TIME-DEPENDENT RADIATIVE TRANSFER

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DEDICATION

To Every 'RESEARCH STUDENT'

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ABSTRACT

The theory of time-dependent radiative transfer is important in the studies of transient phenomena taking place in some of the astrophysical objects. Time-dependence of the radiation field must be considered if the relaxation time of the radiation field is comparable to or longer than the characteristic changes in the properties of the medium. It must also be considered if there is a temporal change in the impinging radiation on the medium.

In chapter I, we discuss the importance of the characteristic time scales which occur in the theory of time-dependent radiative transfer.

In chapter II, a numerical solution for the monochromatic time-dependent transfer equation is presented for the case when the time spent by photon in the absorbed state is significant. Two cases are considered whose boundary conditions are respectively: (1) The surface of a plane-parallel homogeneous medium is illuminated by a pulsed beam, (2) the surface is illuminated by a constant radiation input from time $t = 0$. We investigated the effects of these boundary conditions on the emergent and the reflected radiation from the medium.

In the case (1), we found that the time-dependent reflected radiation falls more rapidly for $\mu = 0.2$ compared to that of $\mu = 0.7$. Here μ is the cosine of the angle made by a ray relative to the normal to the surface . For the case (2), time-dependent reflected radiation reaches steady state faster for $\mu = 0.2$ compared to that of $\mu = 0.7$.

In chapter III, we consider a homogeneous medium where the time spent by a photon between successive acts of scattering is significant. The numerical solution based on the method of characteristics is presented. For the case when the medium is illuminated by an isotropic radiation, the time at which the relaxation of the radiation field to steady state commences depends on the optical thickness of the medium. It also depends on the angle at which the radiation emerges out of the medium.

In chapter IV, we present numerical methods for steady state as well as time-dependent transfer equations in spherically symmetric media. These numerical methods are based on finite-difference methods. Numerical solutions are compared with the known analytical solutions wherever it is possible to do so.

I. INTRODUCTION

1.1 Importance of the time-dependent radiative transfer

New experimental techniques which increased the time resolution of astronomical observations has necessitated the study of time-dependent transfer of radiation. Many of the celestial objects are found to be far from the state of equilibrium. The analysis of the time-dependent characteristics of the observed radiation provides additional insight into the nature of these objects. In particular, time-dependent transfer effects will be important in the study of the objects like atmospheres of supergiant stars, active galactic nuclei, Quasi-stellar objects, supernovae, nova like variables, planetary nebulae, and compact objects with accretion disks. They may also be important when the source of energizing radiation is intrinsically occulted, or reinstated, as in planetary atmospheres.

Two important quantities characterize the time-dependence of the radiation field. One quantity is the time spent by a photon in the absorbed state, t_1 and the other is the time spent by a photon between two consecutive acts of scattering, t_2. Usually one of these characteristic times is dominant and determines the temporal characteristics of the radiation field.

For a resonance line transition, t_1 is usually of the order 10^{-8} sec. t_2 is equal to $\frac{1}{K\eta C}$ where K is the absorption coefficient per particle, η is the number density of the particles and C is the velocity of light. In a low density medium like planetary nebulae, for a resonance line (e.g. $L\alpha$), we have $\eta \simeq 1$, $K \simeq 10^{-12}$, then $t_2 \simeq 10^2$ sec.

Time-dependence of the radiation field must be considered if there is a sudden change in the impinging radiation field on the medium or, if t_1 or t_2 is long compared to the typical time scales in which the atmospheric system is changed.

Time-dependent radiative transfer for an isotropically scattering medium in a planar geometry is given by

$$t_2(z,t)\,\frac{\partial I}{\partial t}(z,\mu,t) + \frac{\mu}{\alpha\,\rho(z,t)}\cdot\frac{\partial I(z,\mu,t)}{\partial z} + I(z,\mu,t) = \left[\frac{\omega}{2}\int_0^t e^{-(t-t')/t_1}\, I(z,\mu',t)\,d\mu'\,\frac{dt'}{t} + (1-\omega)\,B(z,t)\right] \qquad -1<\mu<1 \tag{1}$$

Where $I(z,\mu,t)$ is the specific intensity at position z and at time t in the direction $\cos^{-1}\mu\ (\mu\in[-1,1])$

$\omega(z,t)$ is the albedo for single scattering and $B(z,t)$ represents the thermal sources. α, the mass absorption coefficient and ρ the density of the medium could be functions of time and position and require the determination of the non-steady state populations and dynamics of the medium.

1.2 Descriptions of the problems studied

(a) Transfer in a plane-parallel medium

If the properties of the medium are constant with respect to time and position, the equation (1). is amenable to analytical and semi-analytical treatment under suitable approximations. Most of the techniques are based on (1) first Gaussian approximation or Eddington approximation (Code, 1970, Code & Eason, 1970) (2) principle of invariance or theory of invariant imbedding combined with Lapalce transform technique (Matsumoto, 1974, Bellman et al, 1964), (3) Theory of successive scattering (Matsumoto, 1976, Ganapol, 1979, 1981).

But most of these analytical methods deal with homogeneous semi-infinite or finite media. It is difficult to extend these techniques to time-dependent line transfer problems where the ratio of time intervals to optical depth intervals vary with frequency in the line. Also most of these methods are restricted only to plane parallel media. So there is need to develop numerical techniques which can handle easily the problems of finite inhomogeneous media.

We used numerical techniques to study in a systematic manner the time-dependent radiative transfer in a medium with given properties. In chapter II, we solved equation (1) by considering only the time spent by a photon in the absorbed state. Under this assumption time derivative term vanishes from the equation (1). We developed a numerical method based on discrete space theory of radiative transfer (Grant and Hunt, 1969a). We studied the intensity distributions due to the changes in the impinging radiation on the medium for various optical depths.

In chapter III, we considered a situation where the time spent by a photon between successive acts of scatterings exceeds the time spent by a photon in the absorbed state. Since we have $t_1 \rightarrow 0$ equation (1) reduces to

$$t_2 \frac{\partial I}{\partial t}(z,\mu,t) + \mu \frac{\partial I}{\partial z}(z,\mu,t) + \alpha I(z,\mu,t) = \alpha \left[\frac{\omega}{2} \int_{-1}^{+1} I(z,\mu',t)\, d\mu' + (1-\omega) B(z,t) \right] \tag{2}$$

To solve the equation (2), we have used a finite difference scheme based on the method of characteristics. We considered a homogeneous time-independent slab illuminated by an externally imposed radiation field which enters slab at time t = 0. The medium is assumed to scatter photons isotropically. Mihalas and Klein (1982) showed that a finite difference method with non-constant space and time intervals cannot accurately represent a propogating unscattered wave front. Hence we have distinguished diffuse radiation field due to one or more scattering processes from the directly transmitted radiation. We have shown in a graphical form the relaxation to the steady state of the diffuse emergent radiation, and reflected radiation from a finite slab with a given optical depth. Also the extension of the method to the resonance line transfer under the assumption of complete redistribution is presented.

1.3 Transfer in spherically symmetric medium

The assumption that the medium is stratified in plane parallel layers holds good only when the density scale height in the atmosphere is small compared to the radius of the star. But many stars, such as, supergiants, Wolf Rayet stars have extended atmospheres. Atmospheric extension has important physical and observational implications. The stars with extended envelope exhibit (see Underhill, 1929) features such as dilution effects, presence of large numbers of Balmer lines, forbidden lines etc. We can assume that the atmospheres of these stars are spherically symmetric. We have an additional curvature term $\frac{1-\mu^2}{r} \frac{\partial I}{\partial \mu}$ in the steady state transfer equation. In addition, if time-dependent effects are important, we have the additional time derivative term and an exponential relaxation factor in the scattering integral.

In chapter IV, we presented a method to solve the steady state equation in spherically symmetric medium. Also a first order difference scheme is developed for the time-dependent equation under the assumption that $t_1 << t_2$. We have checked our algorithm for few test cases.

CHAPTER II

EFFECT OF THE TIME SPENT BY THE PHOTON IN THE ABSORBED STATE ON THE TIME DEPENDENT TRANSFER OF RADIATION

2.1 Introduction

In this chapter, we shall present a numerical solution to the time-dependent monochromatic transfer equation when the time spent by a photon in the absorbed state is significant. We have considered the cases where a slab is illuminated by a pulsed beam of radiation and also by a constant source of radiation. We studied the emergent and reflected intensity distributions for the various optical depths of the medium.

Milne (1926) derived the transfer equation in his investigations of the diffusion of imprisoned radiation through a gas. He considered a slab of mercury gas which is illuminated by light for sufficiently long time for the gas to reach a steady state. If the source of illumination is suddenly cut off, the radiation field in the gas will not cease instantaneously due to the fact that the atoms of mercury will decay with a finite mean life time. Chandrashekar (1950)solved this problem and obtained a solution which is expressed in a series form.

Sobolev (1963) obtained the reflection function by considering the time spent by a photon in the absorbed state for semi-infinite media through the probabilistic arguments. Using the time dependent principle of invariance, Matsumoto (1974) studied the reflected intensity distribution from a homogeneous semi-infinite atmosphere when the time-dependence of incident radiation field is expressed by the Heaviside unit step function.

In section 2.2, we shall present a brief description of the Milne's derivation of the transfer equation (cf.Chandrasekhar, 1950), and in section 2.3, we shall present our numerical solution to the transport equation. We shall present the result and discussion in section 2.4.

2.2 Derivation of the transfer equation

Let suffixes 1 and 2 denote respectively the normal and the excited states of the atom. The Einstein coefficients B_{12}, A_{21} and B_{21} can be defined in the following way. $B_{12} I_\nu$ is the probability, per unit time, that an atom exposed to isotropic radiation of intensity $I_\nu d\nu$ in the frequency interval (ν, $\nu + d\nu$) will absorb the quantum $h\nu$ and make a transition to the state 2 . A_{21} is the probability per unit time, that an atom in the state 2 will spontaneously emit a quantum $h\nu$ and pass to the state 1, and B_{21} I_ν is the probability that the same atom will be induced to undergo the same transition. The Einstein coefficients are related by

$$\frac{A_{21}}{B_{21}} = \frac{2h\nu^3}{c^2}\,\frac{g_1}{g_2}$$

and

$$\frac{B_{21}}{B_{12}} = \frac{g_1}{g_2} \qquad (2.1)$$

where g_1 and g_2 are the statistical weights of the states 1 and 2, c is the velocity of light and h is the Planck's constant. The Einstein coefficients are properties of the atom only, and are independent of the radiation field.

If $\sigma(\nu)$ is the atomic absorption coefficient for frequency ν, then

$$\int \sigma(\nu')\,d\nu' = B_{12}\cdot\frac{h\nu}{4\pi} \qquad (2.2)$$

where the integral is extended over the absorption line corresponding to the transition $1 \longrightarrow 2$. By assuming that the absorption throughout the width of the line is uniform, we can approximate the relation.

(2.2) by

$$\sigma(\nu)\Delta\nu = B_{12}\frac{h\nu}{4\pi} \tag{2.3}$$

Let n_1 and n_2 denote the number of atoms per unit volume in the states 1 and 2; Also n_1 and n_2 vary throughout the gas and are time dependent. Let a pencil of radiation of specific intensity I_ν traverse a path length dS through the gas. Counting the gains and losses of the radiation through a path length we get

$$\Delta\nu \frac{dI_\nu}{dS} = \left[n_2(A_{21} + B_{21}I_\nu) - n_1 B_{12} I_\nu\right]\frac{h\nu}{4\pi} \tag{2.4}$$

where the quantities multiplying n_2 and n_1 represent the number of emissions and absorptions (per unit time) of the quantum $h\nu$. Dividing the equation (2.4) by $B_{12}\,h\nu/4\pi$ and making use of the relations (2.1) and (2.3) the above transfer equation becomes

$$\frac{dI_\nu}{\sigma\, dS} = -\left[n_1 - n_2\frac{g_1}{g_2}\right]I_\nu + \frac{2h\nu^3}{c^2}\frac{g_1}{g_2}n_2 \tag{2.5}$$

The excess of the number of absorptions over the number of emissions must equal the rate of increase of the number of atoms in the excited state. Hence

$$n_1 \int B_{12} I_\nu \frac{d\omega}{4\pi} - n_2 \int \left(A_{21} + B_{21} \frac{I_\nu}{4\pi} \right) \frac{d\omega}{4\pi} = \frac{\partial n_2}{\partial t} \quad (2.6)$$

where the integration is extended over the whole solid angle. We shall define the mean intensity of the radiation by

$$J_\nu = \frac{1}{4\pi} \int I_\nu \, d\omega$$

By dividing the equation (2.6) by B_{12} and rearranging the terms, we get

$$\left(n_1 - n_2 \frac{g_1}{g_2} \right) J_\nu - \frac{2h\nu^3}{c^2} \frac{g_1}{g_2} n_2 = \frac{2h\nu^3}{c^2} \frac{g_1}{g_2} \frac{\partial n_2}{\partial t} \quad (2.7)$$

Assuming that $n_1 \gg n_2$ and time independent, equations (2.5) and (2.7) can be written as,

$$\frac{dI_\nu}{n_1 \sigma \, ds} = - I_\nu + \frac{2h\nu^3}{c^2} \frac{g_1}{g_2} \frac{n_2}{n_1} \quad (2.8)$$

and

$$J_\nu = \frac{2h\nu^3}{c^2} \frac{g_1}{g_2} \frac{1}{n_1} \left(n_2 + \frac{\partial n_2}{A_{21} \partial t} \right) \quad (2.9)$$

We assume that the medium is stratified in plane parallel layer in Z-direction, and denote the cosine of the angle made by ray with the normal to the surface by μ

Defining

$$N = \frac{2h\nu^3}{c^2} \frac{g_1}{g_2} \frac{n_2}{n_1}$$

and the optical depth $d\tau = -n_1 \sigma dZ$

equations (2.8) and (2.9) reduce to

$$+\mu \frac{dI(t,\tau,\mu)}{d\tau} = I(t,\tau,\mu) - N(t,\tau) \qquad 0<\mu<1$$

and for the oppositely directed beam,

$$-\mu \frac{dI(t,\tau,-\mu)}{d\tau} = I(t,\tau,-\mu) - N(t,\tau) \qquad 0<\mu<1 \tag{2.10}$$

and

$$J(t,\tau) = N(t,\tau) + \frac{1}{A_{21}} \frac{\partial N}{\partial t} \tag{2.11}$$

Integrating the equation with respect to the time from 0 to t, and simplifying, we get

$$N(t,\tau) = N(0,\tau)\,e^{-A_{21}t} + A_{21}\,e^{-A_{21}t}\int_0^t e^{A_{21}t'}J(t',\tau)\,dt' \qquad (2.12)$$

Assuming that initially all the atoms are in the ground state, we have $N(0,\tau) \doteq 0$. Further writing $t_1 = \frac{1}{A_{21}}$ and substituting for $N(t,\tau)$ in (2.10)

We get

$$\pm\mu\,\frac{dI}{d\tau}(t,\tau,\pm\mu) = I(t,\tau,\pm\mu) - \frac{1}{t_1}\int_0^t e^{-(t-t')/t_1}J(t',\tau)\,dt' \qquad (2.13)$$

Also one can see that $\frac{1}{t_1}e^{-t|t_1}$ is the probability that a photon absorbed at $t = 0$ will be emitted in the time interval $(t, t+dt)$

2.3 Method of solution

In the following section, we shall describe the solution of radiative transfer equation in detail. First, we shall introduce the Interaction principle which explains the relationship between the input and output radiation fields from a given medium. We shall follow closely the two papers of Grant and Hunt (1969 a,b).

C. Interaction Principle

We divide the medium into N shells. At any level we define upward and downward directed intensities $U^+(\tau_n)$, $U^-(\tau_n)$. Let μ be the cosine of the angle made by a ray with the normal to the surface in the direction in which the optical depth increases. We select a finite set of values of μ, $\{\mu_j : 1 \leq j \leq m,\ 0 \leq \mu_1 < \mu_2 \cdots \mu_m \leq 1\}$ and write $U^+(\tau_n)$ and $U^-(\tau_n)$ as vectors in m-dimensional Euclidean space

$$U^+(\tau_n) = \begin{bmatrix} U^+(\tau_n, \mu_1) \\ \vdots \\ U^+(\tau_n, \mu_m) \end{bmatrix}, \quad U^-(\tau_n) = \begin{bmatrix} U^-(\tau_n, \mu_1) \\ \vdots \\ U^-(\tau_n, \mu_m) \end{bmatrix} \tag{2.14}$$

consider a shell bounded by layers n and $n+1$ as shown in Fig.1(a)

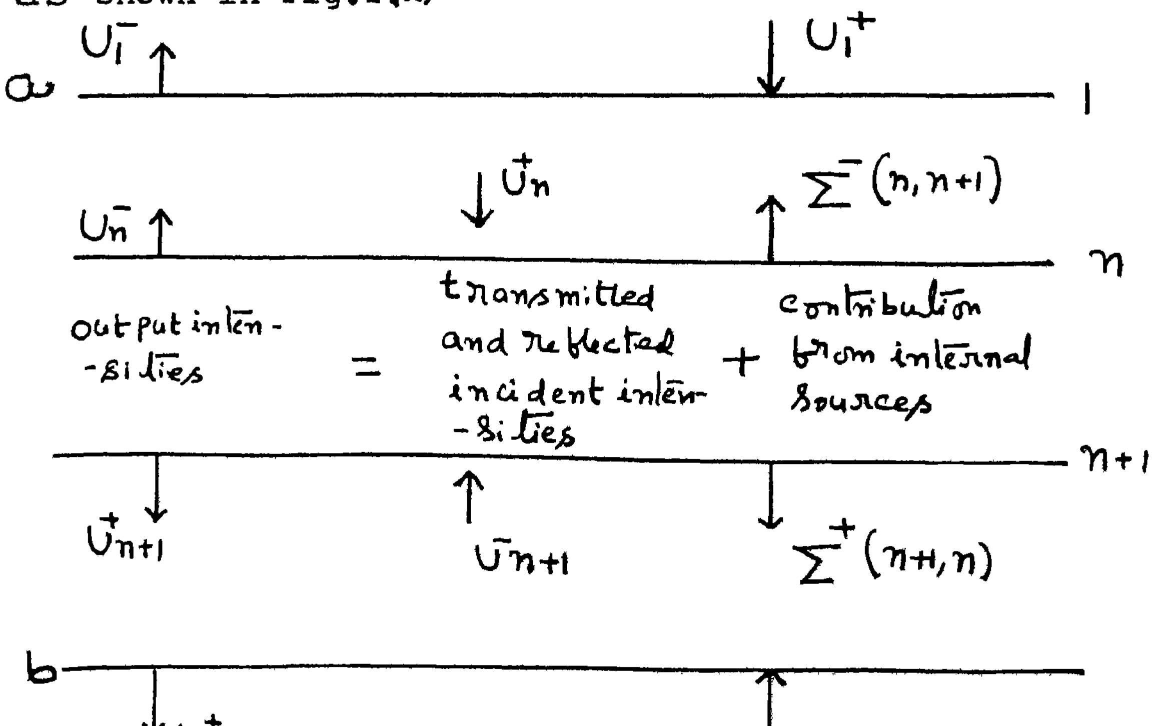


Fig 1(a)

The intensities impinging on this layer are $U^+(\tau_n)$ and $U^-(\tau_{n+1})$. The intensities emerging from the layer $U^+(\tau_{n+1})$, $U^-(\tau_n)$, depend linearly on the incident intensities and on the sources $\Sigma^+(\tau_{n+1},\tau_n)$, $\Sigma^-(\tau_n,\tau_{n+1})$ present within the layer. Then we can write (hereafter, we shall omit τ and retain its subscripts only)

$$U^+_{n+1} = t(n+1,n)\,U^+_n + r(n,n+1)\,U^-_{n+1} + \Sigma^+(n+1,n)$$

$$U^-_n = r(n+1,n)\,U^+_n + t(n,n+1)\,U^-_{n+1} + \Sigma^-(n,n+1) \tag{2.15}$$

or

$$\begin{bmatrix} U^+_{n+1} \\ U^-_n \end{bmatrix} = S(n,n+1)\begin{bmatrix} U^+_n \\ U^-_{n+1} \end{bmatrix} + \Sigma(n,n+1) \tag{2.16}$$

The pair t(n+1,n) and t(n,n+1) are the linear operators of diffuse transmission and r(n,n+1), r(n+1,n) are of diffuse reflection. Equations (2.15) and (2.16) are called the Principle of Interaction.

Now that we have obtained the response function for a layer of specified boundaries, we shall proceed to calculate the response function for two or more consecutive layers, a process termed as "star product". (see also, Redheffer 1962).

II. Star product

Let there be two layers with boundaries τ_n, τ_{n+1} and τ_{n+2} where $a \leq \tau_n \leq \tau_{n+1} \leq \tau_{n+2} \leq b$. Then from equation (2.16), we have

$$\begin{bmatrix} U^+_{n+1} \\ U^-_n \end{bmatrix} = S(n,n+1) \begin{bmatrix} U^+_n \\ U^-_{n+1} \end{bmatrix} + \Sigma(n,n+1)$$

and

$$\begin{bmatrix} U^+_{n+2} \\ U^-_{n+1} \end{bmatrix} = S(n+1,n+2) \begin{bmatrix} U^+_{n+1} \\ U^-_{n+2} \end{bmatrix} + \Sigma(n+1,n+2) \qquad (2.17)$$

As τ_n, τ_{n+1} and τ_{n+2} are arbitrary, we can write again using the interaction principle,

$$\begin{bmatrix} U^+_{n+2} \\ U^-_n \end{bmatrix} = S(n,n+2) \begin{bmatrix} U^+_n \\ U^-_{n+2} \end{bmatrix} + \Sigma(n,n+2) \qquad (2.18)$$

where

$$S(n,n+2) = \begin{bmatrix} t(n+2,n) & r(n,n+2) \\ r(n+2,n) & t(n,n+2) \end{bmatrix}$$

can obtain (2.18) by eliminating U^{+}_{n+1} and U^{-}_{n+1} om (2.17). The relation between S(n,n+1), n+1, n+2) and S(n,n+2) is called 'Star-product' two S-matrices,

$$S(n,n+2) = S(n,n+1) * S(n+1,n+2)$$

so

$$(n+2,n) = t(n+2,n+1)\left[I - r(n,n+1)\, r(n+2,n+1)\right]^{-1} t(n+1,n)$$

$$(n,n+2) = t(n,n+1)\left[I - r(n+2,n+1)\, r(n,n+1)\right]^{-1} t(n+1,n+2)$$

$$\begin{aligned}(n+2,n) = r(n+1,n) + t(n,n+1)\, r(n+2,n+1) \\ \left[I - r(n,n+1)\, r(n+2,n+1)\right]^{-1} t(n+1,n)\end{aligned}$$

$$\begin{aligned}r(n,n+2) = r(n+1,n+2) + t(n+2,n+1)\, r(n,n+1) \\ \left[I - r(n+2,n+1)\, r(n,n+1)\right]^{-1} t(n+1,n+2)\end{aligned} \qquad (2.19)$$

here I is the identity operator.

Let us consider the source term Σ . The esult of adding two layers may be written in terms f two linear operators $\Lambda(n,n+1,n+2)$ and $\Lambda'(n,n+1,n+2)$

$$\begin{aligned}\Sigma(n,n+2) = \Lambda(n,n+1,n+2)\, \Sigma(n,n+1) \\ + \Lambda'(n,n+1,n+2)\, \Sigma(n+1,n+2)\end{aligned}$$

where

$$\Lambda(n,n+1,n+2)=\begin{bmatrix} t(n+2,n+1)\left[I-r(n,n+1)\,r(n+2,n+1)\right]^{-1} & O \\ t(n,n+1)\,r(n+2,n+1)\left[I-r(n,n+1)\,r(n+2,n+1)\right]^{-1} & I \end{bmatrix}$$

and

$$\Lambda'(n,n+1,n+2)=\begin{bmatrix} I & t(n+2,n+1)\,r(n,n+1)\left[I-r(n+2,n+1)\,r(n,n+1)\right]^{-1} \\ O & t(n,n+1)\left[I-r(n+2,n+1)\,r(n,n+1)\right]^{-1} \end{bmatrix} \qquad (2.20)$$

So in practical problem, we divide the medium into N layers and calculate S for each shell and add them by star product. We have for the whole medium,

$$S(1,N)=S(1,2)*S(2,3)*\cdots*S(n,n+1)*\cdots S(N-1,N) \qquad (2.21)$$

A corresponding equation can be written for the source terms. Adding layer by layer at a time one can calculate the complete external response.

III·Calculation of the internal Diffuse radiation field

To calculate the radiation field at any point inside the medium, one has to solve the simultaneous equations

$$\begin{bmatrix} U^{+}_{n+1} \\ U^{-}_{n} \end{bmatrix} = S(n,n+1)\begin{bmatrix} U^{+}_{n} \\ U^{-}_{n+1} \end{bmatrix} + \sum(n,n+1) \qquad (2.22)$$

$$(1 \leq n \leq N)$$

The details of the procedure is given in Grant and Hunt (1968) and we shall quote only the results.

