\mu$ is the gradient of μ, and $D_{[a}D_{b]}\mu = 0$

4. The derivative of the sum of two tensors on $\mathcal{S}$ is the sum of their derivatives.

5. The derivative of the metric is zero.

Using the properties of the covariant derivative (A.9), one can define the Riemann tensor on the three dimensional quotient space $\mathcal{S}$. If k_a is an arbitrary vector field on $\mathcal{S}$ one can show that,

$$\begin{aligned} D_a D_b k_c &\equiv h_a^p h_b^q h_c^r \nabla_p (h_q^s h_r^t \nabla_s k_t) \\ &= h_a^p h_b^s h_c^t \nabla_p \nabla_s k_t - (\xi^m \xi_m)^{-1} h_a^p h_b^q h_c^r (\nabla_p \xi_q) \xi^s \nabla_s k_r \\ &- (\xi^m \xi_m)^{-1} h_a^p h_b^q h_c^r (\nabla_p \xi_r) \xi^t \nabla_q k_t \end{aligned} \tag{A.11}$$

Antisymmetrizing over the indices a and b we get ,

$$D_{[a}D_{b]}k_c = h^p_a h^q_b h^r_c \nabla_{[p}\nabla_{q]}k_r + (\xi^m\xi_m)^{-1} h^p_a h^q_b h^r_c (\nabla_p\xi_q)(\nabla_r\xi_s)k^s + (\xi^m\xi_m)^{-1} h^p_{[a} h^q_{b]} h^r_c (\nabla_p\xi_r)(\nabla_q\xi_t)k^t. \tag{A.12}$$

Here we use the fact that $\pounds_\xi\, k_r = 0$ and $\xi^t k_t = 0$. Taking k_a to be an arbitrary vector field, the Riemann tensor $\mathcal{R}_{abcd}$ of $\mathcal{S}$ is related to the Riemann tensor R_{abcd} of $\mathcal{M}$ by,

$$\mathcal{R}_{abcd} = h^p_{[a} h^q_{b]} h^r_{[c} h^s_{d]} \left[R_{pqrs} + 2(\xi^m\xi_m)^{-1}(\nabla_p\xi_q)(\nabla_r\xi_s) + 2(\xi^m\xi_m)^{-1}(\nabla_p\xi_r)(\nabla_q\xi_s)\right]. \tag{A.13}$$

The equation representing a source free stationary spacetime is given by

$$R_{ab} = 0 \tag{A.14}$$

where R_{ab} is the Ricci tensor. We express the above source free field equations in terms of the norm and the twist of the Killing vector ξ^a and the metric h_{ab} on the quotient space $\mathcal{S}$. The norm λ and twist ω of the Killing vector ξ^a are given by,

$$\lambda = \xi^m\xi_m \tag{A.15}$$

$$\omega_a = \epsilon_{abcd}\xi^b\nabla^c\xi^d \tag{A.16}$$

One can easily see that λ is a scalar and ω_a is a tensor on $\mathcal{S}$. From equation (A.15) and (A.16) one can express the covariant derivative of the Killing vector as,

$$\nabla_a\xi_b = \frac{1}{2}\lambda^{-1}\epsilon_{abcd}\xi^c\omega^d + \lambda^{-1}\xi_{[b}D_{a]}\lambda. \tag{A.17}$$

Since ω^a is a tensor on $\mathcal{S}$, the derivative of ω_a on $\mathcal{S}$ can be related to the Ricci tensor R^a_b as follows,

$$D_{[a}\omega_{b]} = -\epsilon_{abmn}\xi^m R^n_p \xi^p \tag{A.18}$$

In the above we use the identity which is valid for a Killing vector ξ^a,

$$\nabla_a\nabla_b\xi_c = R_{abcd}\xi^d \tag{A.19}$$

The trace of the derivative of ω^a can be simplified to the form,

$$D^a\omega_a = \frac{3}{2}\lambda^{-1}\omega_m D^m\lambda \tag{A.20}$$

Taking the covariant derivative of the equation (A.17) and using the identity (A.19) we get,

$$D^a D_a\lambda = \frac{1}{2}\lambda^{-1}D^m\lambda\, D_m\lambda - \lambda^{-1}\omega^m\omega_m - 2R_{mn}\xi^m\xi^n. \tag{A.21}$$

