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Existence of libration points in the generalized photogravitational restricted problem of three bodies

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Abstract. We prove here the existence of libration points for the generalized photogravitational restricted problem of three bodies. We have assumed the infinitesimal mass to be an oblate spheroid and both the finite masses to be radiating bodies. The effect of their radiation pressure on the motion of the infinitesimal mass has also been taken into account. It is seen that there is a possibility of nine libration points for small values of oblateness, three collinear, four coplanar, and two triangular.

Key words : celestial mechanics—libration points—three body problem

1. Introduction

In stellar systems numerous examples are available where a body is moving under the gravitational field of two radiating bodies. Simmons, McDonald & Brown (1985) have studied the problem when two finite masses are radiating and the infinitesimal mass is spherical. We consider here the case when the infinitesimal mass is an oblate spheroid. We obtain an expression for the radiation pressure, and also study the existence of libration points.

2. General expression for the solar radiation pressure

If the sun's mass be unity, then the force exerted by sunlight pressure on any body is given as $A'P_S/4\pi cmr^2$ in a direction away from the sun along the line joining the sun's centre and the body's centre. Thus the force function is $A'P_S/4\pi cmr$, where A' = the effective cross-section of the satellite; m = satellite mass; P_S = total radiated solar energy (Deutsch 1963). For an oblate spheroidal satellite with equation $(x^2 + y^2)\, a^{-2} + z^2\, c^{-2} = 1$, A' is given by

$$A' = \pi a^2 c\, \{a^2(l^2 + m^2) + c^2 n^2\}^{-1/2},$$

where $lx + my + xz = 0$ is the equation of the plane.

Writing $\sigma = \dfrac{a^2 - c^2}{a^2 + c^2}$, $l^2 + m^2 = 1 - n^2$ and restricting to first order terms in σ, we find that $A' = 5\bar{A}\pi(1 + n^2\sigma)$ where $\bar{A} = m(a^2 + c^2)/5$. Thus $A'P_s/4mc\pi = 5\bar{A}\pi(1 + n^2\sigma)\, P_s/2m.4\pi c = 5\bar{A}P_s(1 + n^2\sigma)/8mc$. If U be the force function due to the gravitational force of the sun on the oblate spheroidal satellite of mass m then

$$U = k^2 m_1 \frac{1}{r} + \frac{\bar{A}\sigma}{2r^3}\,(1 - 3n^2).$$

Hence combining the effect of the solar radiation pressure, the force function U' may be written as

$$U' = U - A'k^2 P_s m_1/4mcr\pi = U - 5\bar{A}P_s(1 + n^2\sigma)\, k^2 m_1/8mcr$$

$$= \left[1 - \frac{5\bar{A}P_s(1 + n^2\sigma)}{8mc}\right]\frac{k^2 m_1}{r} + k^2 m_1 \frac{\bar{A}\sigma}{2r^3}\,(1 - 3n^2).$$

in the notation of Simmons, McDonald & Brown (1985), we have $5\bar{A}P_s/8mc = \beta$ when $\sigma = 0$, and so $U' = [1 - \beta(1 + n^2\sigma)]\,\dfrac{k^2 m_1}{r} + \dfrac{k^2 m_1 \bar{A}\sigma(1 - 3n^2)}{2r^3}$, where β ranges over $0 \leqslant \beta \leqslant \infty$. Since $n^2 \leqslant 1$ and $-1 \leqslant \sigma \leqslant +1$, we get $0 \leqslant 1 + n^2\sigma \leqslant 2$. Thus the introduction of $1 + n^2\sigma$ as a factor produces no change in the range of $\beta : 0 \leqslant \beta \leqslant \infty$. Writing $1 - \beta = \alpha$, we may write the force function U' as

$$U' = \frac{k^2 \alpha m_1}{r} + k^2 m_1 \frac{\bar{A}\sigma}{2r^3}\,(1 - 3n^2). \qquad \text{...(2.1)}$$

Thus the force function due to light pressure is the same whether the satellite is spherical or oblate spheroidal.