Calculate the r and t operators for each shell. Compute, sequentially, for n=1,2,.......,N, the matrices r(1,n) and vectors $V^+_{n+\frac{1}{2}}$, $V^-_{n+\frac{1}{2}}$ from

$$r(1,n) = r(n,n+1) + t(n+1,n)\,r(1,n)\left[I - r(n+1,n)\,r(1,n)\right]^{-1} t(n,n+1) \tag{2.23}$$

$$V^+_{n+\frac{1}{2}} = \hat{t}(n+1,n)\,V^+_{n-\frac{1}{2}} + \Sigma^+(n+1,n) + R_{n+\frac{1}{2}}\,\Sigma^-(n,n+1) \tag{2.24}$$

$$V^-_{n+\frac{1}{2}} = \hat{r}(n+1,n)\,V^+_{n-\frac{1}{2}} + T_{n+\frac{1}{2}}\,\Sigma^-(n,n+1) \tag{2.25}$$

with the initial conditions r(1,1) = 0, $V^+_{\frac{1}{2}} = U^+(a)$

and where

$$\hat{t}(n+1,n) = t(n+1,n)\left[I - r(1,n)\,r(n+1,n)\right]^{-1} \tag{2.26}$$

$$\hat{r}(n+1,n) = r(n+1,n)\left[I - r(1,n)\,r(n+1,n)\right]^{-1} \tag{2.27}$$

and
$$R_{n+\frac{1}{2}} = \hat{t}(n+1,n)\,r(1,n)$$

$$T_{n+\frac{1}{2}} = \left[I - r(n+1,n)\,r(1,n)\right]^{-1}$$

and

$$\hat{t}(n,n+1) = T_{n+\frac{1}{2}}\,t(n,n+1)$$

On this forward sweep, we need to store the quantities r(1,n), t(n,n+1) which represent the diffuse reflection and transmission for each shell and $V^{\pm}_{n+\frac{1}{2}}$, the diffuse source vectors.

Now we shall calculate the intensities at each step by computing sequentially for n=N, N-1, N-2,......, 2,1.

$$U^{+}_{n+1} = r(1, n+1)\, U^{-}_{n+1} + V^{+}_{n+\frac{1}{2}} \tag{2.28}$$

$$U^{-}_{n} = \hat{t}(n, n+1)\, U^{-}_{n+1} + V^{-}_{n+\frac{1}{2}} \tag{2.29}$$

with the initial conditions $U^{-}_{N+1} = U^{-}(b)$.

We have seen in the previous section how to calculate the diffuse radiation field of general physical and geometrical properties. Also the calculation of diffuse field requires the correct estimation of reflection and transmission matrices for each shell or partition of the medium. We shall calculate the r and t matrices for the medium where the time spent by the photon in the absorbed state is significant.

IV. Calculation of Transmission and Reflection operators in a shell of given physical properties

We have seen in the section 2.2, the transfer equation for a plane-parallel medium when the time spent by the photon in the absorbed state is significant is given by

$$\pm\mu\frac{\partial I}{\partial\tau} = I^{\pm}(t,\tau,\pm\mu) - 0.5\int_0^t e^{-(t-t')/t_1}\int_{-1}^{+1} I(t,\tau,\mu')\frac{dt'}{t_1}d\mu'$$

$$0<\mu<1$$

Sometimes, it is convenient to distinguish between the reduced incident radiation $g(t,\tau,\mu)$ which penetrates to the level τ at time t without suffering any scattering or absorption and the diffuse radiation $I(t,\tau,\mu)$ that results as a consequence of one or more scattering processes.

Then the transfer equation is given by

$$\pm\mu\frac{\partial I}{\partial\tau} = I(t,\tau,\pm\mu) - 0.5\int_0^t e^{-(t-t')/t_1}\int_{-1}^{+1} I(t',\tau,\mu')\frac{dt'}{t_1}d\mu' - g(t,\tau,\mu) \qquad (2.30)$$

Though we considered only isotropic phase function, the extension to arbitrary phase function is straight forward. Also we assumed only conservative scattering atmosphere without any thermal sources present.

For a slab atmosphere with no radiation falling on the top, the boundary conditions for the equation (2.30) are

$$I(t, \tau = T, -\mu) = f(t, \mu)$$

$$I(t, \tau = 0, \mu) = 0 \qquad 0 < \mu < 1 \qquad (2.$$

where $f(t, \mu)$ is a given function of t and μ

We shall approximate the angular integral in equation (2.30) as

$$\int_0^1 I(t, \tau, \mu') \, d\mu' \simeq \sum_{j=1}^{J} I(t, \tau, \mu_j) \, c_j \qquad (2.$$

Where the coefficients C_j and cosines μ_j are determined by Gauss-Legendre quadrature of order J. Integral over time is approximated as

$$\int_0^1 e^{-t'/t_1} I(t',\tau,\mu_j)\, dt' \simeq \sum_{i=1}^{I} e^{-t_i/t_1} I(t_i,\tau,\mu_j)\, a_i \qquad (2.33)$$

$$\text{for } j = 1, \cdots J$$

Incorporating these approximations in (2.31),

We get

$$\pm\mu_j \frac{dI}{d\tau}(t_i,\tau,\pm\mu_j) = I(t_i,\tau,\pm\mu_j) - \frac{0.5}{t_1} e^{-t_i} \sum_{i'=1}^{I} \sum_{j'=1}^{J} e^{t_{i'}/t_1} \left[I(t_{i'},\tau,+\mu_{j'}) + I(t_{i'},\tau,-\mu_{j'}) \right] c_{j'} a_{i'} - g(t_i,\tau,\mu_j)$$

$$\text{for } i = 1, 2, \cdots I \qquad j = 1, 2, \cdots J \qquad (2.34)$$

Defining

$$M = \begin{bmatrix} \mu_1 & 0 & \cdots & & \cdots & 0 \\ \vdots & & & & & \\ 0 & \mu_2 & \cdots & & & 0 \\ \vdots & & \ddots & & & \\ 0 & & \mu_J & \cdots & & 0 \\ \vdots & & & & & \\ 0 & \cdots & & \mu_1 & \cdots & 0 \\ \vdots & & & & & \\ 0 & \cdots & & \cdots & \cdots & \mu_J \end{bmatrix}_{IJ \times IJ}$$

Defining

$$T_{11} = \begin{bmatrix} 0 & \cdots & 0 \\ \\ 0 & \cdots & 0 \end{bmatrix}_{J \times J}$$

$$T_{12} = \begin{bmatrix} a_1 e^{t_1} e^{-t_2} & \cdots & a_1 e^{t_1} e^{-t_2} \\ \\ a_1 e^{t_1} e^{-t_2} & \cdots & a_1 e^{t_1} e^{-t_2} \end{bmatrix}_{J \times J}$$

$$T_{ij} = \begin{bmatrix} a_i e^{t_i} e^{-t_j} & \cdots & a_i e^{t_i} e^{-t_j} \\ \\ a_i e^{t_i} e^{-t_j} & \cdots & a_i e^{t_i} e^{-t_j} \end{bmatrix}_{J \times J}$$

and

$$\tilde{C} = \begin{bmatrix} c_1 & & & 0 \\ 0 & c_2 & & 0 \\ 0 & & \ddots & \\ \\ 0 & & & c_J \end{bmatrix}_{J \times J}$$

We have

$$C = \begin{bmatrix} \tilde{C} & \cdots & & 0 \\ 0 & \tilde{C} & \cdots & 0 \\ \\ \\ 0 & & & \tilde{C} \end{bmatrix}_{I \times I}$$

$$T = \begin{bmatrix} T_{11} & 0 & \cdots & & 0 \\ T_{12} & T_{22} & 0 & \cdots & 0 \\ \vdots & & & & \\ T_{1J} & \cdots & & \cdots & T_{JJ} \end{bmatrix}_{I \times I}$$

$$G(\tau) = \begin{bmatrix} g(t_1, \tau, \mu_1) \\ g(t_1, \tau, \mu_2) \\ \vdots \\ g(t_1, \tau, \mu_J) \\ \vdots \\ g(t_I, \tau, \mu_1) \\ g(t_I, \tau, \mu_J) \end{bmatrix}_{IJ \times 1} \cdots$$

and

$$U^{\pm}(\tau) = \begin{bmatrix} I(t_1, \tau, \pm\mu_1) \\ \vdots \\ I(t_1, \tau, \pm\mu_J) \\ \vdots \\ I^{\pm}(t_I, \tau, \mu_1) \\ \vdots \\ I(t_I, t, \mu_J) \end{bmatrix}_{IJ \times 1} \tag{2.35}$$

One can write the system of equations (2.34) as
(By measuring $+\mu$ in the increasing τ direction)

$$M\frac{\partial U^{+}}{\partial \tau}+U^{+}=0.5\left[TC\left(U^{+}+U^{-}\right)\right]-G(\tau)$$

$$-M\frac{\partial U^{-}}{\partial \tau}+U^{-}=0.5\left[TC\left(U^{+}+U^{-}\right)\right]-G(\tau)$$

(2.36)

we shall integrate equations (2.36) from τ_n to τ_{n+1} and write their corresponding discrete equivalents as

$$M\left[U_{n+1}^{+}-U_{n}^{+}\right]+\tau_{n+\frac{1}{2}}U_{n+\frac{1}{2}}^{+}=0.5\left[TC\left(U_{n+\frac{1}{2}}^{+}+U_{n+\frac{1}{2}}^{-}\right)\right]-G\left(\tau_{n+\frac{1}{2}}\right)$$

$$-M\left[U_{n+1}^{-}-U_{n}^{-}\right]+\tau_{n+\frac{1}{2}}U_{n+\frac{1}{2}}^{-}=0.5\left[TC\left(U_{n+\frac{1}{2}}^{+}+U_{n+\frac{1}{2}}^{-}\right)\right]-G\left(\tau_{n+\frac{1}{2}}\right)$$

where

$$\tau_{n+\frac{1}{2}}=\tau_{n+1}-\tau_{n} \qquad (2.37)$$

We shall use the diamond scheme to approximate the quantities $U^{\pm}_{n+\frac{1}{2}}$ as

$$U^{\pm}_{n+\frac{1}{2}} = 0.5\left[U^{\pm}_{n+1} + U^{\pm}_{n}\right] \tag{2.38}$$

Then equations (2.37) become,

$$M\left[\overset{+}{U}_{n+1} - \overset{+}{U}_{n}\right] + 0.5\,\tau_{n+\frac{1}{2}}\left(\overset{+}{U}_{n+1} + \overset{+}{U}_{n}\right) = 0.25\,TC\left(\overset{+}{U}_{n+1} + \overset{+}{U}_{n}\right)\tau_{n+\frac{1}{2}}$$

$$+0.25\,TC\left(\bar{U}_{n+1} + \bar{U}_{n}\right)\tau_{n+\frac{1}{2}} - G_{n+\frac{1}{2}}\,\tau_{n+\frac{1}{2}} \tag{2.39}$$

$$-M\left[\bar{U}_{n+1} - \bar{U}_{n}\right] + 0.5\,\tau_{n+\frac{1}{2}}\left(\bar{U}_{n+1} + \bar{U}_{n}\right) = 0.25\,TC\left(\bar{U}_{n+1} + \bar{U}_{n}\right)\tau_{n+\frac{1}{2}}$$

$$+0.25\,TC\left(\overset{+}{U}_{n+1} + \overset{+}{U}_{n}\right)\tau_{n+\frac{1}{2}} - G_{n+\frac{1}{2}}\,\tau_{n+\frac{1}{2}} \tag{2.40}$$

Rearranging the terms, we get

$$\left[M + 0.5\,\tau_{n+\frac{1}{2}}\,(I - 0.5\,TC)\right] U^{+}_{n+1} - 0.25\,TC\,U^{-}_{n}\,\tau_{n+\frac{1}{2}}$$

$$= 0.25\,TC\,U^{-}_{n+1}\,\tau_{n+\frac{1}{2}} + \left[M - 0.5\,\tau_{n+\frac{1}{2}}\,(I - 0.5\,TC)\right] U^{+}_{n} - G_{n+\frac{1}{2}}\,\tau_{n+\frac{1}{2}}$$

$$\left[M + 0.5\,\tau_{n+\frac{1}{2}}\,(I - 0.5\,TC)\right] U^{-}_{n} - 0.25\,TC\,U^{+}_{n+1}\,\tau_{n+\frac{1}{2}}$$

$$= 0.25\,TC\,U^{+}_{n}\,\tau_{n+\frac{1}{2}} + \left[M - 0.5\,\tau_{n+\frac{1}{2}}\,(I - 0.5\,TC)\right] U^{-}_{n+1} - G_{n+\frac{1}{2}}\,\tau_{n+\frac{1}{2}} \tag{2.41}$$

Here I is the identity matrix of appropriate dimension.

It is now straight forward to put these equations in the canonical form

$$\begin{bmatrix} U^{+}_{n+1} \\ U^{-}_{n} \end{bmatrix} = \begin{bmatrix} t(n+1,n) & r(n,n+1) \\ r(n+1,n) & t(n,n+1) \end{bmatrix} \begin{bmatrix} U^{+}_{n} \\ U^{-}_{n+1} \end{bmatrix} + \begin{bmatrix} \Sigma^{+}_{n+\frac{1}{2}} \\ \Sigma^{-}_{n+\frac{1}{2}} \end{bmatrix} \tag{2.42}$$

Now we shall express the r and t matrices in terms of the following auxiliary matrices.

$$Q^{+}_{n+\frac{1}{2}} = 0.5\, TC$$

$$\overset{+}{S} = M - 0.5\,\tau_{n+\frac{1}{2}}\left(I - Q^{+}_{n+\frac{1}{2}}\right)$$

$$\overset{-}{S} = 0.5\,\tau_{n+\frac{1}{2}}\, Q^{+}_{n+\frac{1}{2}}$$

and

$$\Delta^{+} = \left[M + \tfrac{1}{2}\tau_{n+\frac{1}{2}}\left(I - Q^{+}_{n+\frac{1}{2}}\right)\right]^{-1}$$

$$r^{+} = \Delta^{+} S^{-} \quad ; \quad t^{+} = \left[I - r^{+} r^{+}\right]^{-1}$$

i.e.

$$t(n+1, n) = t^{+}\left[\Delta^{+} S^{+} + r^{+} r^{+}\right]$$

$$t(n, n+1) = t\,(n+1, n)$$

$$r(n+1, n) = 2\, t^{+} r^{+} \Delta^{+} M$$

and

$$r(n, n+1) = r(n+1, n)$$

$$\Sigma^{+}_{n+\frac{1}{2}} = \tau_{n+\frac{1}{2}}\, t^{+}\left[\Delta^{+} G^{+} + r^{+} \Delta^{+} G^{+}\right]$$

$$\Sigma^{-}_{n+\frac{1}{2}} = \Sigma^{+}_{n+\frac{1}{2}} \tag{2.43}$$

From physical considerations, we know that reflection and transmission operators r and t must be non-negative. For this we need $\Delta^{+} \geqslant 0$, $S^{+} \geqslant 0$ and we can achieve this if

$$\tau \leq \tau_{crit} = \min_j \left| \frac{\mu_j}{\frac{1}{2}\left(1 - T_{jj} c_{jj}\right)} \right| \qquad (2.44)$$

So with the help of these transmission and reflection matrices, one can obtain the radiation field as described previously.

Specific cases considered:-

The transfer equation is solved with different types of boundary conditions. The various cases considered are given below.

Case I : Two stream approximation with an incident pulsed beam on the lower boundary of the atmosphere i.e.

$$I(t, \tau = T, -\mu = 1) = \delta(t)$$

where $\delta(t)$ is Dirac-Delta function.

$$I(t, \tau = 0, +\mu = 1) = 0 \qquad (2.45)$$

To avoid including the highly singular function (the δ - function distribution) in the equation, we distinguish the diffuse field due to one or more scatterings from the reduced incident field without scatterings. The equation of transfer for the diffuse field is

$$\pm \frac{\partial I^{\pm}(t,\tau)}{\partial \tau} + I^{\pm}(t,\tau) = 0.5 \int_0^t e^{-(t-t')/t_1} (I^+ + I^-) \frac{dt'}{t_1} + 0.5 \frac{e^{-t/t_1}}{t_1} \cdot e^{-(T-\tau)} \qquad (2.46)$$

where + and - denote the two oppositely travelling beams of radiation.

Case II : If we consider the full angular scattering of the radiation, we have for the diffuse intensity

$$\pm\mu \frac{\partial I^{\pm}(t,\tau,\mu)}{\partial\tau} + I^{\pm}(t,\tau,\mu)$$

$$= 0.5\int_{0}^{t} e^{-(t-t')/t_1} \int_{-1}^{+1} I(t',\tau,\mu')\frac{dt'}{t_1}d\mu' + \frac{0.5}{t_1} e^{-t/t_1}\int_{0}^{1} e^{-(T-\tau)/\mu'} d\mu' \tag{2.47}$$

with the boundary conditions

$$I^{-}(t, \tau = T, \mu) = 0$$

$$I^{+}(t, \tau = 0, \mu) = 0 \qquad 0 < \mu < 1$$

Case III : Search light beam with Dirac-delta time distribution is also considered. The incident field is given by

$$I^{-}(t, \tau = T, \mu) = \delta(t)\,\delta(\mu - \mu_0)$$

$$I^{+}(t, \tau = 0, \mu) = 0 \tag{2.43}$$

The transfer equation for diffuse intensities in this case is

$$\pm \mu \frac{\partial I^{\pm}}{\partial \tau}(t,\tau,\mu) + I^{\pm}(t,\tau,\mu)$$
$$= \frac{0.5}{t_1} \int_0^t e^{-(t-t')/t_1} \int_{-1}^{+1} I(t',\tau,\mu') \, d\mu \, dt'$$
$$+ \frac{0.5}{t'} e^{-t/t_1} \cdot e^{-(T-\tau)/\mu_0} \tag{2.49}$$

with

$$I^{-}(t,\tau=T,\mu) = 0$$
$$I^{+}(t,\tau=0,\mu) = 0$$

For checking the numerical results, we can make use of the following relations

Denote

$$\int_0^{\infty} I^{\pm}(t',\tau,\mu') \, dt' = \tilde{I}^{\pm}(\tau,\mu) \tag{2.50}$$

By integrating equations (2.46), (2.47) and (2.49) with respect to time from 0 to ∞ , we obtain the following steady state equation for various cases.

For the case I,

$$\pm \frac{\partial \tilde{I}^{\pm}}{\partial \tau} + \tilde{I}^{\pm} = 0.5\left(\tilde{I}^{+} + \tilde{I}^{-}\right) + 0.5\, e^{-(T-\tau)} \tag{2.51}$$

with

$$\tilde{I}^{+}(\tau=0) = 0$$
$$\tilde{I}^{-}(\tau=T) = 0$$

For the case II, we have

$$\pm\mu \frac{\partial \tilde{I}^{\pm}}{\partial \tau} + \tilde{I}^{\pm} = 0.5 \int_{-1}^{+1} \tilde{I}(\tau,\mu')\, d\mu' + 0.5 \int_{0}^{1} e^{-(T-\tau)/\mu'}\, d\mu' \tag{2.52}$$

with

$$\tilde{I}^{+}(\tau=0,\mu) = 0$$

$$\tilde{I}^{-}(\tau=T,\mu) = 0$$

for the case III,

$$\pm\mu \frac{\partial I^{\pm}}{\partial \tau} + \tilde{I}^{\pm} = 0.5 \int_{-1}^{+1} \tilde{I}(\tau,\mu')\, d\mu' + 0.5\, e^{-(T-\tau)/\mu_0} \tag{2.53}$$

with

$$\tilde{I}^{+}(\tau=0, \mu) = 0$$

$$\tilde{I}^{-}(\tau=T, \mu) = 0$$

The equations (2.46), (2.47), and (2.49) and the corresponding steady state equations (2.51),(2.52) and (2.53) are solved. The steady state solutions are also obtained by the method of Grant and Hunt (1969a). The time dependent solution I(t, τ, μ) is integrated with respect to time and checked against the steady state solution as given below.

$$\int_{0}^{\infty} I^{\pm}(t', \tau, \mu') \, dt' = I^{\pm}(\tau, \mu) \qquad (2.54)$$

The maximum deviation from the steady state value is 15% which is fact that the above time integration is truncated at a finite time limit.

Case IV: We also considered the situation where the incident radiation distribution with time is given by Heaviside unit step function H (t).

(i) $$\pm \frac{\partial I}{\partial \tau}(t,\tau) = I^{\pm}(t,\tau) - \frac{0.5}{t_1}\int_0^t e^{-(t-t')/t_1} \left(\tilde{I}^{+}(t',\tau) + \tilde{I}^{-}(t',\tau)\right) dt' \qquad (2.55)$$

With

$$I^{-}(t, \tau = T) = H(t)$$

$$I^{+}(t, \tau = 0) = 0$$

where $H(t) = 0$

for $t < 0$

$H(t) = 1$ for $t \geq 0$

(ii) $$\pm \mu \frac{\partial I^{\pm}}{\partial \tau} = I^{\pm}(t,\tau,\mu) - 0.5\int_0^t e^{-(t-t')/t_1}\int_{-1}^{+1} I(t',\tau,\mu')\, \frac{d\mu' dt'}{t_1}$$

$0 < \mu < 1$

with

$$I^{-}(t, \tau = T, \mu) = H(t)$$

$$I^{+}(t, \tau = 0, \mu) = 0 \qquad (2.56)$$

As intensity distributions reach steady state after sufficiently long time, one can check the steady state values from the following equations

$$\pm \frac{\partial \tilde{I}}{\partial \tau} = I^{\pm}(\tau) - 0.5\left(I^{+} + I^{-}\right)$$

with $I^{-}(\tau = T) = 1$

$I^{+}(\tau = 0) = 0$

and

$$\pm \mu \frac{\partial I^{\pm}}{\partial \tau} = I^{\pm}(\tau, \mu) - 0.5 \int_{-1}^{+1} I(\tau, \mu') \, d\mu'$$

with

$I^{+}(\tau = 0, \mu) = 0$

$I^{-}(\tau = T, \mu) = 1$ $\qquad 0 < \mu < 1$

Let t_1 denote the time spent by the photon in the absorbed state and t_2 is the time spent by the photon between two successive acts of scatterings.

Van de Hulst and Irvine (1963) pointed out that the non-stationary problem for $t_2 = 0$ (infinite velocity of propogation) and $t_1 \neq 0$ (time spent by the photon in the absorbed state is significant) is equivalent to the problem of finding the distribution of photon over the number of scatterings.

2.4 Results and Discussion

The numerical results are displayed in graphical forms for all the cases. In all the cases, we have assumed $t_1 = 1.0$.