By contracting equation (A.13) and using the identities given above we get the expression for the Ricci tensor on the three-quotient space as,

$$\begin{aligned}\mathcal{R}_{ab} &= \frac{1}{2}\lambda^{-2}\left[\omega_a\omega_b - h_{ab}\omega_m\omega^m\right] + \frac{1}{2}\lambda^{-1}D_aD_b\lambda \\ &- \frac{1}{4}\lambda^{-2}D_a\lambda\, D_b\lambda \;+\; h_a^m h_b^n R_{mn}\end{aligned} \tag{A.22}$$

In the source-free case ($i.e.$ $R_{ab}=0$), the equation (A.18) implies that ω_a is a gradient of a scalar ω, $i.e.$,

$$\omega_a = D_a\omega \tag{A.23}$$

The source free field equations takes the form,

$$\begin{aligned}\mathcal{R}_{ab} &= \frac{1}{2}\lambda^{-2}\left[D_a\omega\, D_b\omega \;-\; h_{ab}D^m\omega\, D_m\omega\right] \\ &+ \frac{1}{2}\lambda^{-1}D_aD_b\lambda \;-\; \frac{1}{4}\lambda^{-2}D_a\lambda\, D_b\lambda\end{aligned} \tag{A.24}$$

$$D^m D_m\lambda = \frac{1}{2}\lambda^{-1}D^m\lambda\, D_m\lambda \;-\; \lambda^{-1}D^m\omega\, D_m\omega \tag{A.25}$$

$$D^m D_m\omega = \frac{3}{2}\lambda^{-1}D^m\lambda\;\; D_m\omega \tag{A.26}$$

The above equations represent Einstein's equations for a source free stationary spacetime. The field equations with the source function is given by,

$$R_{ab} = \kappa\left[T_{ab} - \frac{1}{2}Tg_{ab}\right] \tag{A.27}$$

where T_{ab} is the energy momentum tensor.

Equation (A.18) now can be written as,

$$D_{[a}\omega_{b]} = -\kappa\epsilon_{abmn}\xi^n T^m_p \xi^p \tag{A.28}$$

and the trace of this equation remains the same as is given in equation (A.20). Equation (A.21) combined with the equation (A.27) can simplified to,

$$D^a D_a\lambda = \frac{1}{2}\lambda^{-1}D^m\lambda\, D_m\lambda - \lambda^{-1}\omega^m\omega_m - 2\kappa T_{mn}\xi^m\xi^n + \kappa\lambda T \tag{A.29}$$

The three-dimensional Ricci tensor given in the equation (A.22) takes the form,

$$\begin{aligned}\mathcal{R}_{ab} &= \frac{1}{2}\lambda^{-2}\left[\omega_a\omega_b - h_{ab}\omega_m\omega^m\right] + \frac{1}{2}\lambda^{-1}D_a D_b\lambda \\ &- \frac{1}{4}\lambda^{-2}D_a\lambda\, D_b\lambda \;+\; \kappa h^m_a h^n_b\left[T_{ab} - \frac{1}{2}g_{ab}T\right]\end{aligned} \tag{A.30}$$

The equations (A.28), (A.29) and (A.36) presents the Einstein's equations with matter.