3. Equations of motion

Let us consider three bodies M_1, M_2 and M with masses $m_1 \geqslant m_2 \geqslant m$ and assume that M_1 and M_2 are homogeneous spheres or bodies with spherical structure or mass points, and each capable of exerting radiation pressure. Let α_1, α_2 be the effects of the radiation pressure from the two finite masses. Let M be a dynamically symmetrical satellite with an equatorial plane of symmetry and let $\bar{A} = \bar{B}$ and $\bar{C}$ denote the equatorial and polar moments of inertia of the body M. Suppose $r_i (i = 1, 2)$ to be the distance between the centre of mass of the bodies M and n_i to be the cosine of the angle between the radius vector and the axis of the satellite.

As in the classical circular restricted problem we shall assume that the bodies M_1 and M_2 are describing circles with their common centre of mass as the centre. Let us adopt a right-handed barycentric coordinate system $(0, x\, y\, z)$ with the origin at the centre of mass 0 of the bodies M_1 and M_2; x and y axes will be assumed to rotate with the angular velocity about the z-axis orthogonal to the orbital plane of the finite bodies. The units are so chosen that $m_1 + m_2 = 1$, the gravitational constant $= 1$ and the distance between the centres of the two bodies $= 1$. With this frame of reference, let the position of the centre of mass of the satellite be specified by the coordinates x, y, z and its orientation by the usual Eulerian angles ψ, θ, ϕ.

In the above coordinate system the equations of motion may be written as (Choudhry 1977)

$$\left.\begin{aligned} &\ddot{x} - 2\dot{y} = \frac{\partial \Omega}{\partial x}, \quad \ddot{y} + 2\dot{x} = \frac{\partial \Omega}{\partial y}, \quad \ddot{z} = \frac{\partial \Omega}{\partial z} \\ &\ddot{\theta} = (\dot{\psi} + 1)^2 \sin\theta \cos\theta - \epsilon(\dot{\psi} + 1)\sin\theta + \frac{1}{A}\frac{\partial \Omega}{\partial \theta} \\ &\ddot{\psi} = -2(\dot{\psi} + 1)\,\dot{\theta}\cot\theta + \frac{\epsilon\dot{\theta}}{\sin\theta} + \frac{1}{A\sin^2\theta}\frac{\partial \Omega}{\partial \psi} \end{aligned}\right\} \quad \ldots(3.1)$$

where the dot denotes the differentiation with respect to t; and

$$\Omega = \tfrac{1}{2}(x^2 + y^2) + U_1 + U_2,$$

$$U_1 = m_1 \frac{\alpha_1}{r_1} + \frac{A\sigma}{2r_1^3}(1 - 3n_1^2),$$

$$U_2 = m_2 \frac{\alpha_2}{r_2} + \frac{A\sigma}{2r_2^3}(1 - 3n_2^2),$$

$$\sigma = (C - A)/A \; (-1 \leqslant \sigma \leqslant 1), \quad mA = \bar{A}, \quad mC = \bar{C}.$$

Here $\bar{A}$ and $\bar{C}$ are principal moments of inertia for the satellite,

$$\epsilon = [(\dot{\psi} + 1)\cos\theta + \dot{\phi}]\, CA^{-1} = \text{const} = C\bar{r}/A,$$

$$\bar{r} = (\dot{\psi} + 1)\cos\theta + \dot{\phi},$$

$$r_i^2 = (x - x_i)^2 + y^2 + z^2, \; n_i = \frac{(x - x_i)\,a_{13} + ya_{23} + za_{33}}{r_i} \; (i = 1, 2) \quad \ldots(3.2)$$

$$a_{13} = \sin\psi \sin\theta, \quad a_{23} = \cos\psi\sin\theta, \quad a_{33} = \cos\theta.$$