Figs 1 and 2 illustrate the reflected intensity distributions for the two stream approximation when the medium is illuminated by a pulsed beam. The reflection function due to Sobolev (1963) for the semi-infinite medium is also plotted in Fig.2. Reflected radiation starts at time t = 0 with the value $0.25\,[1-e^{-2T}]$. We see that it falls more rapidly for T = 1. Also for the semi-infinite medium the radiation drops down gradually compared to a medium with total optical depth T = 2. This is because the photon spends more time in a medium with higher optical depth. Diffuse emergent intensities are plotted in Fig.3 and 4

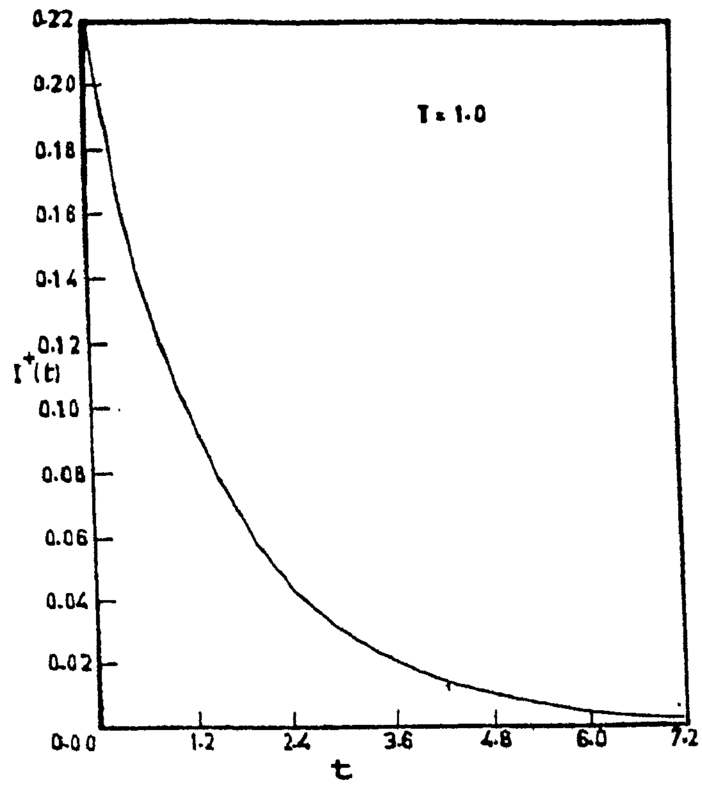


Fig. 1

Reflected intensity distribution when the slab is subjected to a pulse for T = 1.0

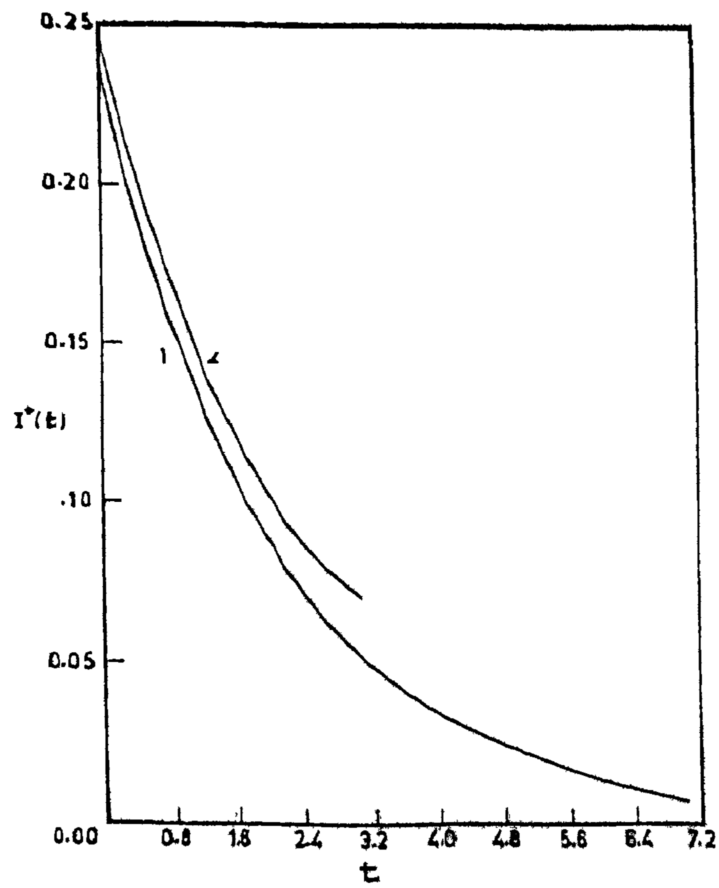
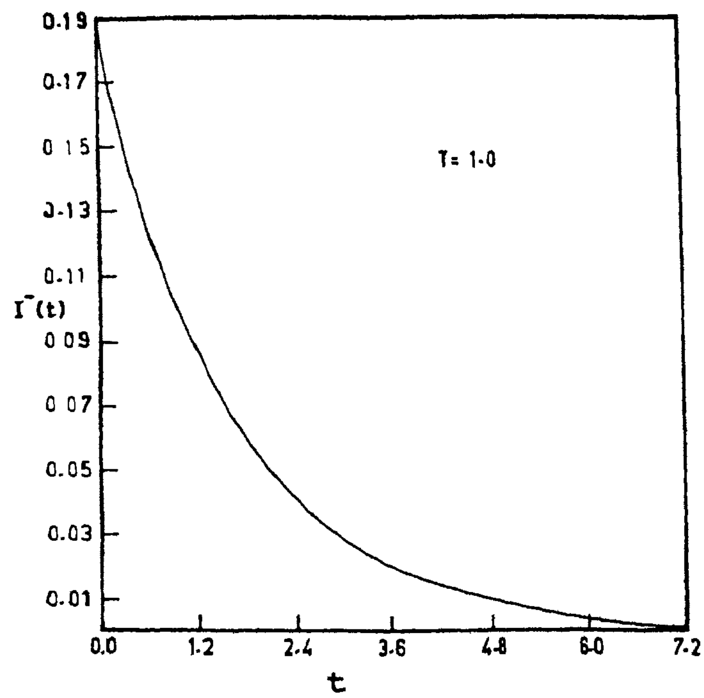


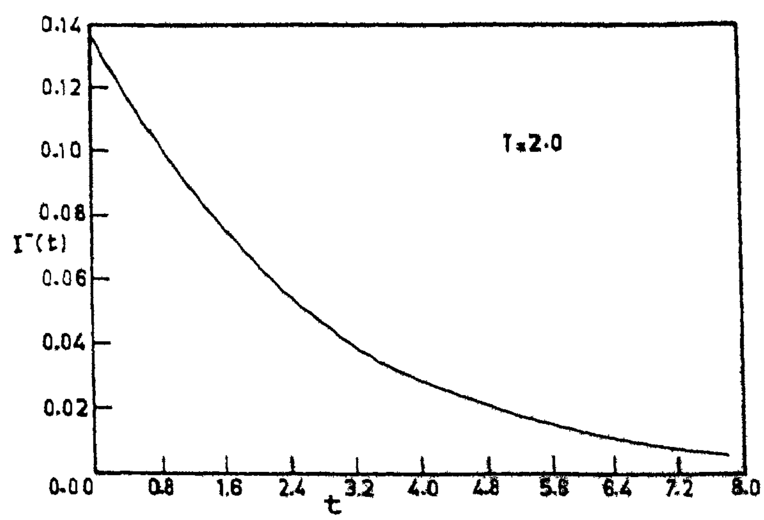
Fig. 2

Same as in Fig. 1 numbers denotes the following cases: (1) T = 2.0; (2) semi-infinite medium.



<u>**Fig.3**</u>

Emergent intensity distribution for the same case as in Fig.2 for T = 1.0



<u>**Fig.4.**</u>

Same as in Fig.3 for T = 2.0

respectively. At time $t = 0$, the intensities starts with the value $0.5\ T e^{-T}$. We see that the time at which it falls by $\frac{1}{e}$ is 1.6 and 2.6 for $T = 1$ and $T = 2$ respectively. From this it is evident that emergent intensity for the medium with $T = 1$ decays faster in comparision to the medium with $T = 2$. Similar interpretation can be given as in the case of reflected intensities.

When isotropic pulsed beam of radiation falls on the medium, the emergent and reflected radiations for optical depth $T = 1$ are displayed in Figs.5 and 6 respectively. A photon reflected at the grazing angle ($\mu = 0.2$) can be regarded as coming from the shallow layers of the medium. Hence it experier few scatterings and spends short time before it reappeares on the surface. On the other hand a photon reflected at an angle nearer to the normal ($\mu = 0.7$) can be regarded as coming from the deeper layers of the medium. Hence it experiences more number of scatterings and spends long time till its reappearance. Due to this reason, the reflected radiation for $\mu = 0.2$ falls more rapidly with time compared to that for $\mu = 0.7$. But the photons emerging from the atmosphere along the direction $\mu = 0.7$,

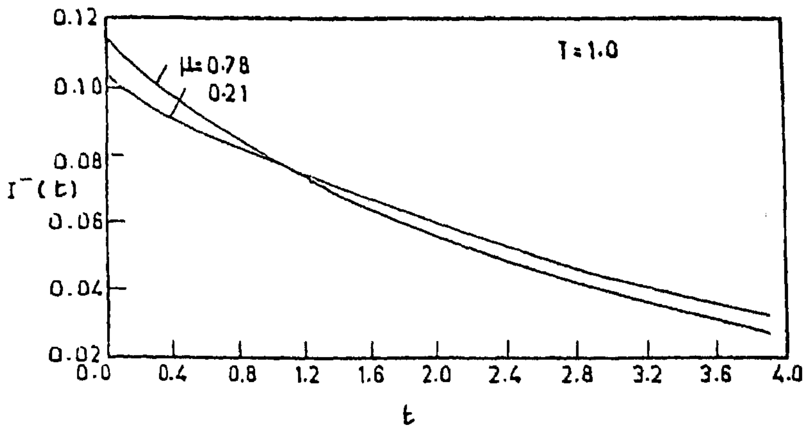


Fig.5

Emergent intensity distribution when the slab subjected to an isotropic pulse for T = 1.0

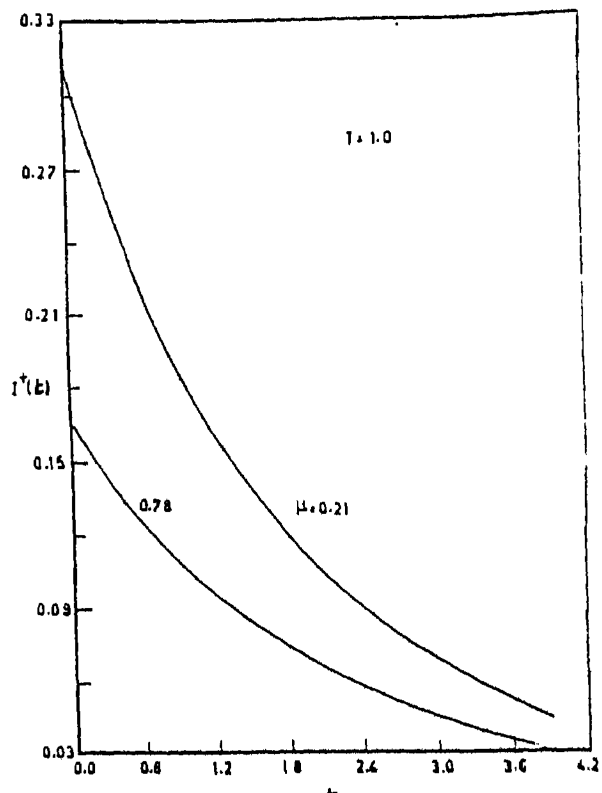
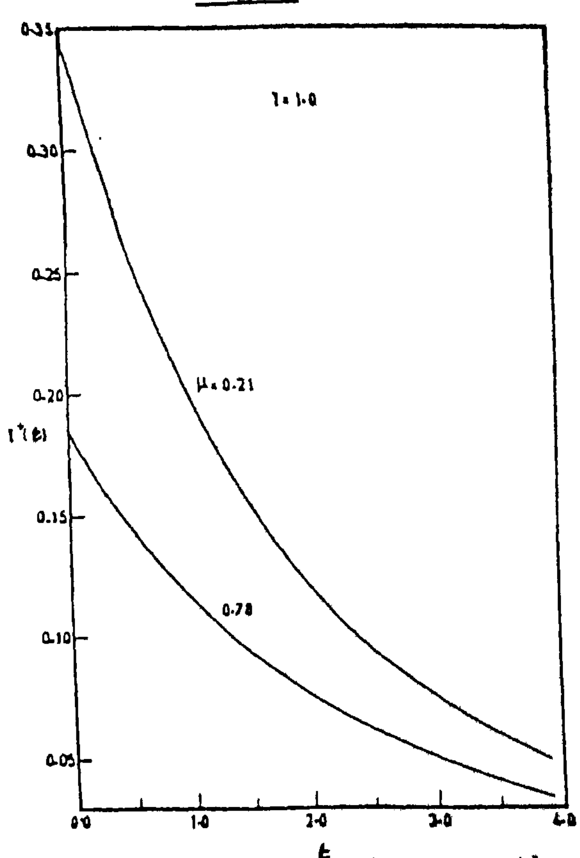


Fig.6

Reflected intensity distribution for the same case in Fig.5.

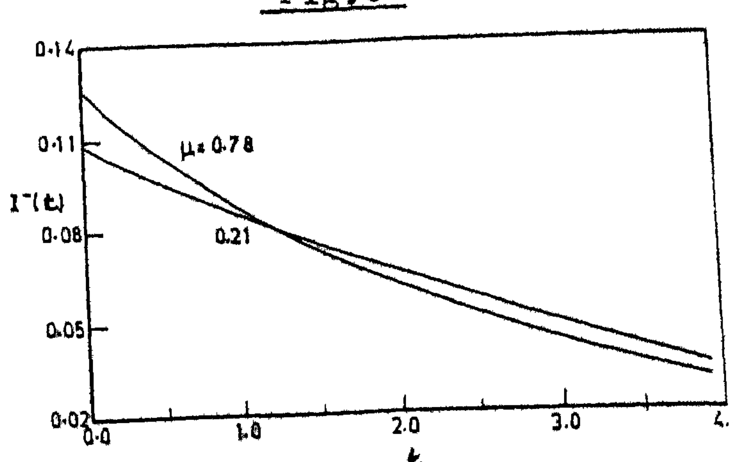
Fig.7



Same as in fig.6 when the slab is subjected to a pulsed beam in a specific direction

$\mu_0 = 0.5$

Fig.8



Emergent intensity distribution for the same case as in fig.7.

experience few scatterings which results in the slow dropping of the emergent intensity distribution for $\mu = 0.7$.

When the pulsed beam in a specified direction angle $\mu_o = 0.5$ is incident on the atmosphere, the corresponding reflected and emergent intensities are displayed in Figs.7 and 8. Even though the decay of the reflected radiation for $\mu = 0.2$ is almost identical to the previous case, we find for $\mu = 0.7$ there is some slight difference.

We also considered a medium illuminated by a constant input of radiation. Once the radiation field reaches steady state, the illumination is cut off. The results are shown in Figs.9 and 10.

Now we shall discuss the cases where the medium is illuminated by a constant radiation starting from time $t = 0$.

When the medium is illuminated by a constant radiation field of intensity 1 from time $t = 0$, the emergent intensities are plotted in Fig.11 and 13 for optical depths $T = 1$ and $T = 2$. Reflected intensities are depicted in Fig. 12 & 14. Emergent intensities start with the value e^{-T} and reach steady state after few time units. At time $t = 0$, the integral in the transfer equation vanishes and the formal solution which we get is $I_o e^{-T}$ for the emergent intensity, where I_o isthe initial condition.

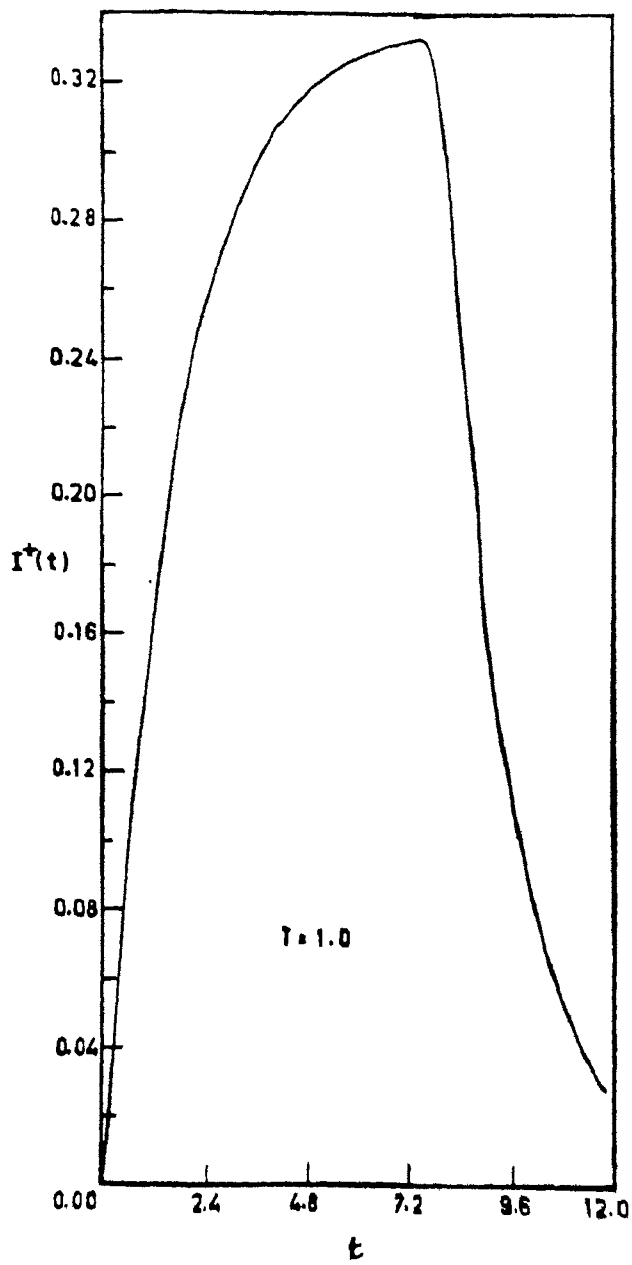


Fig.9

Reflected intensity distribution when the constant source of illumination is cut off.

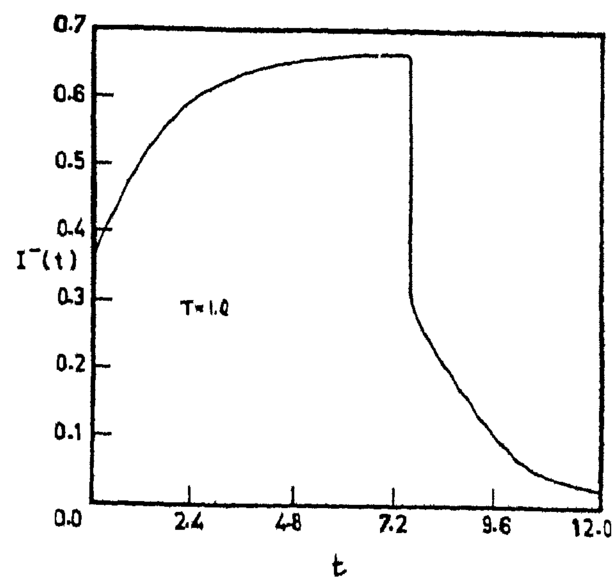


Fig.10

Emergent intensity distribution for the same case as in Fig.9.

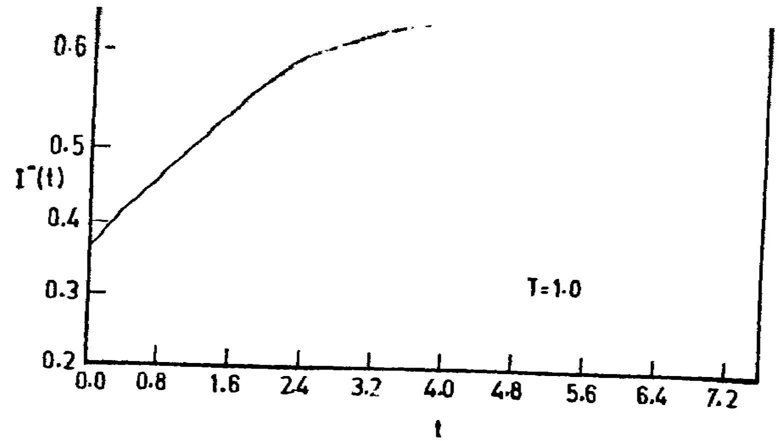


Fig.11

Emergent intensity distribution when the slab is subjected to a constant radiation for T = 1.0.

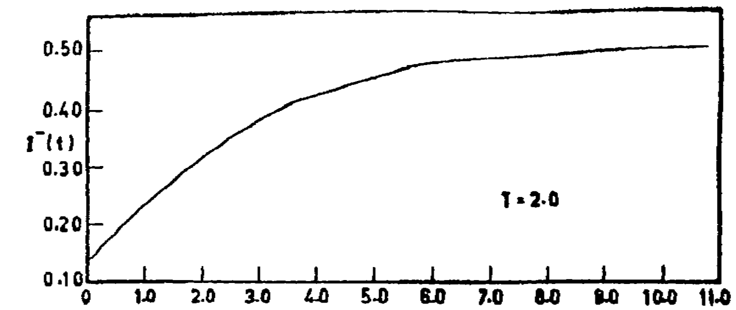


Fig.13

Same as in Fig.11 for T = 2.0

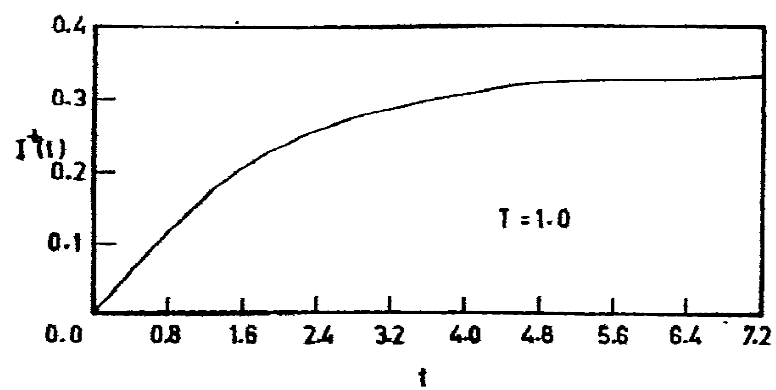


Fig.12.

Reflected intensity distribution for the same case as in Fig.11.

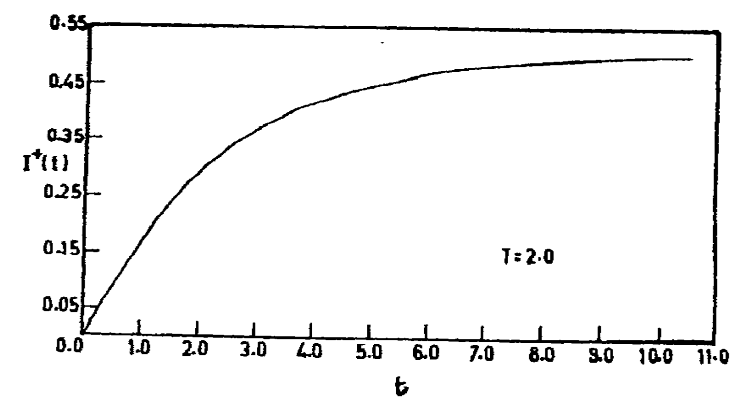


Fig.14.

Same as in Fig.13 for T = 2.0

Time during which the relaxation occurs is roughtly twice for the medium with $T = 2$ compared to that of the medium with $T = 1$.

The angular dependence of the emergent intensities for $T = 0.5$ and $T = 2.0$ are exhibited in Figs.15 and 17, while the reflected intensities are plotted in Figs.16 and 18. If we consider the relative reflected intensity $I(t,0,\mu)/\tilde{I}(0,\mu)$ ($\tilde{I}(0,\mu)$ is the steady state value) the convergence to unity is faster for $\mu = 0.2$ compared to that for $\mu = 0.7$. Similarly, when we consider the relative emergent intensity, the ratio approaces unity faster for $\mu = 0.7$ compared to the case $\mu = 0.2$. This can be explained by the fact that reflected photons coming in the direction $\mu = 0.2$ experience few scatterings and hence spends less time in the medium. Similarly, the emergent photons coming in the direction $\mu = 0.7$ experience few scattering and reach steady state faster. These are in quantitative agreement with that of Matsumoto (1974). Also one can see that for $T = 2.0$ the emergent intensity for $\mu = 0.7$ and reflected intensity for $\mu = 0.2$ show steeper variation with time compared to $T = 0.5$ case.

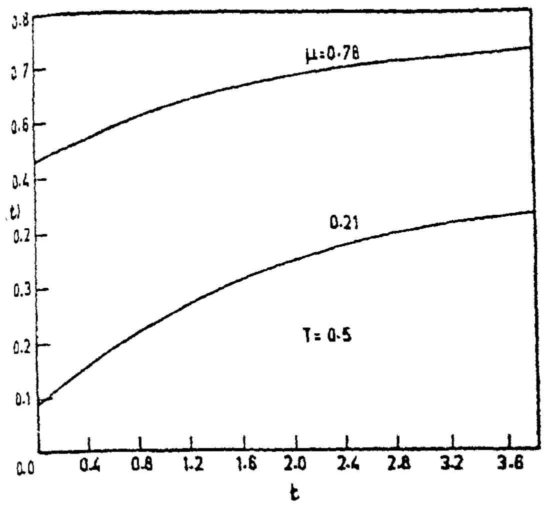


Fig.15.

Emergent intensity distribution when the slab is subjected to constant isotropic radiation for T = 0.5

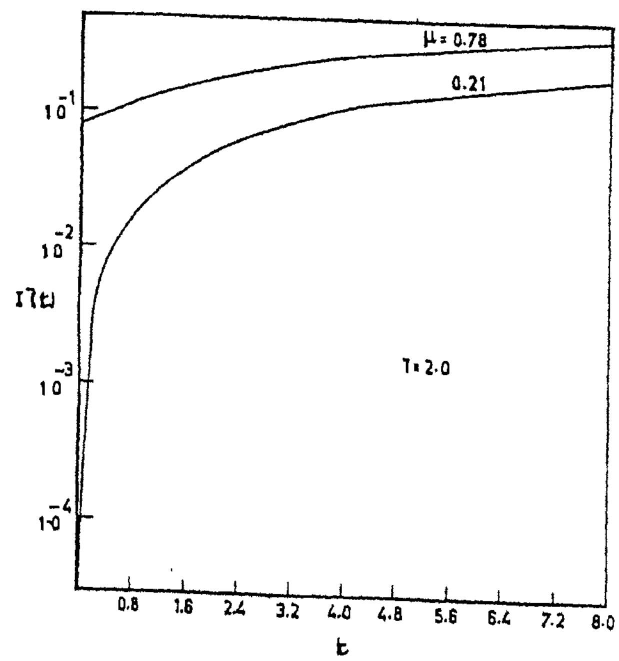


Fig.17

Same as in Fig.15 for T = 2.0

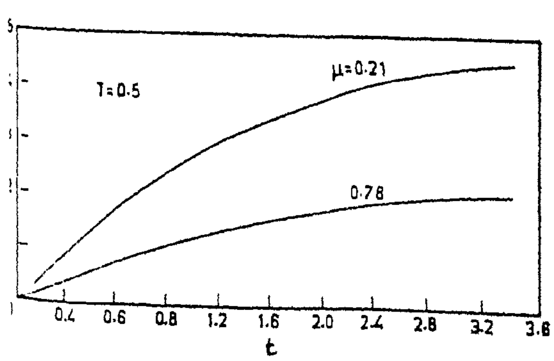


Fig.16.