If the source is assumed to be described by a perfect fluid the energy-momentum tensor is given by,

$$T^{ab} = (\mu + p)u^a u^a + pg_{ab}. \tag{A.31}$$

Here, u^a is the four-velocity of the fluid element. In the case of stationary spacetimes the fluid element is moving along the time like Killing vector ξ^a and the four-velocity can be written as,

$$u^a = e^{\psi}\xi^a; \quad e^{\psi} = (-\lambda)^{-\frac{1}{2}}. \tag{A.32}$$

The equations (A.28), (A.20), (A.29) and (A.36) now can written as

$$D^a\omega_a = \frac{3}{2}\lambda^{-1}\omega_m D^m\lambda \tag{A.33}$$

$$D^a D_a\lambda = \frac{1}{2}\lambda^{-1}D^m\lambda\, D_m\lambda - \lambda^{-1}\omega^m\omega_m - \kappa\lambda(\mu - 3p) \tag{A.34}$$

$$D_{[a}\omega_{b]} = 0 \tag{A.35}$$

$$\begin{aligned} \mathcal{R}_{ab} &= \frac{1}{2}\lambda^{-2}\left[\omega_a\omega_b - h_{ab}\omega_m\omega^m\right] + \frac{1}{2}\lambda^{-1}D_aD_b\lambda \\ &- \frac{1}{4}\lambda^{-2}D_a\lambda\ D_b\lambda \ + \ \frac{\kappa}{2}(\mu - p)h_{ab} \end{aligned} \tag{A.36}$$

This formalism can be easily extended to the case of spacetimes with two Killing vectors. The case of an axially symmetric stationary spacetimes is an example which is described in chapter 5.

Appendix B

Einsteins's Equations with Source

In this appendix we briefly outline the formalism given by Hansen and Wincor[38]. In this formalism Einstein's field equations with a perfect fluid source are simplified for spacetimes admitting two Killing vectors. The detailed derivation is presented in reference[38]. We briefly summarize the main results. Following the notations given in the chapter 5, the field equations can written as follows[38]

$$\begin{aligned} D^a[\tau^{-1} D_a \underset{00}{\lambda}] &= 2\tau^{-3} \underset{00}{\lambda} \left[D^a \underset{00}{\lambda} D_a \underset{11}{\lambda} - D^a \underset{01}{\lambda} D_a \underset{01}{\lambda}\right] \\ &+ 2\tau^{-3} C_0^2 - 2\tau^{-1} R_{mn}\xi^m \xi^n \end{aligned} \quad (B.1)$$

$$\begin{aligned} D^a[\tau^{-1} D_a \underset{01}{\lambda}] &= 2\tau^{-3} \underset{01}{\lambda} \left[D^a \underset{00}{\lambda} D_a \underset{11}{\lambda} - D^a \underset{01}{\lambda} D_a \underset{01}{\lambda}\right] \\ &+ 2\tau^{-3} C_0 C_1 - 2\tau^{-1} R_{mn}\xi^m \eta^n \end{aligned} \quad (B.2)$$

$$\begin{aligned} D^a[\tau^{-1} D_a \underset{11}{\lambda}] &= 2\tau^{-3} \underset{11}{\lambda} \left[D^a \underset{00}{\lambda} D_a \underset{11}{\lambda} - D^a \underset{01}{\lambda} D_a \underset{01}{\lambda}\right] \\ &+ 2\tau^{-3} C_1^2 - 2\tau^{-1} R_{mn}\eta^m \eta^n \end{aligned} \quad (B.3)$$

In this case C_0 and C_1 are functions on the two-manifold $\mathcal{S}$. For the source free case ($R_{ab} = 0$) one can show that C_0 and C_1 are constants. The functions C_0 and C_1 satisfy the equations,