Reflected intensity distribution for the same case as in fig.15.

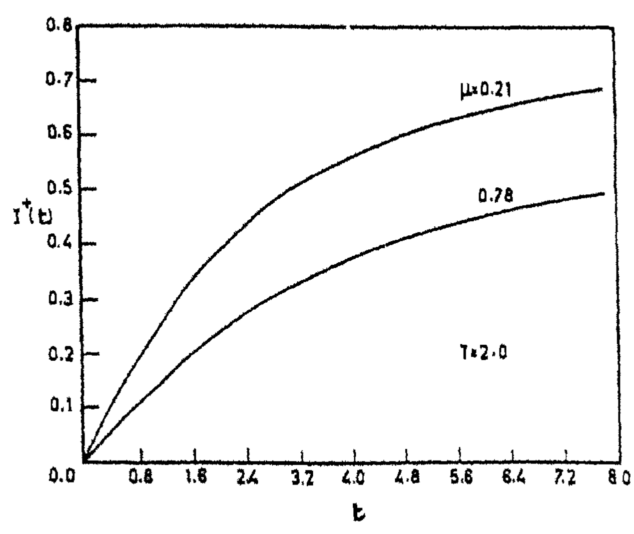


Fig.18

Same as in Fig.17 for T = 2.0

CHAPTER III

A NUMERICAL SOLUTION FOR THE TIME DEPENDENT TRANSFER EQUATION

3.1 Introduction

In this chapter, we shall present the numerical method for the problem of time-dependent transfer in a finite slab in which the material density is sufficiently low so that the time spent by the photon between scatterings exceeds the time spent by the photon in the absorbed state. We have studied the monochromatic transfer problem for a homogeneous slab which is illuminated by a constant beam of radiation from time $t = 0$. We solved the problem when a pulse of radiation (a δ - function in time) impinges on the slab under the two stream approximation. Time-dependent transfer of resonance lines under the assumption of complete redistribution is also investigated.

The factor $\frac{1}{\alpha c}$, where α denotes the absorption coefficient and c, the velocity of light, is the time spent by the photon between emission and reabsorption. Also $1/\alpha c$ connects the time derivative with other terms in the transfer equation. If it is long compared to the typical time scales in which the atmospheric system is changed, the time derivative $\partial I/\partial t$ is important. Klein et al (1976) considered thermal relaxation time which is given by the ratio of the internal energy per unit volume to the emission per unit volume of a gas. They showed that the ratio of thermal relaxation time to the photon's time of flight is less than unity for the typical densities in planetary and crab nebula suggesting that the time derivative may be important in these class of objects.

Bellman et al (1964) solved the time dependent transfer problem by the theory of invariant imbedding and Laplace transform technique. They obtained diffuse reflection function for a finite slab whose surface is irradiated with a constant net flux of radiation. With the aid of time dependent principle of invariance and the inverse method of Bellman (1966), Matsumoto (1974) obtained the solution for semi-infinite homogeneous media by taking into account both $t_1, t_2 \neq 0$ (t_1 is the time spent by a photon in the absorbed state; t_2 is the time spent by the photon between two successive acts of scatterings). Later, he (1976) obtained a convergent series solution by using Laplace transform and the theory of order of scattering developed by Uesugi and Irvine (1970). The series solution is seperable into a time like factor and an angle factor. The angular factors are identical to those developed by Uesugi and Irvine. Ganapol (1981) developed a time-dependent solution directly from stationary solution. Kunasz (1983) proposed an implicit finite difference method for time-dependent line transfer problem.

Recently, Ganapol (1986) presented results for the reflected photon intensity from an anisotropically scattering semi-infinite medium taking into account a mean free time between scatterings and a mean time of temporal capture.

Many different numerical schemes have been proposed for solving the time-dependent transfer problems (see Richmeyer and Morton, 1967). Keller and Wendroff (1957) proposed characteristic S_n method to solve the transfer equation in spherical geometry. Our method presented is similar to their's.

In section 3.2, we outline the method of solution for the monochromatic transfer problem and in section 3.3, we discuss the extension of the method to time-dependent line transfer. The results and discussion are presented in section 3.4.

3.2 Method of Solution

The monochromatic time-dependent transfer equation for the specific intensity $I(Z,\mu,t)$ in slab geometry is given by,

$$\frac{1}{c}\frac{\partial I}{\partial t}(z,\mu,t) + \mu\frac{\partial I}{\partial z}(z,\mu,t) + \alpha(z,t)\,I(z,\mu,t)$$

$$= \alpha(z,t)\left[\frac{\omega(z,t)}{2}\int_{-1}^{+1} P(\mu,\mu')\,I(z,\mu',t)\,d\mu' + (1-\omega)\,B(z,t)\right] \qquad 0<\mu<1$$

and for the oppositely directed beam,

$$\frac{1}{c}\frac{\partial I}{\partial t}(z,-\mu,t) + \mu\frac{\partial I(z,-\mu,t)}{\partial z} + \alpha(z,t)\,I(z,-\mu,t)$$

$$= \alpha(z,t)\left[\frac{\omega(z,t)}{2}\int_{-1}^{+1} P(-\mu,\mu')\,I(z,\mu',t)\,d\mu' + (1-\omega)\,B(z,t)\right] \qquad 0<\mu<1 \quad (3.1)$$

where $\alpha(z,t)$ is the absorption coefficient at spatial coordinate z and time t $\omega(z,t)$is the albedo for single scattering and c is the velocity of light. The phase function $P(\mu,\mu')$ gives the probability that a photon travelling in the direction μ is scattered into the direction μ'

We assume that there is no radiation incident on the top of the atmosphere and initially at time $t = 0$, there is no radiation present within the medium. Then the initial and boundary conditions for the equations (3.1) are given by

$$I(z=0, \mu, t) = f(\mu, t)$$

$$I(z=T, -\mu, t) = 0$$

$$I(z, \pm\mu, t=0) = 0 \quad \text{for } 0 < z < T, \; 0 < \mu < 1 \qquad (3.2)$$

where T is the total geometrical depth of the medium and $f(\mu, t)$ is a given function of μ and t.

We shall approximate the angular integral in equation (3.1) as

$$\int_0^1 P(\mu, \mu') I(z, \mu', t)\, d\mu' \simeq \sum_{j=1}^{J} P(\mu, \mu_j) W_j I(z, \mu_j, t) \qquad (3.3)$$

where W_j and μ_j are the weights and roots of the Gauss-Legendre quadrature of order J.

Incorporating equation (3.3) in equation (3.1) we get

$$\frac{1}{c}\frac{\partial I}{\partial t}(z,\pm\mu_j,t) \pm \mu_j \frac{dI}{dz}(z,\pm\mu_j,t) + \alpha(z,t)\, I(z,\pm\mu_j,t)$$

$$= \Bigg[\alpha(z,t) \Bigg\{ \frac{\omega}{2}(z,t) \sum_{j'=1}^{J} \Big[P(\pm\mu_j,\mu_{j'})\, I(z,\pm\mu_{j'},t)$$

$$+ P(\pm\mu_j,-\mu_{j'})\, I(z,-\mu_{j'},t) \Big] W_{j'}$$

$$+ (1-\omega)\, B(z,t) \Bigg\} \Bigg] \qquad j = 1, 2, \cdots, J \qquad (3.4)$$

In these problems, it is convenient to distinguish between the reduced incident radiation field $g(z,t)$ at spatial position z and time t and the diffuse radiation field $I(z,\pm\mu,t)$ that results as a consequence of one or more scattering processes.

Then the equation (3.4) becomes,

$$\frac{1}{c}\frac{\partial I}{\partial t}(z,\pm\mu_j,t) + \mu_j \frac{dI}{dz}(z,\pm\mu_j,t) +$$

$$\alpha(z,t)\, I(z,\pm\mu_j,t)$$

$$= \alpha(z,t)\Big[\frac{\omega(z,t)}{2}\sum_{j'=1}^{J}\Big\{P(\pm\mu_j,\mu_{j'})\,I(z,\mu_{j'},t) + P(\pm\mu_j,-\mu_{j'})\,I(z,-\mu_{j'},t)\Big\}W_{j'} + [1-\omega(z,t)]\,B(z,t)\Big] + g(z,t) \qquad (3.5)$$

$$j = 1, 2, \cdots J$$

with initial and boundary conditions given by

$$I(z=0,\mu_j,t) = 0$$

$$I(z=T,-\mu_j,t) = 0 \qquad j = 1, 2, \cdots J$$

$$I(z,\mu_j,t=0) = 0 \qquad \text{for } 0 < z < T$$

(3.6)

The equations (3.5) form a hyperbolic system of first order linear partial differential equations. The characteristics are the straight lines in the z,t planes defined by

$$\frac{dz}{ds_j^{\pm}} = \pm\frac{\mu_j}{D_j}$$

(3.7)

$$\frac{dt}{ds_j^{\pm}} = \frac{1}{cD_j}$$

$$D_j = \sqrt{\mu_j^2 + \frac{1}{c^2}} \qquad j = 1, \cdots J$$

where s_j^+ represents the arc-length along the j^{th} characteristic with positive slope and s_j^- represent the arc-length along the j^{th} characteristic with negative slope. Denoting $I(z, \pm\mu_j, t) = I_j^{\pm}$ the equations (3.5) can be written as

$$D_j \frac{dI_j}{ds_j^+} + \alpha I_j^+ = \alpha S_j^+,$$

where

$$S_j^+ = \alpha \left\{ \frac{\omega}{2} \sum_{j'=1}^{J} \left[P(+\mu_j, \mu_{j'}) I(z, \mu_{j'}, t) + P(+\mu_j, -\mu_{j'}) I(z, -\mu_{j'}, t) \right] W_{j'} + (1-\omega) \quad B(z,t) \right\} + g(z,t)$$
$$j = 1, \cdots J \qquad (3.8)$$

and

$$D_j \frac{dI_j^-}{ds_j^-} + \alpha I_j^- = \alpha S_j^-,$$

where

$$S_j^- = \alpha \left\{ \frac{\omega}{2} \sum_{j'=1}^{J} \left[P(-\mu_j, \mu_{j'}) I(z, \mu_{j'}, t) + P(-\mu_j, -\mu_{j'}) I(z, -\mu_{j'}, t) \right] W_{j'} + (1-\omega) \quad B(z,t) \right\} + g(z,t) \qquad j = 1, \cdots J \qquad (3.9)$$

where $\alpha = \alpha(z,t)$ and $\omega = \omega(z,t)$

We shall divide the medium into N layers of equal depth ΔZ. Also the time domain is divided into equal intervals of time duration Δt. With this, we can construct a two dimensional mesh in space and time as shown in Fig.1(a)

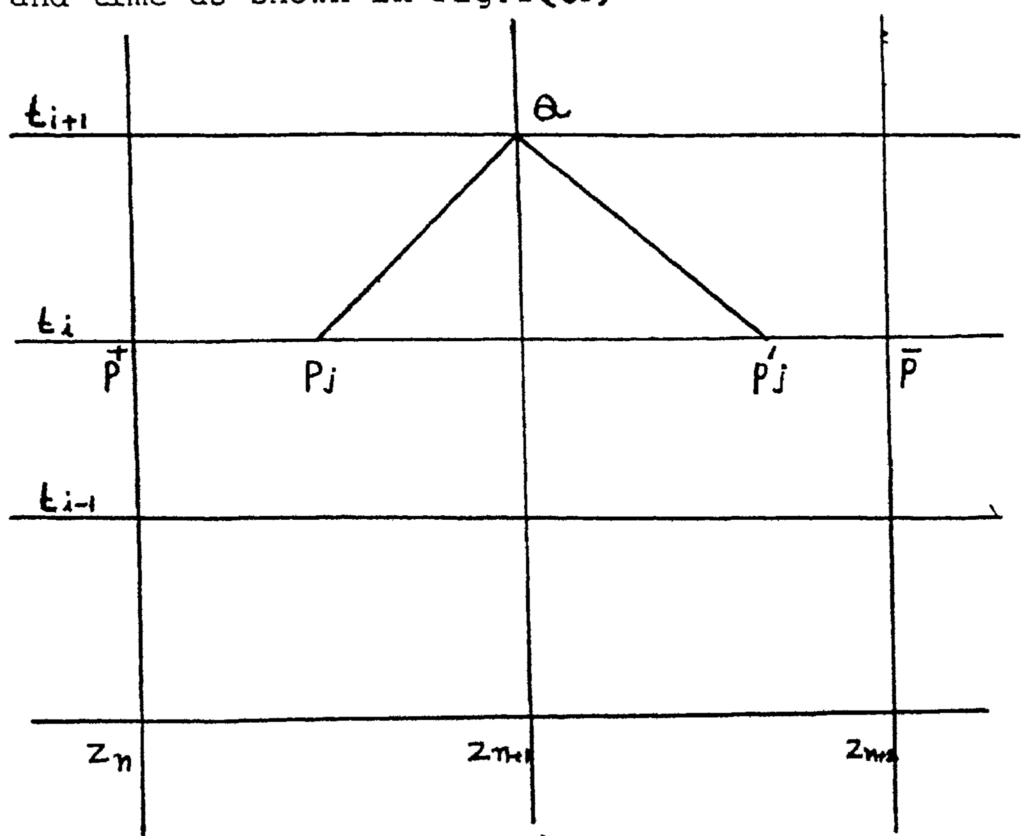


FIG 1(a)

Δt and Δz are chosen such that

$$|\mu_j|\, c\Delta t \leq \Delta Z \quad \text{for all} \quad j = 1, 2, \cdots J$$

We identify Q with an arbitrary point $Q \equiv (Z_{n+1}, t_{i+1})$. Through Q draw a characteristic line whose slope is $\frac{1}{\mu_j c}$. The first intersection of the j^{th} back characteristic with the horizontal mesh line is given by the point $P_j\,(Z_{n+1} - \mu_j c\Delta t,\ t_i)$

Similarly, if we draw through Q a characteristic whose slope is $-\frac{1}{\mu_j C}$, it intersects the mesh line at the point $P_j'(Z_{n+1}+\mu_j C\Delta t, t_i)$

Integrating equation (3.8) along the j^{th} characterisitc from P_j to Q we get

$$\left[I_j^+(Q) - I_j^+(P_j)\right] + \int_{P_j}^{Q} \alpha\, I^+ \frac{ds_j^+}{D_j} = \int_{P_j}^{Q} \alpha\, S_j^+ \frac{ds_j^+}{D_j}$$

(3.10)

Using the modified trapezoidal rule to approximate the integrals we can write equation (3.10) as

$$\left[I_j^+(Q) - I_j^+(P_j)\right] + \frac{\tilde{\alpha}}{2}\left[I_j^+(Q) + I_j^+(P_j)\right]\frac{ds_j^+}{D_j}$$

$$= \frac{\tilde{\alpha}}{2}\left[S_j^+(P_j) + S_j^+(Q)\right]\frac{ds^+}{D_j}$$

(3.11)

where

$$\tilde{\alpha} = \frac{\alpha(Q) + \alpha(P_j)}{2}$$

Similarly integrating equation (3.9) along the j^{th} characteristic from P_j' to Q, we get

$$\left[I_j^-(Q) - I_j^-(P_j') \right] + \frac{\tilde{\alpha}}{2}\left[I_j^-(Q) + I^-(P_j') \right] \frac{ds_j^-}{D_j}$$

$$= \frac{\tilde{\alpha}}{2}\left[S_j^-(P_j') + \bar{S}_j(Q) \right] \frac{ds_j^-}{D_j}$$

where $$\tilde{\alpha} = \frac{\alpha(Q) + \alpha(P_j')}{2} \qquad (3.12)$$

since $\mu_j c \frac{\Delta t}{\Delta z} \leq 1$ we have $\frac{ds_j^{\pm}}{D_j} = c\Delta t$

We can use a formula for linear interpolation to approximate the quantities $I_j^+(P_j)$ and $I_j^-(P_j')$

$$I_j^+(P_j) = I_j^+(z_{n+1} - \mu_j c\Delta t, t_i)$$
$$= \mu_j c \frac{\Delta t}{\Delta z} I_j^+(z_n, t_i) + \left(1 - \mu_j \frac{c\Delta t}{\Delta z}\right) I_j^+(z_{n+1}, t_i) \qquad (3.13)$$

$$I_j^-(P_j') = I_j^-(z_{n+1} + \mu_j c\Delta t, t_i)$$

$$= \mu_j c \frac{\Delta t}{\Delta z} I_j^-(z_{n+2}, t_i) + \left(1 - \mu_j \frac{c\Delta t}{\Delta z}\right) I_j^-(z_{n+1}, t_i) \qquad (3.14)$$

Similarly, $\overset{+}{S}(\rho_j)$, $\bar{S}(\rho_j')$, $\alpha(\rho_j)$, $W(\rho_j)$, $\alpha(\rho_j')$ and $W(\rho_j')$ can be approximated.

Denoting $I^{\pm}_{n+1,j}(t_i) = I^{\pm}_j(z_{n+1}, t_i)$ and incorporating the relations (3.13) and (3.14) in equations (3.11) and (3.12) respectively, we obtain

$$\left[I^{+}_{n+1,j}(t_{i+1}) - \left\{ \left(\mu_j \frac{c\Delta t}{\Delta z}\right) I^{+}_{n,j}(t_i) + \left(1 - \mu_j \frac{c\Delta t}{\Delta z}\right) I^{+}_{n+1,j}(t_i) \right\} \right]$$

$$+ \frac{\tilde{\alpha}}{2} \left[I^{+}_{n+1,j}(t_{i+1}) + \mu_j \frac{c\Delta t}{\Delta z} I^{+}_{n,j}(t_i) + \left(1 - \mu_j \frac{c\Delta t}{\Delta z}\right) I^{+}_{n+1,j}(t_i) \right] * c\Delta t$$

$$= \tilde{\alpha} \Bigg[\frac{\tilde{\omega}}{4} \Bigg\{ \sum_{j'=1}^{J} P(\mu_j, \mu_j') W_{j'} \Big\{ I^{+}_{n+1,j'}(t_{i+1}) + \mu_j \frac{c\Delta t}{\Delta z} I^{+}_{n,j'}(t_i)$$

$$+ \left(1 - \mu_j \frac{c\Delta t}{\Delta z}\right) I^{+}_{n+1,j'}(t_i) \Big\} + P(\mu_j, -\mu_j') W_j' \Big\{ I^{-}_{n+1,j'}(t_{i+1})$$

$$+ \mu_j \frac{c\Delta t}{\Delta z} I^{-}_{n,j'}(t_i) + \left(1 - \mu_j \frac{c\Delta t}{\Delta z}\right) I^{-}_{n+1,j'}(t_i) \Big\} \Bigg\}$$

$$+ \frac{(1-\tilde{\omega})}{2} \Big\{ B_{n+1}(t_{i+1}) + \mu_j c \frac{\Delta t}{\Delta z} B_n(t_i) + \left(1 - \mu_j \frac{c\Delta t}{\Delta z}\right) B_{n+1}(t_i) \Big\} + 0.5 * \Big\{ g_{n+1}(t_{i+1})$$

$$+ \mu_j c \frac{\Delta t}{\Delta z} g_{n,j}(t_i) + \left(1 - \mu_j \frac{c\Delta t}{\Delta z}\right) g_{n+1,j}(t_i) \Big\} \Bigg] c\Delta t \qquad (3.15)$$

$$\left[I^{-}_{n+1,j}(t_{i+1}) - \left\{ \left(\mu_j c \frac{\Delta t}{\Delta z}\right) I^{-}_{n+2,j}(t_i) + \left(1-\mu_j \frac{c\Delta t}{\Delta z}\right) I^{-}_{n+1,j}(t_i) \right\} \right]$$

$$+ \frac{\tilde{\tilde{\alpha}}}{2} \left\{ I^{-}_{n+1,j}(t_{i+1}) + \mu_j \frac{c\Delta t}{\Delta z} I_{n+2,j}(t_i) + \left(1-\mu_j \frac{c\Delta t}{\Delta z}\right) I^{-}_{n+1,j}(t_i) \right\} * c\Delta t$$

$$= \tilde{\tilde{\alpha}} \left[\frac{\tilde{\tilde{\omega}}}{4} \left\{ \sum_{j'=1}^{J} \left[p(-\mu_j, \mu_j') W_{j'} \left(I^{+}_{n+1,j'}(t_{i+1}) + \mu_j \frac{c\Delta t}{\Delta z} I^{+}_{n+2,j'}(t_i) \right.\right.\right.\right.$$

$$\left. + \left(1-\mu_j \frac{c\Delta t}{\Delta z}\right) I^{+}_{n+1,j'}(t_i) \right) + p(-\mu_j, -\mu_j') W_j' \left(I^{-}_{n+1,j'}(t_{i+1}) + \right.$$

$$\left.\left.\left. \mu_j \frac{c\Delta t}{\Delta z} I^{-}_{n+2,j'}(t_i) + \left(1-\mu_j \frac{c\Delta t}{\Delta z}\right) I^{-}_{n+1,j'}(t_i) \right) \right] c\Delta t \right\} +$$

$$\frac{(1-\tilde{\tilde{\omega}})}{2} \left\{ B_{n+1}(t_{i+1}) + \mu_j \frac{c\Delta t}{\Delta z} B_{n+2}(t_i) + \left(1-\mu_j \frac{c\Delta t}{\Delta z}\right) B_{n+1}(t_i) \right\}$$

$$c\Delta t + \left\{ g_{n+1,j}(t_{i+1}) + \mu_j \frac{c\Delta t}{\Delta z} g_n(t_i) + \left(1-\mu_j \frac{c\Delta t}{\Delta z}\right) g_{n+1}(t_i) \right\}$$

$$\left. c\Delta t \right] \tag{3.16}$$

Defining the following matrices

$$B = \begin{bmatrix} \mu_1 \frac{\Delta t}{\Delta z} & & & \\ & \mu_2 \frac{\Delta t}{\Delta z} & & \\ & & \ddots & \\ & & & \mu_J \frac{\Delta t}{\Delta z} \end{bmatrix}_{J \times J}$$

$$A = \begin{bmatrix} 1-\mu_1 \frac{\Delta t}{\Delta z} & & \\ & \ddots & \\ & & 1-\mu_J \frac{\Delta t}{\Delta z} \end{bmatrix}_{J \times J}$$

$$U^{\pm}_{n+1}(t_{i+1}) = \begin{bmatrix} I^{\pm}_{n+1,1}(t_{i+1}) \\ \vdots \\ I^{\pm}_{n+1,J}(t_{i+1}) \end{bmatrix}_{J \times 1}$$

$$P^{++} = \begin{bmatrix} p(\mu_1,\mu_1) & p(\mu_1,\mu_2) & \cdots & p(\mu_1,\mu_J) \\ p(\mu_2,\mu_1) & p(\mu_2,\mu_2) & & p(\mu_2,\mu_J) \\ \vdots & & & \\ p(\mu_J,\mu_1) & \cdots & & p(\mu_J,\mu_J) \end{bmatrix}_{J \times J}$$

$$\overline{P}^{+} = \begin{bmatrix} p(-\mu_1,\mu_1) & \cdots & p(-\mu_1,\mu_J) \\ p(-\mu_2,\mu_1) & & p(-\mu_2,\mu_J) \\ \vdots & & \vdots \\ p(-\mu_J,\mu_1) & & p(-\mu_J,\mu_J) \end{bmatrix}_{J \times J}$$

$$W = \begin{bmatrix} W_1 & & \\ & \ddots & \\ & & W_J \end{bmatrix}_{J \times J}$$

$$G_{n+1}(t_i) = \begin{bmatrix} g_{n+1}(t_i) \\ \vdots \\ g(n+1)(t_i) \end{bmatrix}_{J \times 1}$$

$$\tilde{B}_{n+1}(t_i) = \begin{bmatrix} B_{n+1}(t_i) \\ \vdots \\ B_{n+1}(t_i) \end{bmatrix}_{J \times 1} \qquad (3.17)$$

We can write equations (3.15) and (3.16) as

$$\left[U^+_{n+1}(t_{i+1}) - \left\{ A U^+_{n+1}(t_i) + B U^+_n(t_i) \right\} \right] + \frac{\tilde{\alpha}}{2} \left\{ U^+_{n+1}(t_{i+1}) + A U^+_{n+1}(t_i) + B U^+_n(t_i) \right\} c\Delta t =$$

$$\tilde{\alpha} \frac{\tilde{\omega}}{4} \left[p^{++} W \left(U^+_{n+1}(t_{i+1}) + A U^+_{n+1}(t_i) + B U^+_n(t_i) \right) + p^{+-} W \left(U^-_{n+1}(t_{i+1}) + A U^-_{n+1}(t_i) + B U^-_n(t_i) \right) \right] c\Delta t$$

$$+ \frac{(1-\tilde{\omega})}{2} \tilde{\alpha} \left\{ \tilde{B}_{n+1}(t_{i+1}) + A \tilde{B}^+_{n+1}(t_i) + B \tilde{B}^+_n(t_i) \right\} c\Delta t$$

$$0.5 c\Delta t \left\{ G_{n+1}(t_{i+1}) + B G_n(t_i) + A G_{n+1}(t_i) \right\} \qquad (3.18)$$

$$\left\{\bar{U}_{n+1}(t_{i+1}) - \left(A\bar{U}_{n+1}(t_i) + B\bar{U}_{n+2}(t_i)\right)\right\} + \frac{\tilde{\alpha}}{2}\left\{\bar{U}_{n+1}(t_{i+1}) + A\bar{U}_{n+1}(t_i) + B\bar{U}_{n+2}(t_i)\right\} c\Delta t = \tilde{\alpha}\left[\frac{\tilde{\omega}}{4}\left\{\bar{P}^{+}W\left(U^{+}_{n+1}(t_{i+1}) + AU^{+}_{n+1}(t_i) + BU^{+}_{n+2}(t_i)\right) + P^{-}W\left(\bar{U}_{n+1}(t_{i+1}) + B\bar{U}_{n+2}(t_i) + A\bar{U}_{n+1}(t_i)\right)\right\} c\Delta t + \frac{(1-\tilde{\omega})}{2}\left\{\tilde{B}_{n+1}(t_{i+1}) + A\tilde{B}_{n+1}(t_i) + B\tilde{B}^{+}_{n+2}(t_i)\right\} c\Delta t\right] + 0.5\, c\Delta t\left\{G_{n+1}(t_{i+1}) + BG_{n+2}(t_i) + A\,G_{n+1}(t_i)\right\} \tag{3.19}$$

Rearranging the terms we have

$$\left[I + \frac{\tilde{\alpha}}{2} c\Delta t \cdot I - \tilde{\alpha}\frac{\tilde{\omega}}{4} P^{++} W c\Delta t\right] U^{+}_{n+1}(t_{i+1})$$

$$= \left[I - \frac{\tilde{\alpha}}{2} c\Delta t\, I + \tilde{\alpha}\frac{\tilde{\omega}}{4} P^{++} W c\Delta t\right] A\, U^{+}_{n+1}(t_i)$$

$$+ \left[I - \frac{\tilde{\alpha}}{2} c\Delta t\, I + \tilde{\alpha}\frac{\tilde{\omega}}{4} P^{++} W c\Delta t\right] B\, U^{+}_{n}(t_i)$$

$$+ \tilde{\alpha}\frac{\tilde{\omega}}{4} c\Delta t\, P^{+-} W\,\bar{U}_{n+1}(t_{i+1}) + \tilde{\alpha}\frac{\tilde{\omega}}{4} c\Delta t\, P^{+-} W A\,\bar{U}_{n+1}(t_i) + \tilde{\alpha}\frac{\tilde{\omega}}{4} c\Delta t\, P^{+-} W B\bar{U}_{n}(t_i) + \frac{(1-\tilde{\omega})}{2}\tilde{\alpha}\left\{\tilde{B}_{n+1}(t_{i+1}) + A\tilde{B}_{n+1}(t_i) + B\tilde{B}^{+}_{n}(t_i)\right\} c\Delta t + 0.5\, c\Delta t\left\{G_{n+1}(t_{i+1}) + BG_{n}(t_i) + A\,G_{n+1}(t_i)\right\} \tag{3.20}$$

and

$$\left[I + \frac{\tilde{\tilde{\alpha}}}{2} c\Delta t\, I - \tilde{\tilde{\alpha}} \frac{\tilde{\tilde{\omega}}}{4} P^{--} W c\Delta t\right] U^{-}_{n+1}(t_{i+1})$$

$$= \left[I - \frac{\tilde{\tilde{\alpha}}}{2} c\Delta t\, I + \tilde{\tilde{\alpha}} \frac{\tilde{\tilde{\omega}}}{4} P^{--} W c\Delta t\right] A\, U^{-}_{n+1}(t_i)$$

$$+ \left[I - \frac{\tilde{\tilde{\alpha}}}{2} c\Delta t\, I + \tilde{\tilde{\alpha}} \frac{\tilde{\tilde{\omega}}}{4} P^{--} W c\Delta t\right] B\, U^{-}_{n+2}(t_i)$$

$$+ \tilde{\tilde{\alpha}} \frac{\tilde{\tilde{\omega}}}{4} c\Delta t\, P^{-+} W\, U^{+}_{n+1}(t_{i+1}) + \tilde{\tilde{\alpha}} \frac{\tilde{\tilde{\omega}}}{4} c\Delta t\, P^{-+} W A\, U^{+}_{n+1}(t_i) + \tilde{\tilde{\alpha}} \frac{\tilde{\tilde{\omega}}}{4} c\Delta t\, P^{-+} W B\, U^{+}_{n+2}(t_i)$$

$$+ \frac{(1-\tilde{\omega})}{2} \tilde{\tilde{\alpha}} \left\{ \tilde{B}_{n+1}(t_{i+1}) + A \tilde{B}_{n+1}(t_i) + B \tilde{B}_{n+2}(t_i) \right\} c\Delta t$$

$$+ 0.5\, c\Delta t \left\{ G_{n+1}(t_{i+1}) + B\, G_{n+2}(t_i) + A\, G_{n+1}(t_i) \right\}$$

(3.21)

where I is the identity matrix.

By setting

$$\Delta^{+} = \left[I + \frac{\tilde{\alpha}}{2} c\Delta t\, I - \tilde{\alpha} \frac{\tilde{\omega}}{4} P^{++} W c\Delta t\right]^{-1}$$

$$\Delta^{-} = \left[I + \frac{\tilde{\tilde{\alpha}}}{2} c\Delta t\, I - \tilde{\tilde{\alpha}} \frac{\tilde{\tilde{\omega}}}{4} P^{--} W c\Delta t\right]^{-1}$$

$$S^{++} = \left[I - \frac{\tilde{\alpha}}{2} c\Delta t\, I + \tilde{\alpha} \frac{\tilde{\omega}}{4} P^{++} W c\Delta t\right]$$

$$S^{--} = \left[I - \frac{\tilde{\tilde{\alpha}}}{2} c\Delta t\, I + \tilde{\tilde{\alpha}} \frac{\tilde{\tilde{\omega}}}{4} P^{--} W c\Delta t\right]$$

$$\overset{+-}{Q} = \tilde{\alpha}\,\frac{\tilde{\omega}}{4}\,c\Delta t\,\overset{+-}{P}W\ ,\quad \overset{-+}{Q} = \tilde{\tilde{\alpha}}\,\frac{\tilde{\tilde{\omega}}}{4}\,c\Delta t\,\overset{-+}{P}W$$

and

$$\overset{+}{\Delta}\,\overset{++}{S} = \overset{+}{T} \qquad \overset{+}{\Delta}\,\overset{+-}{Q} = \overset{-}{R}$$

$$\overset{-}{\Delta}\,\overset{--}{S} = \overset{-}{T} \qquad \overset{-}{\Delta}\,\overset{-+}{Q} = \overset{+}{R}$$

$$\overset{+}{\sum_{n+\frac{1}{2}}} = \tilde{\alpha}\left(\frac{1-\tilde{\omega}}{2}\right)\overset{+}{\Delta}\left\{\tilde{B}_{n+1}(t_{i+1}) + A\,\tilde{B}_{n+1}(t_i) + B\,\tilde{B}_n(t_i)\right\}c\Delta t$$
$$+\ 0.5\,c\Delta t\,\overset{+}{\Delta}\left\{G_{n+1}(t_{i+1}) + B\,G_n(t_i) + A\,G_{n+1}(t_i)\right\}$$

$$\overset{-}{\sum_{n+\frac{3}{2}}} = \tilde{\tilde{\alpha}}\left(\frac{1-\tilde{\tilde{\omega}}}{2}\right)\overset{-}{\Delta}\left\{\tilde{B}_{n+1}(t_{i+1}) + A\,\tilde{B}_{n+1}(t_i) + B\,\tilde{B}_{n+2}(t_i)\right\}c\Delta t$$
$$+\ 0.5\,c\Delta t\,\overset{-}{\Delta}\left\{G_{n+1}(t_{i+1}) + B\,G_{n+2}(t_i) + A\,G_{n+1}(t_i)\right\} \qquad (3.22)$$

we get

$$U^{+}_{n+1}(t_{i+1}) = \overset{+}{T}A\,U^{+}_{n+1}(t_i) + \overset{+}{T}B\,U^{+}_n(t_i)$$
$$+\ \overset{-}{R}\,U^{-}_{n+1}(t_{i+1}) + \overset{-}{R}A\,U^{-}_{n+1}(t_i)$$
$$+\ \overset{-}{R}B\,U^{-}_n(t_i) + \overset{+}{\sum_{n+\frac{1}{2}}}(t_i) \qquad (3.23)$$

$$U^-_{n+1}(t_{i+1}) = T^- A\, U^-_{n+1}(t_i) + T^- B\, U^-_{n+2}(t_i) + R^+ U^+_{n+1}(t_{i+1}) + R^+ A\, U^+_{n+1}(t_i) + R^+ B\, U^+_{n+2}(t_i) + \sum\nolimits^-_{n+3/2}(t_i) \qquad (3.24)$$

Eliminating $U^+_{n+1}(t_{i+1})$ from the equation (3.24) using equation (3.23) we get,

$$\bar{U}_{n+1}(t_{i+1}) = \left[I - R^+ R^-\right]^{-1}\Big[(R^+ T^+ + R^+) A\, U^+_{n+1}(t_i) + (R^+ R^- + T^-) A\, U^-_{n+1}(t_i) + R^+ T^+ B\, U^+_n(t_i) + R^+ R^- B\, U^-_n(t_i) + R^+ \sum\nolimits^+_{n+\frac{1}{2}}(t_i) + T^- B\, U^-_{n+2}(t_i) + R^+ B\, U^+_{n+2}(t_i) + \sum\nolimits^-_{n+3/2}(t_i)\Big] \qquad (3.25)$$

From the initial values, one can calculate the $\bar{U}_{n+1}(t_2)$ for $n = N-1, \cdots\cdots 0$. And then these values are used to obtain $U^+_{n+1}(t_2)$ for $n = 1, \cdots N-1$.

This procedure is repeated for $t_3, t_4, \cdots$, etc.

3.3. Extension of the method to calculate the time-dependent line profiles.

The time-dependent transfer equation for a two level atom under the assumption of complete re-distribution is given by

$$\frac{1}{c}\frac{\partial I}{\partial t}(z,\pm\mu,x,t) \pm \mu \frac{\partial I}{\partial z}(z,\pm\mu,x,t)$$

$$+K_L(z,t)\,\phi(x)\,I(z,\pm\mu,x,t)$$

$$= k_L(z,t)\,\phi(x)\left[\frac{1-\epsilon}{2}\int_{-\infty}^{+\infty}\int_{-1}^{+1}\phi(x')\,I(z,\mu',x',t)\,d\mu'\,dx'\right.$$

$$\left. +\,\epsilon\,B(x,z,t)\right] \qquad 0<\mu<1 \tag{3.26}$$

We have neglected continuous absorption in writing the equation (3.26). X is the frequency measured in Doppler units and is defined by $X = (\nu-\nu_0)/\Delta S$, ΔS being some standard frequency interval. $K_L(z,t)$ is the line centre absorption coefficient and $\phi(X)$ is the absorption profile function.

$$\epsilon = \frac{C_{21}}{C_{21} + A_{21}\left(1 - e^{\frac{-h\nu}{kT_e}}\right)^{-1}}$$

(3.27)

is the probability per scatter that a photon will be destroyed by collisional de-excitation. C_{21} is the rate coefficient for collisional de-excitation of the atom and A_{21} is the Einstein coefficient for spontaneous emission. h and k are the Planck and Boltzman constants and T_e is the electron temperature of the gas.

Other symbols in equation (3.26) have the usual meaning. The assumption of complete redistribution supposes that there is no correlation between the frequencies of the absorbed and the emitted photons. We have assumed that the profile function is the Voigt function $H(a, x)$ given by

$$\phi(x) = H(a,x) = \frac{a}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{e^{-u^2}}{(x-u)^2 + a^2} du$$

(3.28)

where a is the damping constant for the upper level of the transition. We have considered isotropic scattering.

We discretized the angular integral by Gauss-quadrature of order J . We discretized the frequency integral also by Gauss-quadrature of order K.

$$\int_0^\infty \phi(x') I(z, \pm\mu_j, x') \simeq \sum_{k=1}^{K} \phi(x_k) I(z, \pm\mu_j, x_k) \omega_k \qquad j = 1, 2, \cdots J \tag{3.29}$$

Since the problem has symmetric solution with respect to the line centre, we have considered only positive frequency grid. Using (3.29) in (3.26), we get

$$\frac{1}{c} \frac{\partial I}{\partial t}(z, \pm\mu_j, x_k, t) \pm \mu_j \frac{\partial I}{\partial z}(z, \pm\mu_j, x_k, t)$$

$$+ K_L(z,t)\, \phi(x_k)\, I(z, \pm\mu_j, x_k, t)$$

$$= K_L(z,t)\, \phi(x_k) \left[\left\{ (1-\epsilon) \sum_{k'=1}^{K} \phi(x_{k'}) \sum_{j'=1}^{J} \left[I(z, +\mu_{j'}, x_{k'}, t) \right.\right.\right.$$

$$\left.\left.\left. + I(z, -\mu_{j'}, x_{k'}, t)\, c_{j'} \right] \omega_{k'} \right\} + \epsilon B(z,t) \right] \tag{3.30}$$

$$k = 1, 2, \cdots K$$
$$j = 1, 2, \cdots J$$

As these equations form hyperbolic system of linear partial differential equations, we use the same computational procedure as described in section 3.2.

3.4 Results and discussion

We have set $\alpha = C = 1$ in eqn. ~~3.1.~~ Since $\alpha = 1$, the total optical depth of the medium is same as the geometrical depth T.

Case I To check the numerical algorithm, we considered a pure absorption case. The transfer equation solved is

$$\frac{\partial I}{\partial t} + \mu \frac{\partial I}{\partial z} + I = B$$

with

$$I(z=0, \mu, t) = 0; \quad t \geq 0$$

$$I(z, \mu, t=0) = 0; \quad 0 < z < T$$

and $B = 1$ for $t \geq 0$

$$0 \leq z \leq T$$

(3.31)

The analytical solution when α and B are constant with respect to the time and position z is given by

$$I(z,\mu,t) = B(1-e^{-t}) \; ; \; t \leq \frac{z}{\mu}$$
$$= B(1-e^{-z/\mu}) \; ; \; t \geqslant \frac{z}{\mu} \qquad (3.32)$$

The numerical solution agrees with the above solution. The results for $\mu = 0.21$ and 0.78 are plotted in fig.1.

Case II   A slab is constantly illuminated by a beam of radiation in a specified direction $\mu_0 = 0.5$ from time $t = 0$. The transfer equation for diffuse intensity is

$$\frac{\partial I}{\partial t} + \mu \frac{\partial I}{\partial z} + I(z,\mu,t) = \left\{ 0.5 \int_{-1}^{+1} I(z,\mu',t)\, d\mu' + 0.5\, H\left(t - \frac{z}{\mu_0}\right) e^{-z/\mu_0} \right\} \qquad (3.33)$$

where $H(t)$ is a Heaviside unit step function.

The initial and boundary conditions for equation (3.32) are

$$\begin{aligned} I(z=0,\mu,t) &= 0 \\ I(z=T,-\mu,t) &= 0 \end{aligned} \qquad t \geqslant 0 \qquad (3.34)$$
$$I(z,\pm\mu,0) = 0 \quad \text{For } 0 < z < T$$

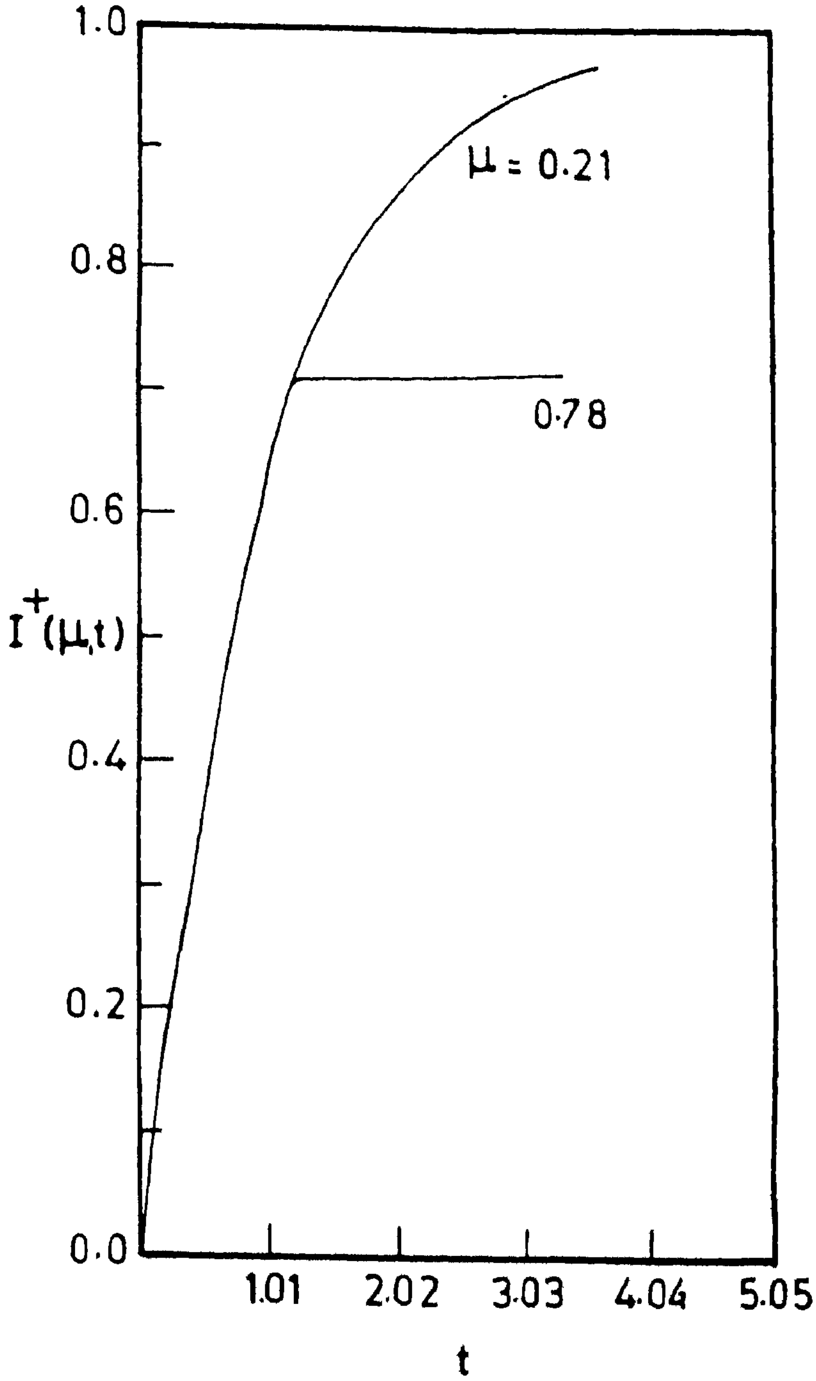


Fig.1

The intensity distribution for a pure absorbing medium with a constant thermal source.

As the intensity distributions reach steady state after sufficiently long time, the steady state solutions are checked by solving the equation

$$\pm\mu \frac{\partial \tilde{I}}{\partial z} + \tilde{I}(z,\pm\mu,t) = \left\{0.5\int_{-1}^{+1} I(z,\mu',t)\,d\mu' + 0.5\, e^{-z/\mu_0}\right\}$$

with

$$\tilde{I}(z=0,\mu)=0 \qquad 0<\mu\leq 1$$
$$\tilde{I}(z=T,-\mu)=0 \tag{3.35}$$

Emergent intensity distributions for total optical depths 1 and 2 are plotted in Figs 2 and 3 respectively. Van de Hulst and Irvine (1962) pointed out that the non-stationary problem for $t_1 = 0$ is equivalent to the problem of finding the distribution of photons over the pathlengths in a homogeneous medium.

We know that in a medium with optical depth T, the photon path length is at least few multiples of T. For $T=2$ one can see that the time at which the relaxation to steady state commences is nearly twice that of T = 1.

Reflected radiation distributions are plotted in figs 4 and 5 respectively. The reflected radiation

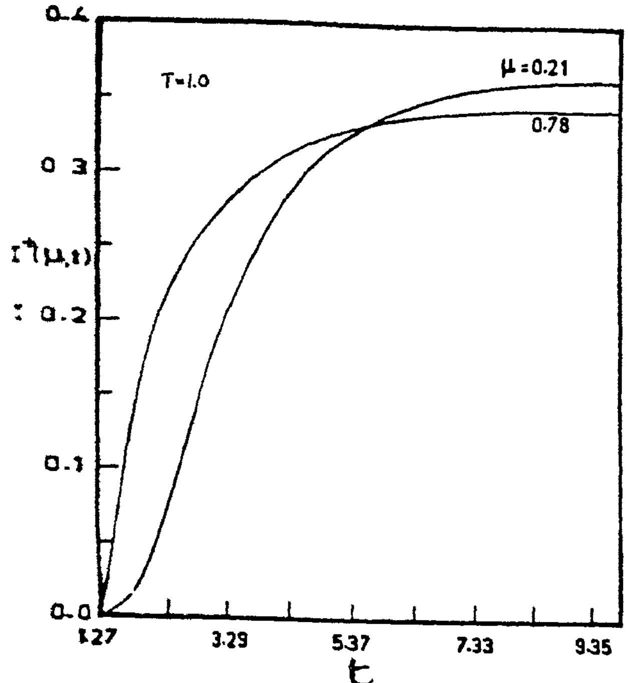


Fig.2

The emergent intensity distribution with respect to the time for $\mu = 0.21$ and 0.78 when a slab of total optical depth $T=1.0$ is illuminated constantly in a direction $\mu_0 = 0.5$.

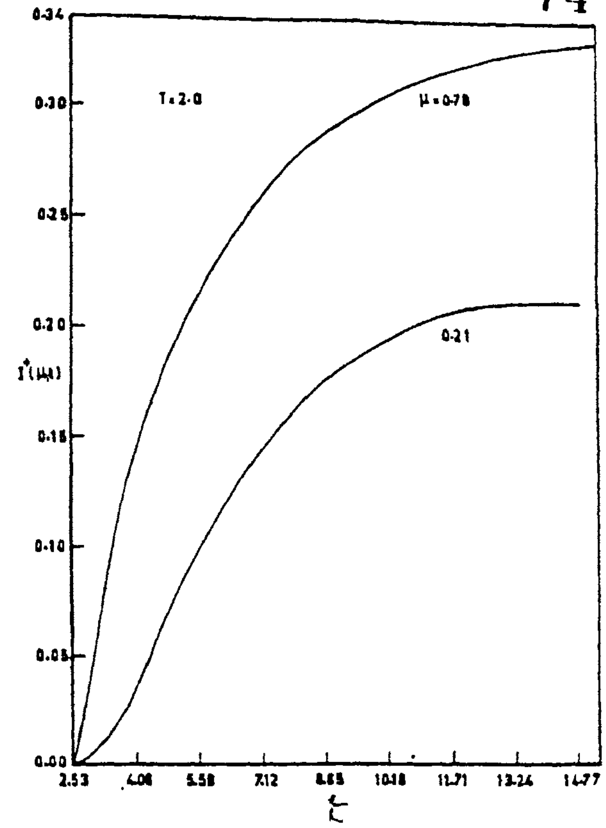
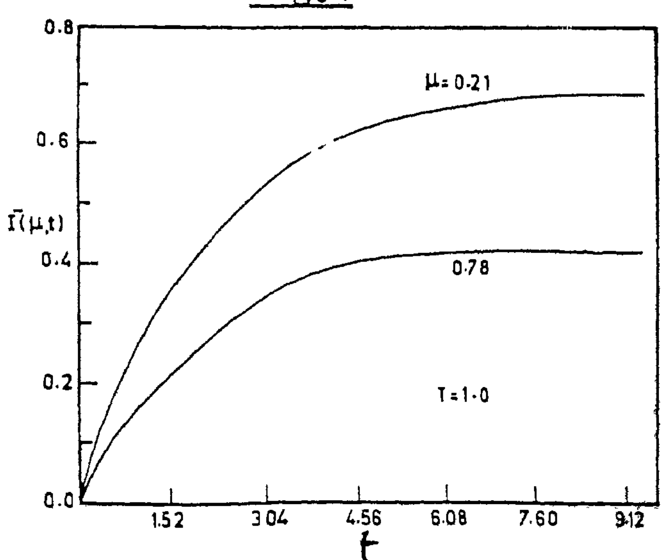


Fig.3

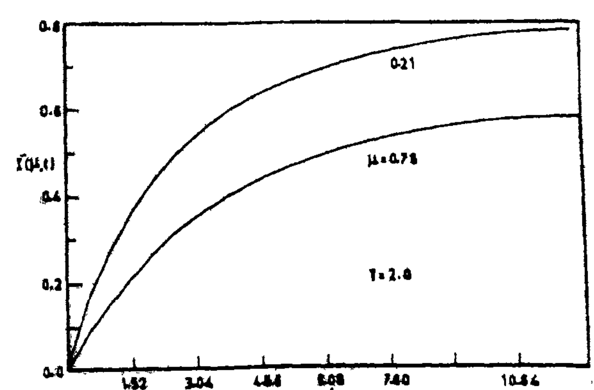
Same as in fig.2 for a slab with $T = 2.0$

Fig.4



The corresponding reflected intensity for the same case as in fig.2.