$$D_a C_0 = -\sqrt{2}\tau \epsilon_a{}^m R_{mn} \xi^n, \quad (B.4)$$

$$D_a C_1 = -\sqrt{2}\tau\epsilon_a{}^m R_{mn}\eta^m. \qquad \text{(B.5)}$$

The two-dimensional Ricci tensor can be written as,,

$$\begin{aligned} \mathcal{R}_{ab} &= \tau^{-2}\left[D_a \underset{00}{\lambda} D_b \underset{11}{\lambda} - D_a \underset{01}{\lambda} D_b \underset{01}{\lambda}\right] + \tau^{-1} D_a D_b \tau \\ &+ 2\tau^{-4} h_{ab} \left[2C_0 C_1 \underset{01}{\lambda} - C_0^2 \underset{11}{\lambda} - C_1^2 \underset{00}{\lambda}\right] + h_a^m h_b^n R_{mn}. \end{aligned} \qquad \text{(B.6)}$$

where R_{ab} is the Ricci tensor on $\mathcal{M}$. We express two-dimensional Ricci tensor in terms of the trace and the trace free part as follows,

$$\begin{aligned} \mathcal{R} &= \tau^{-2}\left[D^a \underset{00}{\lambda} D_a \underset{11}{\lambda} - D^a \underset{01}{\lambda} D_a \underset{01}{\lambda}\right] \qquad \text{(B.7)} \\ &+ 6\tau^{-4}\left[2C_0 C_1 \underset{01}{\lambda} - C_0^2 \underset{11}{\lambda} - C_1^2 \underset{00}{\lambda}\right] + h^{mn} R_{mn} \\ \mathcal{R}_{ab} - \frac{1}{2} h_{ab}\mathcal{R} &= \tau^{-2}\left[D_a \underset{00}{\lambda} D_b \underset{11}{\lambda} - D_a \underset{01}{\lambda} D_b \underset{01}{\lambda}\right] \qquad \text{(B.8)} \\ &- \frac{1}{2}\tau^{-2}\left[D^a \underset{00}{\lambda} D_a \underset{11}{\lambda} - D^a \underset{01}{\lambda} D_a \underset{01}{\lambda}\right] \\ &+ \tau^{-1} D_a D_b \tau - \frac{1}{2}\tau^{-1} h_{ab} D^m D_m \tau \\ &+ h_a^m h_b^n R_{mn} - \frac{1}{2} h_{ab} h^{mn} R_{nm} \end{aligned}$$

In order to simplify the equation (B.7) further, we compute,

$$\begin{aligned} D^m D_m \tau &= 4\left[C_0 C_1 \underset{01}{\lambda} - C_0^2 \underset{11}{\lambda} - C_1^2 \underset{00}{\lambda}\right] \\ &+ 4\tau^{-1}\left[2 \underset{01}{\lambda} R_{mn}\xi^m\eta^n - \underset{11}{\lambda} R_{mn}\xi^m\xi^n - \underset{00}{\lambda} R_{mn}\eta^m\eta^n\right]. \end{aligned} \qquad \text{(B.9)}$$

Substituting into equation (B.7) we get,

$$\begin{aligned} \mathcal{R} &= \tau^{-2}\left[D^a \underset{00}{\lambda} D_a \underset{11}{\lambda} - D^a \underset{01}{\lambda} D_a \underset{01}{\lambda}\right] \\ &- 6\tau^{-4}\left[2C_0 C_1 \underset{01}{\lambda} - C_0^2 \underset{11}{\lambda} - C_1^2 \underset{00}{\lambda}\right] + h^{mn} R_{mn} \\ &+ 2\tau^{-2}\left[2 \underset{01}{\lambda} R_{mn}\xi^m\eta^n - \underset{11}{\lambda} R_{mn}\xi^m\xi^n - \underset{00}{\lambda} R_{mn}\eta^m\eta^n\right] \end{aligned} \qquad \text{(B.10)}$$