Fig.5



Same as in fig.4 for a slab with $T = 2.0$

reach steady state faster than the emergent radiation. This can be explained by the fact that most of the reflected photons emerge from the shallow layers of the medium and hence travel less pathlength.

Also if we examine the steady state values for the emergent radiation, the value for 0.21 is slightly larger compared to that of $\mu = 0.7$ for the medium with $\tau = 1.0$. This is in qualitative agreement with Chandrasekhar's result (See, Chandrasekar 1950).

<u>Case III</u> A slab is constantly illuminated by an isotropic radiation field. We set $I(z=0, \mu, t) = H(t)$ at one boundary of the medium and zero incidence at the other boundary. Transfer equation for the diffuse intensity is

$$\frac{\partial I}{\partial t}(z, \pm\mu, t) \pm \mu \frac{\partial I}{\partial z}(z, \pm\mu, t) + I(z, \pm\mu, t)$$
$$= \left\{ 0.5 \int_{-1}^{+1} I(z, \mu', t)\, d\mu' + 0.5 \int_{0}^{1} H\left(t - \frac{z}{\mu'}\right) e^{-z/\mu'} d\mu' \right\} \tag{3.36}$$

The steady state values are checked by solving the equation

$$\mu \frac{\partial \tilde{I}}{\partial z} + \tilde{I}(z, \pm\mu) = \left\{ 0.5 \int_{-1}^{+1} I(z, \mu') d\mu' + 0.5 \int_{0}^{1} e^{-z/\mu'} d\mu' \right\}$$

(3.37)

With

$$\tilde{I}(z=0, \mu) = 0$$

$$\tilde{I}(z=T, -\mu) = 0 \qquad (3.38)$$

The reflected intensity distributions are plotted in Figs 6,7 and 8 for T = 1,2 and 5. The corresponding emergent intensity distributions are plotted in Figs. 9,10,and 11.

The behavior of the reflected radiation is almost identical in all the cases except for the fact that the time at which the relaxation commences is more for the medium with higher optical thickness. Same qualitative reasons hold good as in the case II .

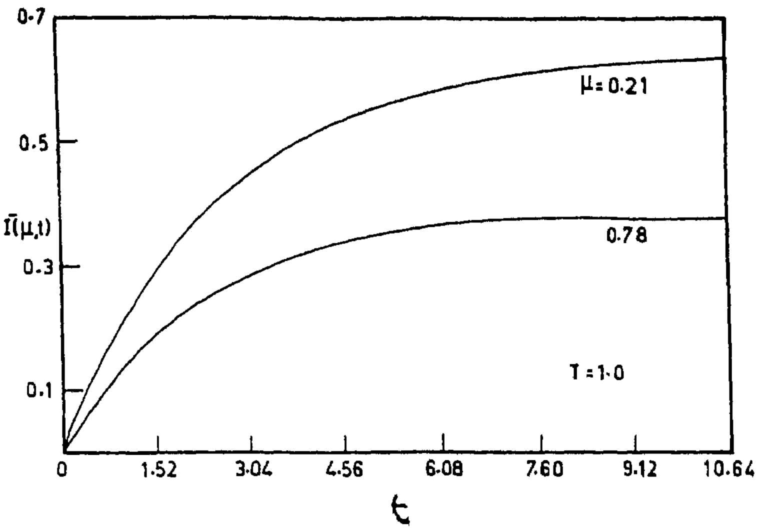
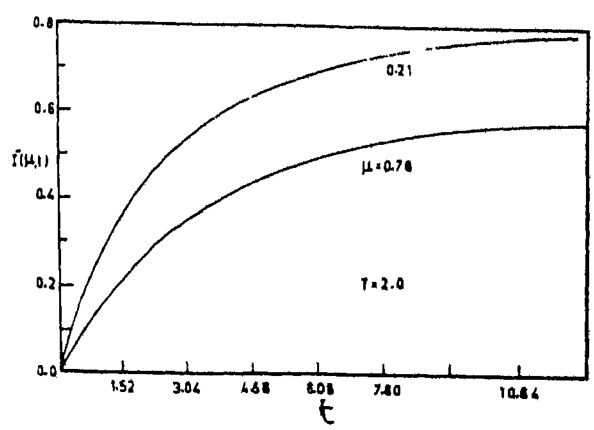


FIG.6

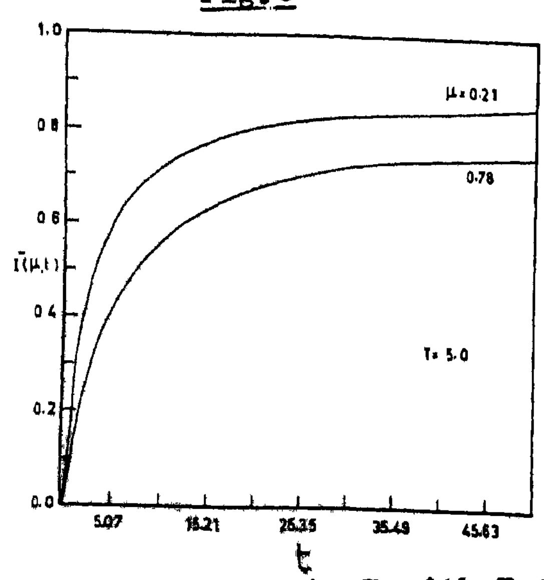
The reflected intensity distribution when a slab with T = 1.0 is illuminated by an isotropic radiation field.

Fig.7



Same as in fig.6 with T=2.0

Fig.8



Same as in fig.7 with T=5.0

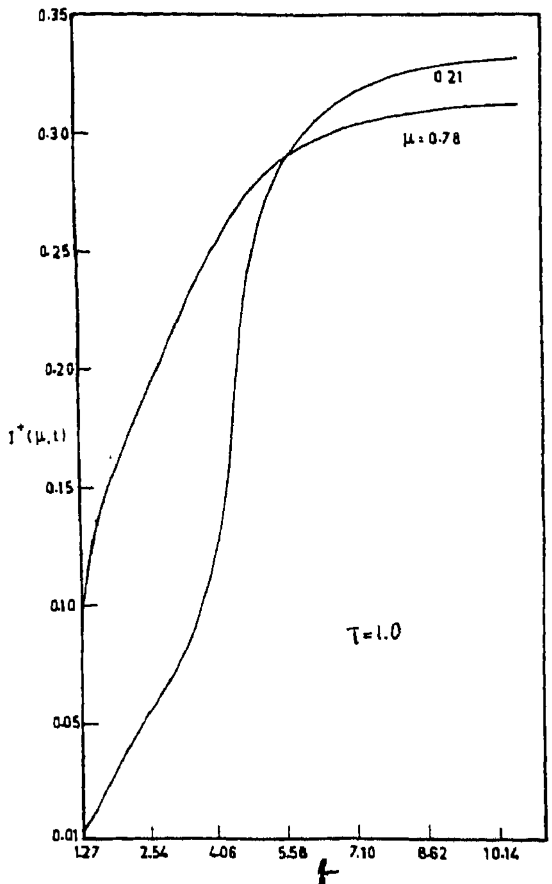


Fig.9

The corresponding emergent intensity distribution for the same case as in fig.6

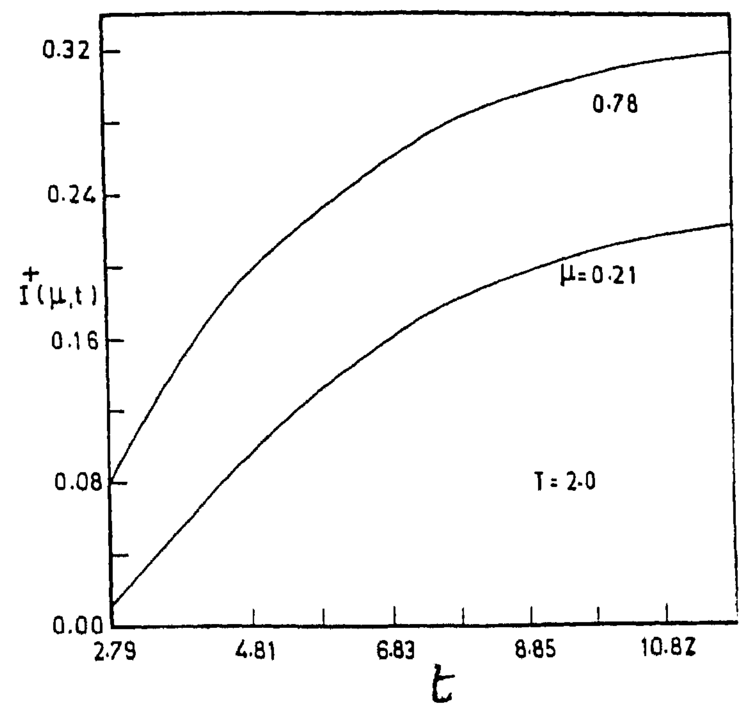


Fig.10

Same as in fig.9 with T = 2.0

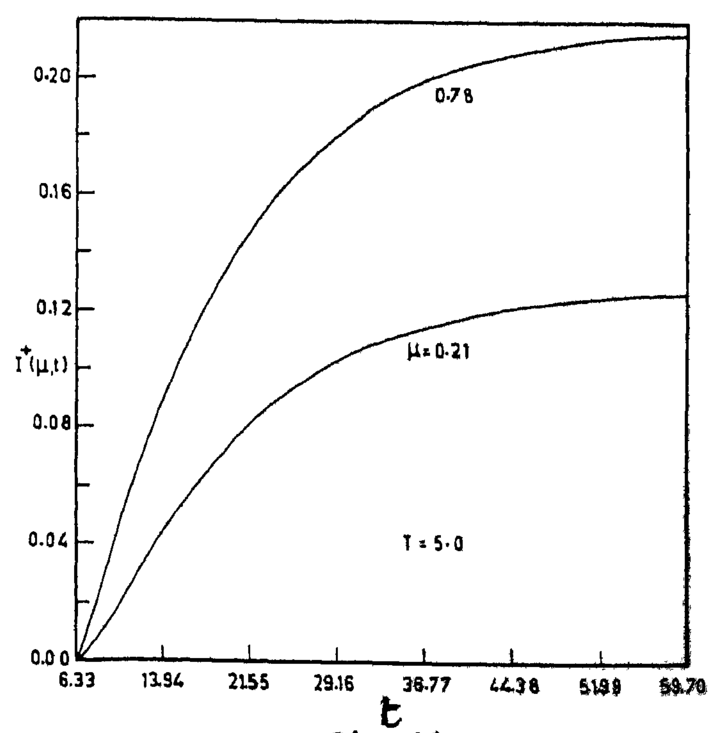


fig.11

same as in fig.10 with T = 5.0

<u>Case IV</u> A slab is illuminated by a pulsed beam. The transfer equation under the two stream approximation is

$$\frac{\partial I^{\pm}}{\partial t} \pm \mu \frac{\partial I^{\pm}}{\partial Z} + I^{\pm} = 0.5\left(I^{+} + I^{-}\right)$$

with

$$I^{+}(0,t) = \delta(t)$$
$$I^{-}(T,t) = 0 \qquad \text{and } \mu = 1/\sqrt{3} \tag{3.39}$$

Here I^{+} and I^{-} refer to two oppositely directed streams of radiation. By setting $\chi = \sqrt{3}/_2\, z$; $y = t/2$ we can transform the equations (3.39) to

$$\frac{\partial I^{+}}{\partial y} + \frac{\partial I^{+}}{\partial x} + I^{+} = I^{-} \tag{3.40}$$

$$\frac{\partial I^{-}}{\partial y} - \frac{\partial I^{-}}{\partial x} + I^{-} = I^{+} \tag{3.41}$$

with $\quad I^{+}(x_0, y) = \delta(y)$

$$I^{-}(X, Y) = 0 \tag{3.42}$$

where

$$X = \sqrt{3/2}\; T, \quad x_0 = 0$$

Code (1970) **solves** the above set of **equations** (3.40) (3.41) and (3.42) using a technique developed by Chandrasekhar (1950). Also note that

$$\bar{I}^{\pm}(t) = \frac{1}{2} I^{\pm}(y) \tag{3.43}$$

To check the numerical results, we can derive some relations which connect the time-dependent solutions to the steady state solution.

(i) The characteristic of the equation (3.40) is given by

$$\frac{dy}{dx} = 1 \tag{3.44}$$

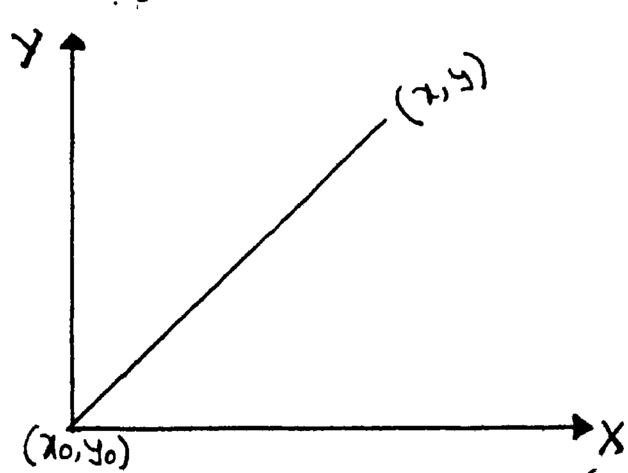


FIG 2(a)

Integrating equation (3.40) from $(x_0, y-x+x_0)$ to (x,y) along the characteristic, (**see Fig 2 (a)**), **we obtain**

$$\overset{+}{I}(x,y) = \overset{+}{I}(x_0, y-x+x_0)\, e^{-y} + e^{-y} \int_{y-x+x_0}^{y} e^{y'} \bar{I}(y'-y+x, y')\, dy'$$

(3.45)

Boundary condition (3.42), is

$$\overset{+}{I}(x_0, y-x+x_0) = \delta(y-x+x_0)$$

Integrating the relation (3.45) from T(x) to ∞ where $T(x)$ is the travel time for the pulse to reach the depth point x of the medium, we obtain

$$\int_{T(x)}^{\infty} \overset{+}{I}(x,y')\, dy' = e^{-T(x)} + \int_{x_0}^{x} e^{(x'-x)} \int_{T(x)}^{\infty} \bar{I}(x', x'+y-x)\, dy\, dx' \qquad (3.46)$$

If we set

$$\int_{T(x)}^{\infty} \overset{+}{I}(x,y')\, dy' = \tilde{I}^{+}(x) \qquad (3.47)$$

and

$$\int_{T(x)}^{\infty} \bar{I}(x', x'+y-x)\, dy = \tilde{I}^{-}(x') \qquad (3.48)$$

we obtain

$$\tilde{I}^{+}(x) = e^{-T(x)} + \int_{x_0}^{x} e^{-(x-x')}\, \tilde{I}^{-}(x')\, dx' \tag{3.49}$$

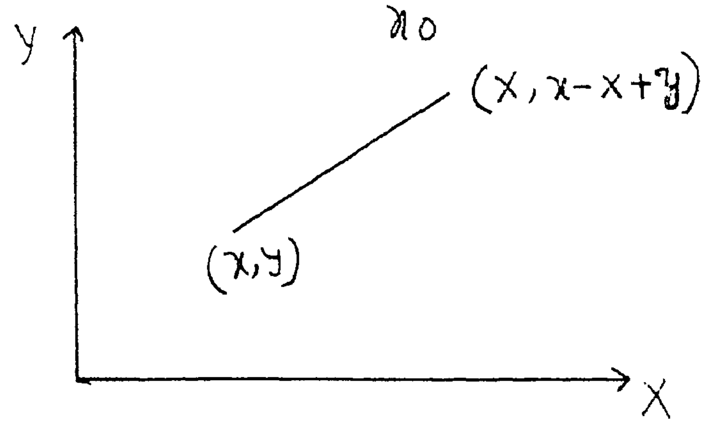


FIG:3(a)

(ii) The characteristic of the equation (3.41) is given by, (see fig 3(b))

$$\frac{dx}{dy} = -1$$

Integrating the equation (3.41) from (x, y) to $(X, x-X+y)$, we get

$$\bar{I}(x,y) = \int_{x}^{X} e^{-x'}\, I^{+}(x', x-x'+y)\, dx' \tag{3.50}$$

Since the **reflected** radiation at any point x of the medium starts only after the arrival of the pulse, we integrate (3.50) from y to ∞ where $y >> x$

Then

$$\int_{y}^{\infty} \bar{I}(x,y')\,dy' = \int_{x}^{X} e^{-x'} \int_{y}^{\infty} \overset{+}{I}(x',\ x-x'+y')\,dy'\,dx' \tag{3.51}$$

If we set $\displaystyle\int_{y}^{\infty} I^{+}(x',\ x-x'+y')\,dy' = \tilde{I}^{+}(x')$

and

$$\int_{y}^{\infty} \bar{I}(x,y')\,dy' = \tilde{I}^{-}(x) \tag{3.52}$$

we obtain

$$\tilde{I}^{-}(x) = \int_{x}^{X} e^{-x'}\,\tilde{I}^{+}(x')\,dx' \tag{3.53}$$

The equations (3.49) and (3.53) are the integral solutions of the steady state equations

$$\frac{\partial \tilde{I}^{+}}{\partial x} + \tilde{I}^{+} = \tilde{I}^{-}$$

$$-\frac{\partial \tilde{I}^{-}}{\partial x} + \tilde{I}^{-} = \tilde{I}^{+}$$

with

$$\tilde{I}^{+}(0) = 1$$

$$\tilde{I}^{-}(X) = 0.$$

one can note that

$$\tilde{I}^{+}(x) = \frac{1+X-x}{1+X}$$

$$\tilde{I}^{-}(x) = \frac{X-x}{1+X} \tag{3.54}$$

The relations (3.47) and (3.49) i.e.

$$\int_{T(x)}^{\infty} I^{+}(x, y')\, dy' = \tilde{I}^{+}(x)$$

and

$$\int_{y}^{\infty} I^{-}(x, y')\, dy' = \tilde{I}^{-}(x)$$

are useful in checking the time-dependent solution at each depth point of the atmosphere.

Since the Dirac-delta distribution is **difficult** to treat numerically, for the reflected radiation we imposed the conditions $\bar{U}_{n+1}(t_{n+1}) = 0.5\, e^{-t_{n+1}}$ and added a source term $0.5\, e^{-t_{n+2}}$ to the equation (3.25) to obtain $\bar{U}_{n+1}(t_{n+2})$.

Two cases which are identical to Code's (1970) are considered to check our numerical results. The emergent intensities are plotted in Fig. 12. for $X = 0.433$ and $X = 0.866$ respectively.

A pulse of radiation with a value e^{-X} emerges out of the medium at time $Y = X$. Following this pulse, the multiple scattered radiation falls off approximately as $e^{-Y/X}$. Reflected intensity distributions are plotted in Fig 13. Reflected radiation commences at $Y = 0$ and initially decays as $0.5\, e^{-Y}$. Its behavior is identical for all values of X until $Y = 2X$ which corresponds to twice the transit time. Then the radiation falls off to a low value.

Also the results for $\mu = \frac{1}{\sqrt{3}}$, $T = 3.0$ are plotted in Figs 14 and 15. We see that the sudden drop of the reflected radiation at twice the transit time reduces gradually and smoothen out at higher optical depths. One can note that Code's methods works only for $T < 1.8138$ but our method does not have any such restriction.

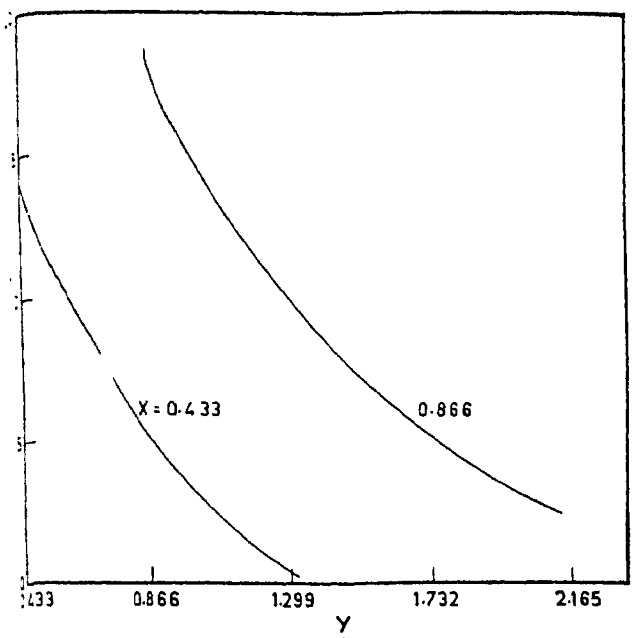


Fig.12

The emergent intensity distributions when a pulse impinges on the medium with X = 0.433 and X = 0.866.

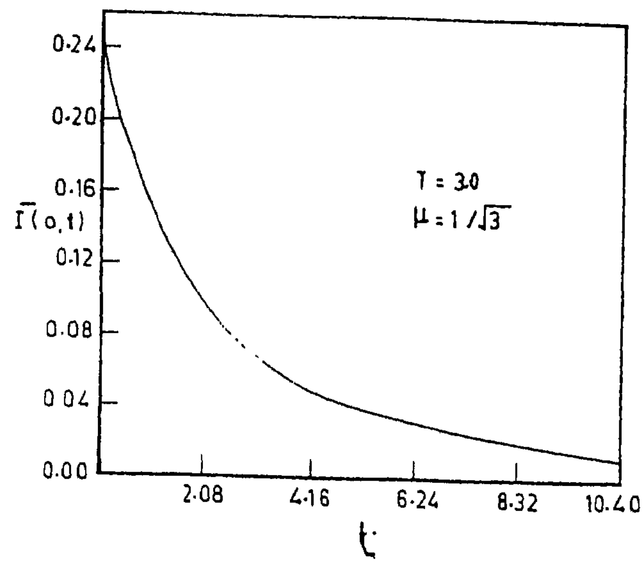
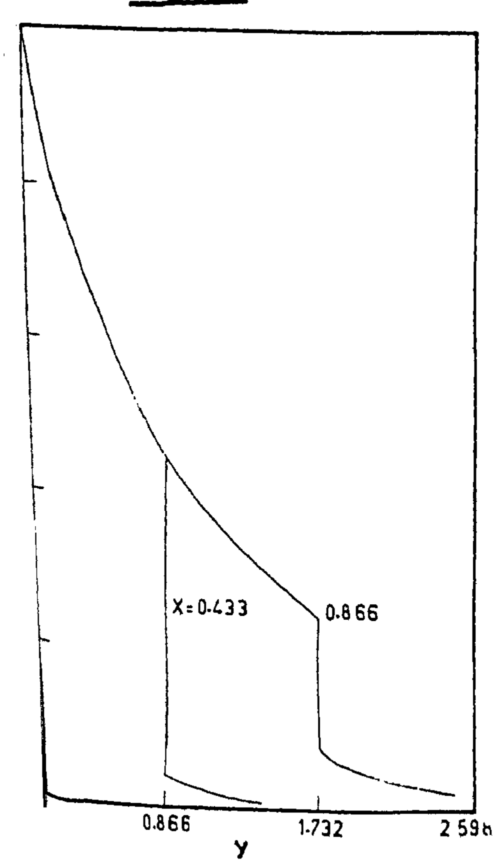


Fig.14

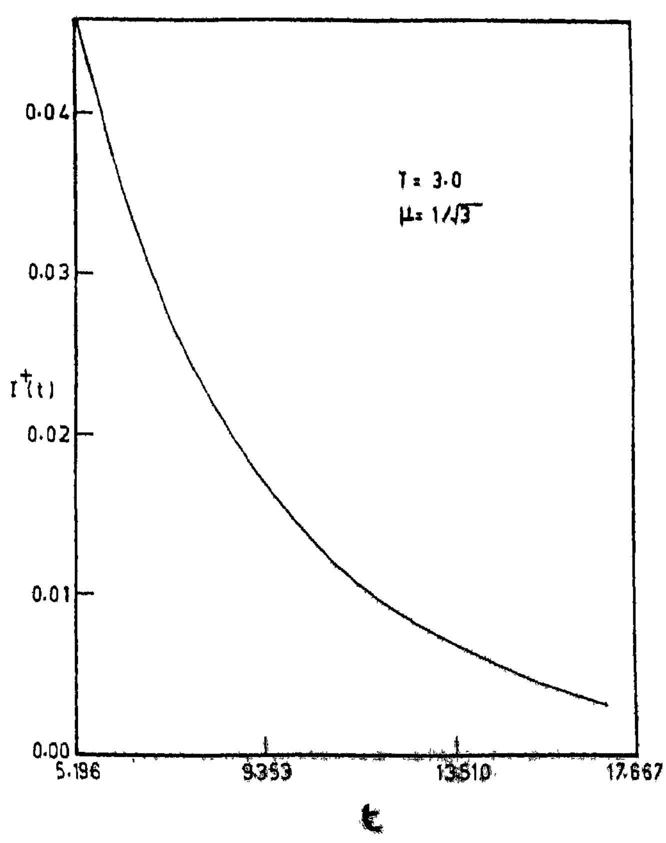
The same as in Fig.13 for a medi with T = 3.0 and $\mu = \frac{1}{\sqrt{3}}$

Fig.13



flected intensity distri-
ition for the same case as in
.g.12.

Fig.15



Same as in fig.14 for a medium with T = 3.0 and $\mu = \frac{1}{\sqrt{3}}$

Case V A slab is subject to an isotropic white radiation field. The boundary conditions imposed are

$$I(z=T, -\mu, x, t) = 0 \qquad t \geq 0$$

$$I(z=0, \mu, x, t) = H(t), \quad t \geq 0$$

$$I(z, \pm\mu, x, 0) = 0$$

The transfer equation for diffuse intensity is

$$\frac{\partial I^{\pm}}{\partial t} \pm \mu \frac{\partial I^{\pm}}{\partial z} + \phi(x) I^{\pm} = \phi(x) \left[0.5 \int_{-1}^{+1}\int_{-\infty}^{+\infty} \phi(x') I(z', \mu', x', t) \, dx' d\mu' \right.$$

$$\left. + 0.5 \int_{-\infty}^{\infty} \phi(x') \int_{0}^{1} H\left(t - \frac{z}{\mu'}\right) e^{-z/\mu' \cdot \phi(x')} d\mu' dx' \right]$$

with zero incidence at the boundaries.

Since the computational algorithm is time consuming, we have calculated the line profiles only for two cases. After adding the contribution of the directly transmitted light $e^{-\frac{\tau \phi(x)}{\mu}}$ to the diffuse intensity, we have plotted the emergent line profiles for $T = 5.0$ and $T = 25.0$ in Figs 16 and 17. At earlier times, we see deeper absorption profiles. Since the optical depth in the wing is very small (of order $\simeq 10^{-4}$), one can see the immediate convergence to the steady state in the wing region of the line profile.

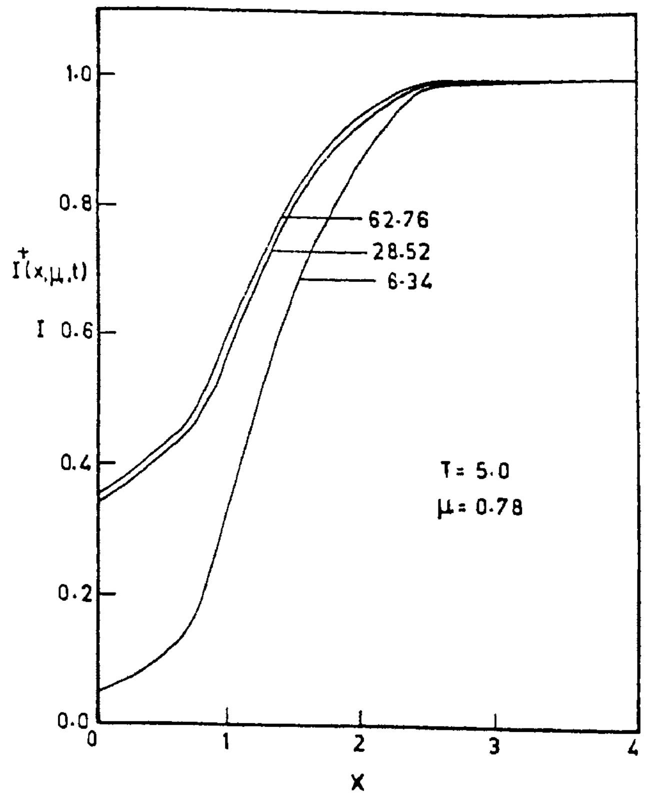


Fig.16.

The emergent time-dependent line profles when an isotropic white light falls on the medium with T = 5.0. Numbers denote different time steps.

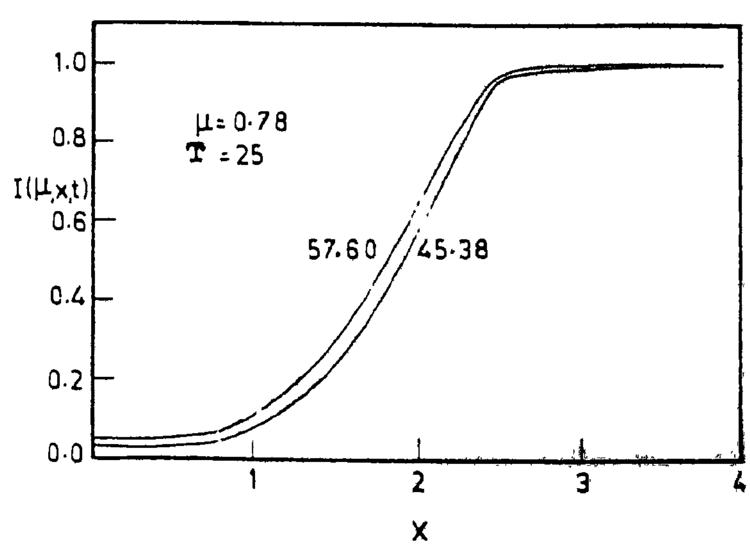


Fig.17.

The same as in fig.16 for T = 25.0

CHAPTER IV

TIME DEPENDENT TRANSFER IN SPHERICALLY SYMMETRIC MEDIA

4.1 Introduction

In this chapter, we shall present the numerical methods for the steady state and time-dependent transfer equations in spherically symmetric media. Reflected intensity distributions with respect to time are illustrated for the various ratios B/A where B is the outer and A is the inner radius of the atmosphere.

The assumption of plane-parallel stratification of the atmosphere holds good only when the actual thickness of the atmosphere is very small, i.e. $\delta r / r \simeq 0$ where δr is the thickness of the atmosphere and r is the radius of the star. However, many stars, such as the supergiant stars and Wolf-Rayet stars, have extended atmospheres whose thicknesses are an appreciable, fraction of a stellar radius. As a first approximation, one can assume that these atmospheres are spherically symmetric.

Hummer and Rybicki (1971) used the variable Eddington factor method to solve the steady state transfer equation in spherical symmetry. Peraiah and Grant (1973) have proposed a numerical method based on discrete space theory of Grant and Hunt (1968,1969a,1969 b). Peraiah and Grant derived simple conditions for their method to be stable and give non-negative solutions. In section 4.2, we present our numerical method to solve the steady state transfer equation.

There are several numerical methods proposed to solve the time-dependent transfer equation in spherically symmetric media. Carlson (1953) proposed S_n method for neutron transport calculations. Keller and Wendroff (1957) suggested the variant of S_n method and they also discussed the stability and the convergence of the method. Grant (1968) has solved the time-dependent transfer equation in purely absorbing media using a method developed by Lathrop and Carlson (1967). He has written difference equations in a matrix form and studied the stability and the non-negativity of the solutions.

The method to solve the time-dependent transfer equation is given in section 4.3. Results are discussed in section 4.4.

4.2 A numerical method for solving steady state transfer equation in spherical geometry

Transfer equation is given by

$$\mu \frac{\partial I}{\partial r}(r,\mu) + \frac{1-\mu^2}{r}\frac{\partial I}{\partial \mu}(r,\mu) + \sigma(r)\, I(r,\mu)$$
$$= \sigma(r)\left[\left\{1-\omega(r)\right\} B(r) + \frac{1}{2}\,\omega(r)\int_{-1}^{+1} I(r,\mu')\,d\mu'\right] \qquad (4.1)$$
$$0<\mu<1$$

for the outward going ray, and

$$-\mu \frac{\partial I(r,-\mu)}{\partial r} - \frac{1-\mu^2}{r}\frac{\partial I}{\partial \mu}(r,-\mu) + \sigma(r)\, I(r,-\mu)$$
$$= \sigma(r)\left[\left\{1-\omega(r)\right\} B(r) + \frac{\omega(r)}{2}\int_{-1}^{+1} I(r,\mu')\,d\mu'\right] \qquad (4.2)$$
$$0<\mu<1$$

for the inward going rays, where we restricted μ to lie in the interval $[0,1]$. We have assumed isotropic scattering in writing the equations (4.1) and (4.2). The integral over μ is approximated by Radau-quadrature formula based on the zeros of polynomial of degree 2J over $[-1,1]$

$$\int_{-1}^{+1} I(r,\mu')\,d\mu' \simeq \sum_{j=1}^{J} I(r,\mu_j)W_j + \sum_{j=1}^{J} I(r,-\mu_j)W_j \qquad (4.3)$$

We shall denote $I(r,\pm\mu_j) = I_j^{\pm}$. We replace the μ derivation in the equation (4.1) and (4.2) by $(I_{j+1}-I_j)/\Delta\mu$ centered at $\mu_{j+\frac{1}{2}} = \mu_j + \frac{1}{2}\Delta\mu$

By approximating the other terms centered at the same point, we obtain

$$\frac{1}{2} a_j \frac{\partial I^+_{j+1}}{\partial r} + \frac{1}{2} \bar{a}_j \frac{\partial I^+_j}{\partial r} + \frac{b_j}{r}\left(I^+_{j+1} - I^+_j\right) + \frac{\sigma}{2}\left(I^+_j + I^+_{j+1}\right)$$
$$= \sigma \left\{ \frac{\omega}{2} \sum_{j=1}^{J} W_j \left(I^+_j + I^-_j\right) + (1-\omega) B(r) \right\} \quad j = 1, \cdots J \tag{4.4}$$

and

$$-\frac{1}{2} a_j \frac{\partial I^-_{j+1}}{\partial r} - \frac{1}{2} \bar{a}_j \frac{\partial I^-_j}{\partial r} - \frac{b_j}{r}\left(I^-_{j+1} - I^-_j\right) + \frac{\sigma}{2}\left(I^-_j + I^-_{j+1}\right)$$
$$= \sigma \left\{ \frac{\omega}{2} \sum_{j=1}^{J} W_j \left(I^+_j + I^-_j\right) + (1-\omega) B(r) \right\} \quad j = 1, \cdots J \tag{4.5}$$

The quantities $a_j, \bar{a}_j$ and b_j are determined in such a way that the approximations,

$$\left(\mu \frac{\partial I^{\pm}}{\partial r}\right)_{\mu = \mu_{j+\frac{1}{2}}} = \frac{1}{2}\left(a_j \frac{\partial I^{\pm}_{j+1}}{\partial r} + \bar{a}_j \frac{\partial I^{\pm}_j}{\partial r}\right)$$

$$\left(\frac{1-\mu^2}{r} \frac{\partial I^{\pm}}{\partial \mu}\right)_{\mu = \mu_{j+\frac{1}{2}}} = \frac{b_j}{r}\left(I^{\pm}_{j+1} - I^{\pm}_j\right) \tag{4.6}$$

ave a minimum truncation error. Keller and
Wendroff (1957) listed some possible choice of
the coefficients $\omega_j, \overline{\omega_j}$ and b_j . The co-
efficients ω_j, $\overline{\omega_j}$ are listed in Table 1 and
for the b_j Table 2.

Table 1

$j \neq 0$	A	B	C
ω_j	$\frac{1}{2}(\mu_j+\mu_{j+1})$	$\frac{1}{3}(2\mu_{j+1}+\mu_j)$	μ_{j+1}
$\overline{\omega_j}$	$\frac{1}{2}(\mu_j+\mu_{j+1})$	$\frac{1}{3}(\mu_{j+1}+2\mu_j)$	μ_j

Table 2

$j \neq 0$	A	B	C
b_j	$\frac{1}{\Delta\mu}\left[1-\left(\frac{\mu_j+\mu_{j+1}}{2}\right)^2\right]$	$\frac{1}{\Delta\mu}\left[1-\left(\frac{\mu_j^2+\mu_j\mu_{j+1}+\mu_{j+1}^2}{3}\right)\right]$	$\frac{1}{\Delta\mu}\left[1-\frac{\mu_j^2+\mu_{j+1}^2}{2}\right]$

Keller and Wendroff (1957) have chosen the coefficients
A in the tables which reduce the equations (4.6) and
(4.7) to the normal form. The entries in the column B
of the tables 1 and 2 are used by Carlson (1955) and

are determined by assuming that intensities vary linearly with μ over the sub-intervals and integrating across each interval. The coefficients C in the tables can be determined by integrating equations (4.1) and (4.2) over μ_j to μ_{j+1} and approximating

$$\int_{\mu_j}^{\mu_{j+1}} \pm\mu \, \frac{\partial I^{\pm}}{\partial r} \, d\mu = 0.5 \left[\pm\mu_j \frac{\partial I_j^{\pm}}{\partial r} \pm \mu_{j+1} \frac{\partial I_{j+1}^{\pm}}{\partial r} \right] \Delta\mu$$

$$\int_{\mu_j}^{\mu_{j+1}} \left(\frac{1-\mu^2}{r} \, \frac{\partial I^{\pm}}{\partial \mu} \right) d\mu \simeq \frac{1}{r} \left[1 - \frac{(\mu_{j+1}^2 + \mu_j^2)}{2} \left\{ I_{j+1}^{\pm} - I_j^{\pm} \right\} \right]$$

and

$$\int_{\mu_j}^{\mu_{i+1}} I^{\pm} \, d\mu' \simeq 0.5 \left[I_{j+1}^{\pm} + I_j^{\pm} \right] \qquad (4.7)$$

At $\mu = \pm 1$, the curvature terms vanish from the equations (4.4) and (4.5)

Writing

$$U^{\pm} = \left[I_1^{\pm} \cdots\cdots I_J^{\pm} \right]^T$$

$$M = \begin{bmatrix} \frac{1}{2}a_1 & \frac{1}{2}\bar{a}_1 & \cdots & \cdots & 0 \\ 0 & \frac{1}{2}a_2 & \frac{1}{2}\bar{a}_2 & \cdots & 0 \\ \vdots & & & & \\ 0 & \cdots & \cdots & \cdots & a_J \end{bmatrix}_{J \times J}$$

$$T = \begin{bmatrix} 0.5 & 0.5 & \cdots & \cdots & 0 \\ 0 & 0.5 & 0.5 & \cdots & 0 \\ & & & & \\ & & & & 1 \end{bmatrix}_{J \times J}$$

$$\Lambda = \begin{bmatrix} -b_1 & b_1 & \cdots & & 0 \\ 0 & -b_2 & b_2 & & 0 \\ \vdots & & & & \vdots \\ \cdots & \cdots & \cdots & \cdots & 0 \end{bmatrix}_{J \times J}$$

$$W = \begin{bmatrix} w_1 & \cdots & & w_J \\ & & & \\ & & & \\ w_1 & \cdots & & w_J \end{bmatrix}_{J \times J}$$

$$\tilde{B} = \begin{bmatrix} B(r) \\ \\ \\ B(r) \end{bmatrix}_{J \times 1}$$

(4.8)

We have for the outward going rays,

$$M\frac{\partial U^+}{\partial r}+\frac{1}{r}\Lambda U^+ + \sigma T U^+ = \sigma\left\{0.5\,\omega(r)\,W\left(U^+ + U^-\right)+(1-\omega)\tilde{B}\right\} \tag{4.9}$$

and for the inward going rays,

$$-M\frac{\partial U^-}{\partial r}-\frac{1}{r}\Lambda U^- + \sigma T U^- = \sigma\left\{0.5\,\omega(r)\,W\left(U^+ + U^-\right)+(1-\omega)\tilde{B}\right\} \tag{4.10}$$

We shall divide the medium into N spherical shells. To perform discretization with respect to the radial coordinate, we integrate equations (4.9) and (4.10) from r_n to r_{n+1} giving

$$M\left[U^+_{n+1}-U^+_n\right]+\tau_{n+\frac{1}{2}}\,T\,U_{n+\frac{1}{2}} = \tau_{n+\frac{1}{2}}\left[\left(1-\omega_{n+\frac{1}{2}}\right)\tilde{B}_{n+\frac{1}{2}}+\left(\frac{1}{2}\omega_{n+\frac{1}{2}}W-\frac{\rho\Lambda}{\tau_{n+\frac{1}{2}}}\right)U^+_{n+\frac{1}{2}} + \left(\frac{1}{2}\,\omega_{n+\frac{1}{2}}W\right)U^-_{n+\frac{1}{2}}\right] \tag{4.11}$$

and

$$M\left[U_n^- - U_{n+1}^-\right] + \tau_{n+\frac{1}{2}} T\, U_{n+\frac{1}{2}}^-$$

$$= \tau_{n+\frac{1}{2}}\left[\left(1-\omega_{n+\frac{1}{2}}\right)\bar{B}_{n+\frac{1}{2}} + \left(\frac{1}{2}\omega_{n+\frac{1}{2}} W + \frac{\rho\Lambda}{\tau_{n+\frac{1}{2}}}\right) U_{n+\frac{1}{2}}^- \right.$$
$$\left. + \left(\frac{1}{2}\omega_{n+\frac{1}{2}} W\right) U_{n+\frac{1}{2}}^+ \right] \tag{4.12}$$

Here $U_n^{\pm} = U^{\pm}(r_n)$ while $U_{n+\frac{1}{2}}^{\pm}$, $\tau_{n+\frac{1}{2}}$, $\omega_{n+\frac{1}{2}}$ and $B_{n+\frac{1}{2}}^{\pm}$ are the suitable averages over the cell. We shall define $\Delta r_{n+\frac{1}{2}} = r_{n+1} - r_n$, $\tau_{n+\frac{1}{2}} = \sigma_{n+\frac{1}{2}}\Delta r_{n+\frac{1}{2}}$ and $\rho = \Delta r_{n+1/2}/r_{n+\frac{1}{2}}$ where $r_{n+1/2}$ is mean radius $\frac{1}{2}(r_{n+1}+r_n)$. We shall use the conventional "Diamond" scheme to approximate the quantities $U_{n+\frac{1}{2}}^{\pm}$ as

$$U_{n+\frac{1}{2}}^{\pm} = \frac{1}{2}\left[U_{n+1}^{\pm} + U_n^{\pm}\right] \tag{4.13}$$

Now we can arrange these equations (4.11) and (4.12) in the canonical form

$$\begin{bmatrix} U_{n+1}^+ \\ U_n^- \end{bmatrix} = \begin{bmatrix} t(n+1,n) & r(n,n+1) \\ r(n+1,n) & t(n,n+1) \end{bmatrix} \begin{bmatrix} U_n^+ \\ U_{n+1}^- \end{bmatrix} + \begin{bmatrix} \Sigma_{n+\frac{1}{2}}^+ \\ \Sigma_{n+\frac{1}{2}}^- \end{bmatrix} \tag{4.14}$$

where $\mathfrak{r}$ and $\mathfrak{t}$ are the reflection and transmission operators defined below:

writing

$$\mathcal{Q}^{+}_{n+\frac{1}{2}} = \frac{\omega}{2}W - \frac{\rho\Lambda}{\tau_{n+\frac{1}{2}}}$$

$$\tilde{\mathcal{Q}}_{n+\frac{1}{2}} = \frac{\omega}{2}W$$

$$\overset{+}{S} = M - \frac{1}{2}\tau_{n+\frac{1}{2}}\left(T - \mathcal{Q}^{+}_{n+\frac{1}{2}}\right)$$

$$\bar{\mathcal{Q}}_{n+\frac{1}{2}} = \frac{\omega}{2}W + \frac{\rho\Lambda}{\tau_{n+\frac{1}{2}}}$$

$$S^{-} = M - \frac{1}{2}\tau_{n+\frac{1}{2}}\left(T - \bar{\mathcal{Q}}_{n+\frac{1}{2}}\right)$$

$$\tilde{S} = \frac{1}{2}\tau_{n+\frac{1}{2}}\tilde{\mathcal{Q}}_{n+\frac{1}{2}}$$

and

$$\overset{+}{\Delta} = \left[M - \frac{1}{2}\tau_{n+\frac{1}{2}}\left(T - \mathcal{Q}^{+}_{n+\frac{1}{2}}\right)\right]^{-1}$$

$$\bar{\Delta} = \left[M - \frac{1}{2}\tau_{n+\frac{1}{2}}\left(T - \bar{\mathcal{Q}}_{n+\frac{1}{2}}\right)\right]^{-1}$$

$$\mathfrak{r}^{+} = \overset{+}{\Delta}\,\tilde{S} \tag{4.15}$$

$$\bar{\mathfrak{r}} = \bar{\Delta}\,\tilde{S}$$

$$\mathfrak{t}^{+} = \left[I - \mathfrak{r}^{+}\bar{\mathfrak{r}}\right]^{-1}$$

$$\mathfrak{t}^{-} = \left[I - \bar{\mathfrak{r}}\,\mathfrak{r}^{+}\right]^{-1}$$

Then

$$t(n+1,n) = \overset{+}{t}\left[\overset{+}{\Delta}\overset{+}{S}+\overset{+}{r}\,\overset{-}{r}\right]$$

$$t(n,n+1) = \overset{-}{t}\left[\overset{-}{\Delta}\overset{-}{S}+\overset{-}{r}\,\overset{+}{r}\right]$$

$$r(n+1,n) = 2\,\overset{-}{t}\,\overset{-}{r}\,\overset{+}{\Delta}\,M$$

$$r(n,n+1) = 2\,\overset{+}{t}\,\overset{+}{r}\,\overset{-}{\Delta}\,M$$

and

$$\overset{+}{\Sigma}_{n+\frac{1}{2}} = \tau_{n+\frac{1}{2}}\left(1-\omega_{n+\frac{1}{2}}\right)\overset{+}{t}\left[\overset{+}{\Delta}B+\overset{+}{r}\,\overset{-}{\Delta}B\right]$$

$$\overset{-}{\Sigma}_{n+\frac{1}{2}} = \tau_{n+\frac{1}{2}}\left(1-\omega_{n+\frac{1}{2}}\right)\overset{-}{t}\left[\overset{-}{\Delta}B+\overset{-}{r}\,\overset{+}{\Delta}B\right] \tag{4.16}$$

We have solved the set of equations (4.14) using the method outlined in chapter 2. We tested the method for a pure absorbing medium with constant thermal source where we know the analytic expression for the radiation field. For a pure scattering medium, we checked the solutions with that of Peraiah and Grant (1973). Our method does not seem to impose any restrictions on the optical thickness τ of an elementary shell, and on the " aspect ratio " $\rho = \Delta r / r$. Also we found that the choice of the coefficients c in the tables 1 and 2 give better results compared to the coefficients A and B. This may be due to the Radau quadrature points which we have employed.