The left-hand side of equation (B.8) vanishes identically; then we have

$$D_a D_b \tau - \frac{1}{2} h_{ab} D^m D_m \tau \;\; +$$

$$\frac{1}{2}\tau^{-1}\left[D_a \underset{00}{\lambda} D_b \underset{11}{\lambda} + D_a \underset{11}{\lambda} D_b \underset{00}{\lambda} - 2D_a \underset{01}{\lambda} D_b \underset{01}{\lambda}\right] - \frac{1}{2}\tau^{-1}h_{ab}\left[D^m \underset{00}{\lambda} D_m \underset{11}{\lambda} - D^m \underset{01}{\lambda} D_m \underset{01}{\lambda}\right] + \tau\left[h_a{}^m h_b{}^n R_{mn} - \frac{1}{2}h_{ab}\, h^{mn}R_{mn}\right] = 0 \tag{B.11}$$

Now we consider the Bianchi identity,

$$\nabla^m \left(R_{am} - \frac{1}{2}h_{am}R\right) = 0. \tag{B.12}$$

By projecting on to two-manifold $\mathcal{S}$ and simplifying, the Bianchi identity takes the following form,

$$D^m\left(\tau h^n_m\, R_{np}\xi^p\right) = 0, \tag{B.13}$$

$$D^m\left(\tau h^n_m\, R_{np}\eta^p\right) = 0, \tag{B.14}$$

$$D^m\left[\tau\left(h^n_a h^p_m\, R_{np} - \frac{1}{2}h_{am}h^{np}\, R_{np}\right)\right] + \frac{1}{2}h^{nm}R_{nm}\, D_a\tau + \tau^{-1}\left[\underset{11}{\lambda}\, D_a\left(R_{nm}\xi^m\xi^n\right) - 2\underset{01}{\lambda}\, D_a\left(R_{nm}\xi^m\eta^n\right) + \underset{00}{\lambda}\, D_a\left(R_{nm}\eta^m\eta^n\right)\right] + 2\sqrt{2}\tau^{-2}\left[\underset{11}{\lambda}\, C_0\epsilon^m_a R_{mn}\xi^n - \underset{01}{\lambda}\, C_0\epsilon^m_a R_{mn}\eta^n - \underset{01}{\lambda}\, C_1\epsilon^m_a R_{mn}\xi^n + \underset{00}{\lambda}\, C_1\epsilon^m_a R_{mn}\eta^n\right] = 0 \tag{B.15}$$

Now we split the energy momentum tensor into terms on the two-manifold $\mathcal{S}$ and along the Killing vectors ξ^a and η^a as follows,

$$\underset{00}{T} = T_{mn}\xi^m\xi^n \tag{B.16}$$

$$\underset{01}{T} = T_{mn}\xi^m\eta^n \tag{B.17}$$

$$\underset{11}{T} = T_{mn}\eta^m\eta^n \tag{B.18}$$

$$\underset{0}{\Theta}_a = h^m_a T_{mn}\xi^m \tag{B.19}$$

$$\underset{1}{\Theta}_a = h^m_a T_{mn}\eta^m \tag{B.20}$$

$$t_{ab} = h^m_a\, h^n_b\, T_{mn} \tag{B.21}$$

With the above, the Einstein equations take the form,

$$D^m\left[\tau^{-1}D_m \underset{00}{\lambda}\right] = 2\tau^{-3}\underset{00}{\lambda}\left[D^m \underset{00}{\lambda} D_m \underset{11}{\lambda} - D^m \underset{01}{\lambda} D_m \underset{01}{\lambda}\right] + 2\tau^{-3}C_0^2 - 16\pi\tau^{-1}\left[\underset{00}{T} - \frac{1}{2}\underset{00}{\lambda} T\right] \tag{B.22}$$

$$D^m\left[\tau^{-1}D_m \underset{01}{\lambda}\right] = 2\tau^{-3}\underset{01}{\lambda}\left[D^m \underset{00}{\lambda} D_m \underset{11}{\lambda} - D^m \underset{01}{\lambda} D_m \underset{01}{\lambda}\right] + 2\tau^{-3}C_0C_1 - 16\pi\tau^{-1}\left[\underset{01}{T} - \frac{1}{2}\underset{01}{\lambda} T\right] \tag{B.23}$$