4.3 Numerical Solution of the time-dependent transfer equation in spherically symmetric media

Time-dependent transfer equation is given by

$$\frac{1}{c}\frac{\partial I}{\partial t}(r,\mu,t) + \mu\frac{\partial I}{\partial r}(r,\mu,t) + \frac{1-\mu^2}{r}\frac{\partial I}{\partial \mu}(r,\mu,t) + \sigma(r,t)\,I(r,\mu,t) = \sigma(r,t)\left[\frac{\omega(r,t)}{2}\int_{-1}^{+1} I(r,\mu',t)\,d\mu' + (1-\omega(r,t))\,B(r,t)\right] \quad -1<\mu<1 \tag{4.17}$$

Again we use the coefficients c in tables 1 and 2 discussed in section 4.2 to discretize the μ

variable.. Using the same notation which we employed in section 4.21 we can write equations (4.17) as

$$\frac{1}{c} T \frac{\partial U^{+}(r,t)}{\partial t} + M \frac{\partial U^{+}(r,t)}{\partial r} + \frac{\Lambda}{r} U^{+}(r,t)$$
$$+\sigma(r,t)\, T\, U^{+}(r,t) = \sigma(r,t)\left[\frac{\omega(r,t)}{2} W(U^{+}+\bar{U}) + (1-\omega(r,t))\tilde{B}(r,t)\right] \tag{4.18}$$

and

$$\frac{1}{c} T \frac{\partial \bar{U}(r,t)}{\partial t} - M \frac{\partial \bar{U}(r,t)}{\partial r} - \frac{\Lambda}{r} \bar{U}(r,t)$$
$$+\sigma(r,t)\, T \bar{U}(r,t) = \sigma(r,t)\left[\frac{\omega(r,t)}{2} W(U^{+}+\bar{U}) + (1-\omega(r,t))\tilde{B}(r,t)\right] \tag{4.19}$$

We divide the medium into N layers. We use forward time differences to represent $\frac{\partial U^{\pm}}{\partial t}$. For $\mu > 0$, we use backward space differences and for $\mu < 0$, we use forward one. Then we shall obtain

$$\frac{1}{c} T \left\{ \frac{U^{+}_{n+1}(t_{i+1}) - U^{+}_{n+1}(t_i)}{\Delta t} \right\} + M \left\{ \frac{U^{+}_{n+1}(t_i) - U^{+}_{n}(t_i)}{\Delta r} \right\} +$$

$$0.5 \cdot \frac{\Lambda}{r_{n+1}} \left[U^{+}_{n+1}(t_{i+1}) + U^{+}_{n+1}(t_i) \right] + 0.5\, \tilde{\sigma} T \left[U^{+}_{n+1}(t_i) + U^{+}_{n+1}(t_{i+1}) \right]$$

$$= \tilde{\sigma} \Big\{ 0.25\, \tilde{\omega} \Big[W \big(U^{+}_{n+1}(t_{i+1}) + U^{+}_{n+1}(t_i) \big)$$

$$+ W \big(U^{-}_{n+1}(t_{i+1}) + U^{-}_{n+1}(t_i) \big) \Big] + 0.5 (1 - \tilde{\omega})$$

$$\Big[B^{+}_{n+1}(t_i) + B^{+}_{n+1}(t_{i+1}) \Big] \Big\} \quad \text{for } n = 1,\ldots,N-1 \qquad (4.20)$$

and

$$\frac{1}{c} T \left\{ \frac{\bar{U}_{n+1}(t_{i+1}) - \bar{U}_{n+1}(t_i)}{\Delta t} \right\} - M \left\{ \frac{\bar{U}_{n+2}(t_{i+1}) - \bar{U}_{n+1}(t_i)}{\Delta r} \right\}$$

$$- \frac{0.5 \Lambda}{r_{n+1}} \Big[U^{-}_{n+1}(t_{i+1}) + U^{-}_{n+1}(t_i) \Big] + 0.5\, \tilde{\sigma} T$$

$$\Big[U^{-}_{n+1}(t_i) + \bar{U}_{n+1}(t_{i+1}) \Big] = \tilde{\sigma} \Big\{ 0.25\, \tilde{\omega} \Big[W \big(U^{+}_{n+1}(t_i) + U^{+}_{n+1}(t_{i+1}) \big)$$

$$+ W \big(U^{-}_{n+1}(t_i) + \bar{U}_{n+1}(t_{i+1}) \big) \Big] + 0.5 (1 - \tilde{\omega})$$

$$\Big[B^{+}_{n+1}(t_i) + B^{+}_{n+1}(t_{i+1}) \Big] \Big\}$$

for $n = N-1,\ldots,0.$ (4.21)

where $\tilde{\sigma} = 0.5 \left\{ \sigma(r_{n+1}, t_{i+1}) + \sigma(r_{n+1}, t_i) \right\}$

$$\tilde{\omega} = 0.5 \left\{ \omega(r_{n+1}, t_{i+1}) + \omega(r_{n+1}, t_i) \right\}$$

Since we have used only first order differences in space and time, the truncation error of the scheme is $O(\Delta t) + O(\Delta r)$. As the method follows the characteristics in a certain sense, there is a cancellation of the errors coming from the derivative terms. To make other terms accurate up to second order, we replaced the un-differentiated terms $U^{\pm}_{n+1}(t_{i+1})$ etc. by the averages

$$0.5\left[\; U^{\pm}_{n+1}(t_{i+1}) + U^{\pm}_{n+1}(t_i)\right]$$

Denoting

$$\Delta^{+} = \left[T + \frac{0.5\,\Lambda c\Delta t}{r_{n+1}} + 0.5\,\tilde{\sigma}\,Tc\Delta t - 0.25\,\tilde{\omega}\,Wc\Delta t\right]^{-1}$$

$$S^{+} = T - \frac{Mc\Delta t}{\Delta r} - \frac{0.5\,\Lambda c\Delta t}{r_{n+1}} - 0.5\,\tilde{\sigma}\,Tc\Delta t + 0.25\,\tilde{\omega}\,Wc\Delta t$$

$$D = \frac{Mc\Delta t}{\Delta r}$$

$$Q = 0.25\,\tilde{\omega}\,Wc\Delta t$$

$$\Sigma^{+}_{n+\frac{1}{2}}(t_i) = 0.5\,(1-\tilde{\omega})\,\Delta^{+}\left[B^{+}_{n+1}(t_i) + B^{+}_{n+1}(t_{i+1})\right]c\Delta t$$

$$\Delta^{-} = \left[T - \frac{0.5\,\Lambda c\Delta t}{r_{n+1}} + 0.5\,\tilde{\sigma}\,Tc\Delta t - 0.25\,\tilde{\omega}\,Wc\Delta t\right]^{-1}$$

$$S^{-} = T - \frac{Mc\Delta t}{\Delta r} + \frac{0.5\,\Lambda c\Delta t}{r_{n+1}} - 0.5\,\tilde{\sigma}\,Tc\Delta t + 0.25\,\tilde{\omega}\,Wc\Delta t$$

$$\overline{\sum}_{n+\frac{1}{2}}(t_i) = 0.5\,(1-\tilde{\omega})\,\bar{\Delta}\left[\bar{B}_{n+1}(t_i) + \bar{B}_{n+1}(t_{i+1})\right] c\,\Delta t$$

(4.22)

we have

$$\overset{+}{U}_{n+1}(t_{i+1}) = \overset{+}{\Delta}\,\overset{+}{S}\,\overset{+}{U}_{n+1}(t_i) + \overset{+}{\Delta} D\, U_n^{+}(t_i) + \overset{+}{\Delta} Q\, \bar{U}_{n+1}(t_{i+1}) + \overset{+}{\Delta} Q\, \bar{U}_{n+1}(t_i) + 0.5\,\overset{+}{\Delta}(1-\tilde{\omega})\left\{B_{n+1}^{+}(t_i) + B_{n+1}^{+}(t_{i+1})\right\} c\Delta t.$$

(4.23)

and

$$\bar{U}_{n+1}(t_{i+1}) = \bar{\Delta} D\, \bar{U}_{n+2}(t_i) + \bar{\Delta}\,\bar{S}\,\bar{U}_{n+1}(t_i) + \bar{\Delta} Q\, U_{n+1}^{+}(t_i) + \bar{\Delta} Q\, \overset{+}{U}_{n+1}(t_{i+1}) + 0.5\,\bar{\Delta}\,(1-\tilde{\omega})\, c\Delta t\left[B_{n+1}^{-}(t_i) + B_{n+1}^{-}(t_{i+1})\right]$$

(4.24)

Eliminating $U_{n+1}^{+}(t_{i+1})$ from equation (4.24) using equation (4.23), we get

$$\bar{U}_{n+1}(t_{i+1}) = \left[I - \bar{\Delta} Q\, \overset{+}{\Delta} Q\right]^{-1} \Big[\bar{\Delta} D\, U_{n+2}^{-}(t_i) + \bar{\Delta} Q\, U_{n+1}^{+}(t_i) + \bar{\Delta}\,\bar{S}\,\bar{U}_{n+1}(t_i) + 0.5\,(1-\tilde{\omega})\, c\Delta t\, \bar{\Delta}\left\{B_{n+1}^{-}(t_i) + B_{n+1}^{-}(t_{i+1})\right\} + \bar{\Delta} Q\,\Big\{\Delta^{+} S^{+} U_{n+1}^{+}(t_i) + \Delta^{+} D\, U_n^{+}(t_i) + \Delta^{+} Q\, \bar{U}_{n+1}(t_i) + 0.5\,(1-\tilde{\omega})\, c\Delta t\, \Delta^{+}\left\{B_{n+1}^{+}(t_i) + B_{n+1}^{+}(t_{i+1})\right\}\Big\}\Big]$$

(4.25)

From the initial values at t_1 one can use the equation (4.25) to obtain the values $U^-_{n+1}(t_2)$ for n = N-1,.......,0. Using these values in (4.23), we get $U^+_{n+1}(t_2)$ for $n = 1, 2, \cdots N-1$. We repeat the procedure to obtain the radiation field at the time points $t_2, t_3 \cdots$ etc; Since the matrices M and T are not normal matrices, it is difficult to perform Von-neumann stability analysis.

4.4 Results and Discussion

We have considered the following cases. In all the cases, we have set C = 1.

Case I We have considered a purely absorbing medium with constant thermal source. The ratio of outer to inner radius is chosen as 2 and the absorption coefficient α as 1.

The source function B is given by

$$B(t) = 1 \quad \forall\, t \geq 0$$

The analytical solution is given by

$$\begin{aligned} I(r, -\mu, t) &= \left[1 - e^{-\alpha t}\right] \quad \text{for } s \leq t \\ &= \left[1 - e^{-s}\right] \quad \text{for } s \geq t \\ & \qquad\qquad 0 \leq \mu \leq 1 \end{aligned} \tag{4.26}$$

where

$$s = r\mu - \sqrt{R^2 - r^2(1-\mu^2)}$$

The numerical solutions for μ = 0.59,0.87 and 1.0 agree quite well with the analytical solutions and they are plotted in Fig 1. Solutions for μ = 0.2 show slight instability.

<u>Case II</u> We considered a homogeneous atmosphere with conservative scattering. The boundary conditions chosen are

$$I(r = r_A, \mu, t) = H(t)$$

$$I(r = r_B, -\mu, t) = 0$$

where

$$H(t) \quad = 1 \text{ for } t \geq 0$$
$$\quad = 0 \text{ for } t < 0$$

We distinguish the diffuse field due to one or more scatterings from the reduced incident field without scatterings. Since the directly transmitted intensity at position r and at time t in the direction μ is

$$H\left[t - \left(r\mu - \sqrt{r_A^2 - r^2(1-\mu^2)}\right)\right] \cdot e^{-\alpha\left[r\mu - \sqrt{r_A^2 - r^2(1-\mu^2)}\right]}$$

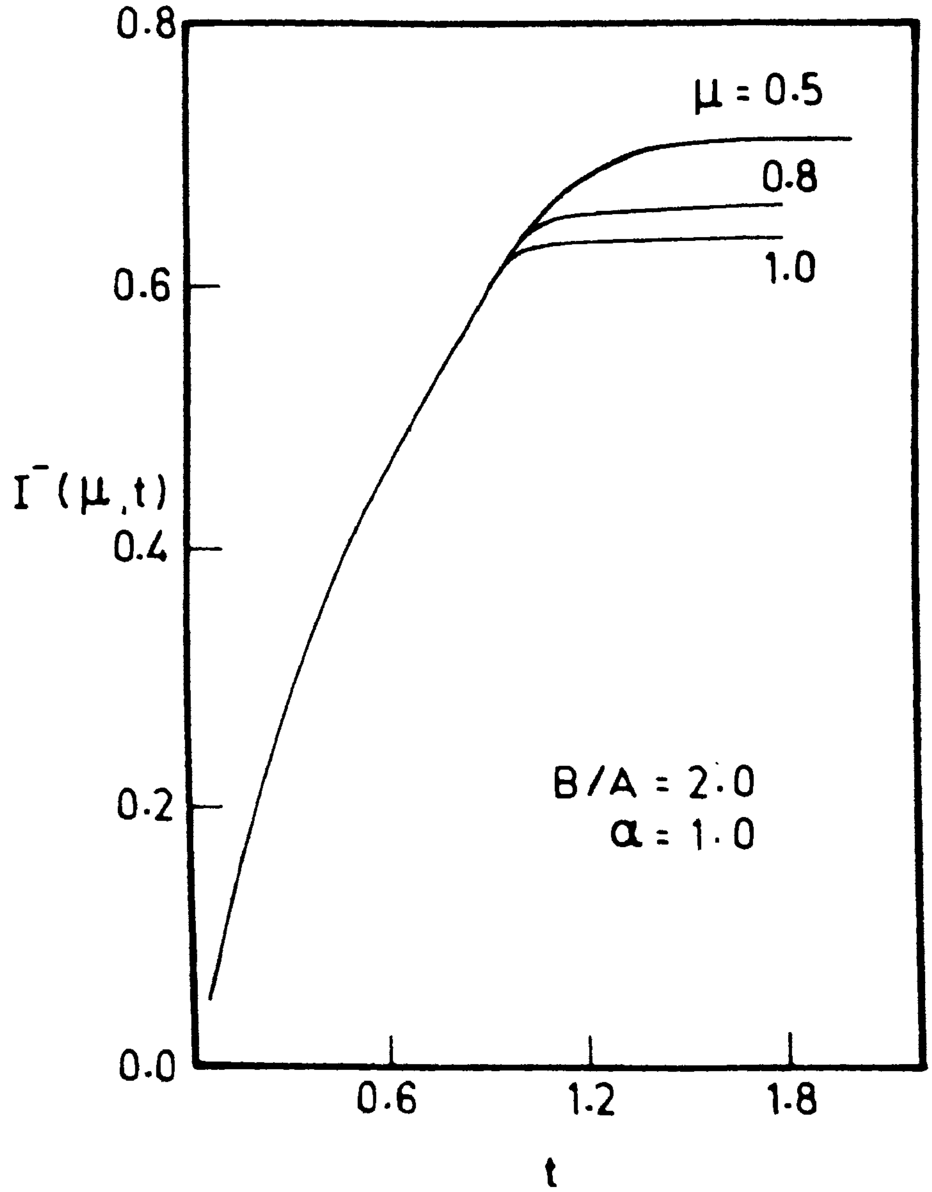


Fig.1

Fig.1. Intensity distribution for a homogeneous purely absorbing medium with constant thermal source.

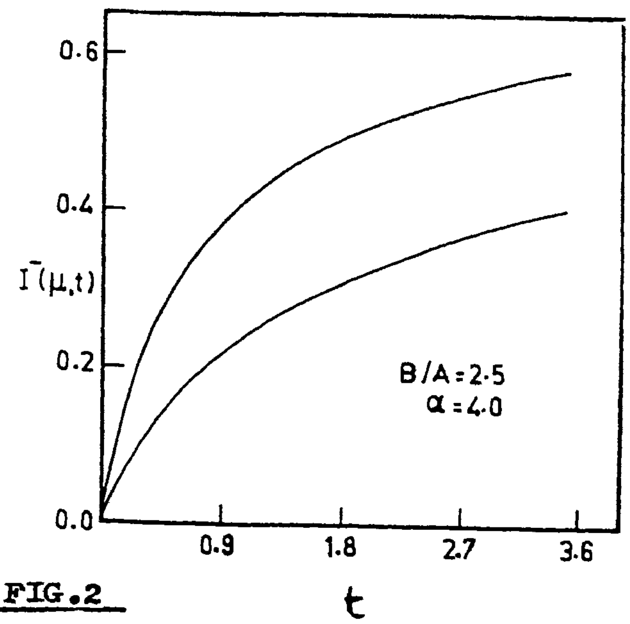


FIG.2

Fig.2. Reflected intensity distribution when a sphere is subjected to a constant radiation for B/A = 2.5, α = 4.0.

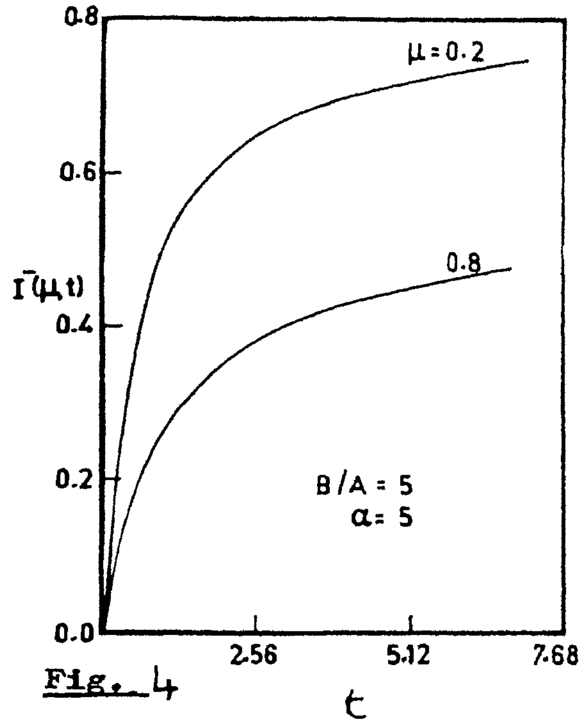


Fig. 4

Same as in Fig 2 with B/A = 5, α = 5.0

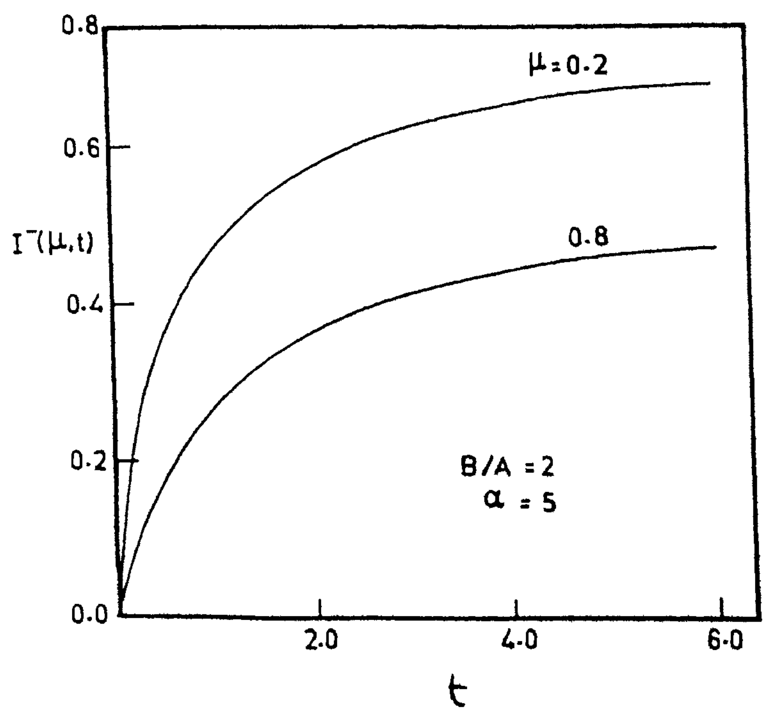


Fig. 3

Same as in Fig.2 with α = 5.0, B/A=2

(Where $\mu^2 \geqslant 1 - \frac{r_A^2}{r^2}$), the transfer equation for the diffuse intensity is

$$\frac{\partial I^{\pm}}{\partial t} \pm \mu \frac{\partial I^{\pm}}{\partial r} \pm \frac{1-\mu^2}{r} \frac{\partial I}{\partial \mu} + \alpha \cdot I(r,\mu,t)$$

$$= 0.5\alpha \int_{-1}^{+1} I(r,\mu',t)\, d\mu' + 0.5\alpha \int_{0}^{1} H\left[t - \left(r\mu' - \sqrt{r_A^2 - r^2(1-\mu'^2)}\right)\right] \cdot e^{-\alpha\left(r\mu' - \sqrt{r_A^2 - r^2(1-\mu'^2)}\right)}\, d\mu'$$

(4.27)

Since we are solving for the diffuse field, the boundary conditions are

$$I\,(r = r_A, \mu, t) = 0$$

$$I\,(r = r_B, -\mu, t) = 0$$

Reflected intensity distribution is plotted in Fig 2 for B/A = 2.5 and α = 4.0. For α = 5.0, Figs 3 and 4 give the corresponding results for B/A = 2.0 and B/A = 5.0. The ratios of the time-dependent reflected intensity at a particular time to the steady values for various ratios are given in Tables 3,4 and 5. Steady state values are calculated from the algorithm described in section 4.2.

Table 3

μ	$\frac{\bar{I}(t, r_A, \mu)}{\bar{I}(r_A, \mu)}$ for $\alpha = 4$, $B/A = 5.$ at time $t = 3.45$
0.21	0.96
0.59	0.87
0.87	0.87
1.00	0.87

Table 4

μ	$\frac{\bar{I}(t, r_A, \mu)}{\bar{I}(r_A, \mu)}$ for $\alpha = 5.0$, $B/A = 2.0$	
	$t = 2.0$	$t = 6.0$
0.21	0.84	0.99
0.59	0.69	0.85
0.87	0.67	0.86
1.00	0.67	0.86

Table 5

μ	$\frac{\bar{I}(t, r_A, \mu)}{\bar{I}(r_A, \mu)}$	for $\alpha = 5.0$, $B/A = 5.$
	$t = 2.0$	$t = 6.7$
0.21	0.80	0.97
0.59	0.56	0.72
0.87	0.52	0.71
1.00	0.51	0.70

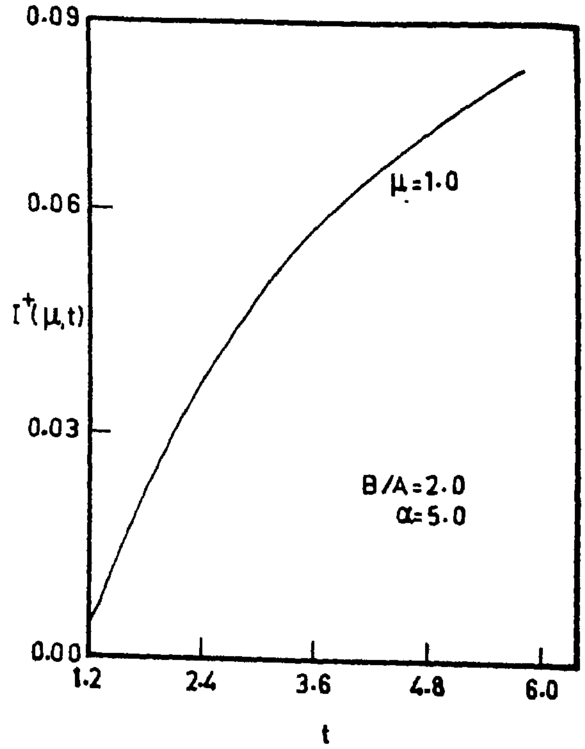


Fig.5

Emergent intensity for the same case as in fig.3

In all the cases, convergence to unity increases for the photons reflected at the grazing angle (i.e. $\mu \simeq 0.2$). From tables 4 and 5, we see that at time $t = 2.0$, the ratio is less at $\mu \simeq 1$ for $B/A = 5.0$ compared to that of $B/A = 2.0$. This is due to the increase in the path length for the photons reflected in the normal direction in a medium with larger sphericity.

We have plotted the diffuse emergent intensity distributions at $\mu = 1.0$ for $B/A = 2.0$ and $\alpha = 5.0$ in Fig 5 up to the time point $t = 6.0$.

CHAPTER V

CONCLUSION AND FUTURE WORK

In this chapter, we shall briefly summarise our results.

We found that the time-dependent problem for $t_1 \gg t_2$ (t_1 is the time spent by the photon in the absorbed state, t_2 is the time spent by the photon between two consecutive acts of scatterings), is equivalent to the problem of finding distribution of photons over the number of scatterings. When a pulsed beam of radiation falls on the medium, the reflected intensity distribution falls gradually with time as the reflected ray approaches the normal direction. If the medium is subjected to an isotropic radiation of constant intensity, the reflected intensity distribution reaches the steady state faster as the angle of reflection approaches the grazing angle. The emergent intensity distribution reaches steady state faster when the emergent ray approaches the normal direction.

The method of characteristics which we proposed is stable and solves the problem when $t_2 \gg t_1$. The time dependent problem for $t_1 = 0, t_2 \neq 0$ is equivalent to the problem of finding the distribution of photons over the path lengths in a homogeneous medium. For $t_2 \gg t_1$, when a slab is illuminated by an isotropic radiation, the behaviour of the reflected radiation is nearly the same, except for the fact that the time at which the relaxation to steady state commences is more for the medium with high optical depth. When the slab is subjected to a pulsed beam of radiation, under the two stream approximation, we found that the sudden drop of the radiation at twice the transit time reduces gradually and smootheness out at higher optical depths.

The method, which we presented to solve the steady state equation in spherical media is stable and gives non-negative solutions. The results agree well with that of Peraiah and Grant (1973).

For $t_2 \gg t_1$, the reflected radiation in the normal direction reaches steady state slowly in a medium with a larger sphericity.

FUTURE WORK

Using the numerical methods presented in the previous chapter, we can consider the situations where the properties of medium vary with time and position. Behaviour of the radiation field in line as well as continuum in planar and spherical media when both t_1 and t_2 are important has to be studied. Also one has to consider the realistic atmospheric models to compare the theoretical results with the observations.

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