$$D^m\left[\tau^{-1}D_m \underset{11}{\lambda}\right] = 2\tau^{-3}\underset{11}{\lambda}\left[D^m \underset{00}{\lambda} D_m \underset{11}{\lambda} - D^m \underset{01}{\lambda} D_m \underset{01}{\lambda}\right] + 2\tau^{-3}C_1^2 - 16\pi\tau^{-1}\left[\underset{11}{T} - \frac{1}{2}\underset{11}{\lambda} T\right] \tag{B.24}$$

$$D_aC_0 = -8\sqrt{2}\,\pi\,\tau\,\epsilon_a^m \underset{0}{\Theta}_m \tag{B.25}$$

$$D_aC_1 = -8\sqrt{2}\,\pi\,\tau\,\epsilon_a^m \underset{1}{\Theta}_m \tag{B.26}$$

$$\begin{aligned}\mathcal{R} &= \tau^{-2}\left[D^m \underset{00}{\lambda} D_m \underset{11}{\lambda} - D^m \underset{01}{\lambda} D_m \underset{01}{\lambda}\right] \\ &- 6\tau^{-4}\left[\underset{11}{\lambda} C_0^2 + \underset{00}{\lambda} C_1^2 - 2\underset{01}{\lambda} C_0C_1\right] \\ &+ 8\pi\left[T + 4\tau^{-2}\left(\underset{11}{\lambda}\underset{00}{T} + \underset{00}{\lambda}\underset{11}{T} - 2\underset{01}{\lambda}\underset{01}{T}\right)\right]\end{aligned} \tag{B.27}$$

$$\begin{aligned} & D^m(\tau t_{am}) - \tau\left[\underset{00}{T} D_a\left(\tau^{-2}\underset{11}{\lambda}\right) - \right. \\ & 2\underset{01}{T} D_a\left(\tau^{-2}\underset{01}{\lambda}\right) + \left.\underset{11}{T} D_a\left(\tau^{-2}\underset{11}{\lambda}\right)\right] + \\ & 2\sqrt{2}\tau^{-2}\left[\underset{11}{\lambda} C_0\epsilon_a^m \underset{0}{\Theta}_m - \underset{01}{\lambda} C_0\epsilon_a^m \underset{1}{\Theta}_m - \right. \\ & \left.\underset{01}{\lambda} C_1\epsilon_a^m \underset{0}{\Theta}_m + \underset{00}{\lambda} C_1\epsilon_a^m \underset{1}{\Theta}_m\right] = 0 \end{aligned} \tag{B.28}$$

$$\begin{aligned} & D_aD_b\tau + \frac{1}{2}\tau^{-1}\left[D_a \underset{00}{\lambda} D_b \underset{11}{\lambda} + \right. \\ & \left. D_a \underset{11}{\lambda} D_b \underset{00}{\lambda} - 2D_a \underset{01}{\lambda} D_b \underset{01}{\lambda}\right] - \\ & \frac{1}{2}\tau^{-1} h_{ab}\left[D^m \underset{00}{\lambda} D_m \underset{11}{\lambda} - D^m \underset{01}{\lambda} D_m \underset{01}{\lambda}\right] + \\ & \tau^{-3} h_{sb}\left[\underset{11}{\lambda} C_0^2 + \underset{00}{\lambda} C_1^2 - 2\underset{01}{\lambda} C_0C_1\right] + \\ & 8\,\pi\,\tau\, t_{ab} = 0 \end{aligned} \tag{B.29}$$

where

$$T \equiv T^m_m = h^{mn} t_{nm} - 2\tau^{-2} \left[\underset{11}{\lambda} \underset{00}{T} - 2 \underset{01}{\lambda} \underset{01}{T} + \underset{00}{\lambda} \underset{11}{T} \right] \qquad \text{(B.30)}$$

These form the Einstein equations with two Killing vector fields. In chapter 5, this formalism is specialized for a perfect fluid source in axially symmetric stationary spacetimes.

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