Letting $\mu = m_2/(m_1 + m_2)$, we may write $m_1 = 1 - \mu$, $m_2 = \mu$, $x_1 = -\mu$, $x_2 = 1 - \mu$. Differentiating Ω, we get

(a) $$\frac{\partial \Omega}{\partial x} = x - \sum_{i=1}^{2} m_i \left\{ \frac{\alpha_i(x - x_i)}{r_i^3} + \tfrac{3}{2}A\sigma\frac{x - x_i}{r_i^5}(1 - 3n_i^2) + \frac{3\sigma n_i A}{r_i^3}\left(-\frac{n_i(x - x_i)}{r_i^2} + \frac{a_{13}}{r_i}\right)\right\},$$

(b) $$\frac{\partial \Omega}{\partial y} = y - \sum_{i=1}^{2} m_i \left\{ \frac{\alpha_i y}{r_i^3} + \tfrac{3}{2}A\sigma\frac{y}{r_i^5}(1 - 3n_i^2) + \frac{3A\sigma n_i}{r_i^3}\left(-\frac{n_i y}{r_i^2} + \frac{a_{23}}{r_i}\right)\right\},$$

(c) $$\frac{\partial\Omega}{\partial z} = -\sum_{i=1}^{2} m_i \left\{\frac{z\alpha_i}{r_i^3} + \tfrac{3}{2}A\sigma\frac{z}{r_i^5}(1-3n_i^2) + \frac{3A\sigma n_i}{r_i^3}\left(-\frac{n_i z}{r_i^2} + \frac{a_{33}}{r_i}\right)\right\},$$

(d) $$\frac{\partial\Omega}{\partial\psi} = -\sum_{i=1}^{2} \alpha_i m_i \frac{3A\sigma n_i}{r_i^4}[-a_{23}(x-x_i) + ya_{13}],$$

(e) $$\frac{\partial\Omega}{\partial\theta} = -\sum_{i=1}^{2} \alpha_i m_i \frac{3A\sigma n_i}{r_i^4}[(x-x_i)\sin\psi\cos\theta - y\cos\psi\cos\theta - z\sin\theta]. \quad ...(3.3)$$

4. Conditions for the existence of collinear libration points

For a libration point, we have

$$\frac{\partial\Omega}{\partial x} = 0 = \frac{\partial\Omega}{\partial y} = \frac{\partial\Omega}{\partial z} = \frac{\partial\Omega}{\partial\psi} = \frac{\partial\Omega}{\partial\theta}. \quad ...(4.1)$$

Here we find that $\frac{\partial\Omega}{\partial z} = 0$ is satisfied by $z = 0$ if $a_{33} = \cos\theta = 0$ or when $\sin\theta = 0$ which gives $n_i = \pm z/r_i$. Thus we find that a plane motion is admissible when $\theta = \pi/2$ as well as when $\theta = 0$.

From the condition $\frac{\partial\Omega}{\partial y} = 0$, we find that it is satisfied by $y = 0$ if $\theta = \pi/2$, $\psi = \pi/2, 3\pi/2$ and also if $\theta = 0$, $\psi = 0, \pi$ because in the latter case

$$n_i\left(-\frac{n_i y}{r_i} + a_{23}\right) = \frac{z}{r_i}\left(-\frac{zy}{r_i^2}\right) = 0,$$

which is identically satisfied with $z = 0$. The case $\theta = 0$, $\psi = \pi/2, 3\pi/2$ gives $y = 0$ and it needs no separate treatment. Lastly, let us take up the case $\theta = \pi/2$, $\psi = 0, \pi$. In this case $n_i = y/r_i$ and so $\frac{\partial\Omega}{\partial y} = 0$ is identically satisfied by $y = 0$ and $n_i = 0$.

Thus we find that collinear libration points are possible under the following conditions :

$$\left.\begin{array}{ll} \text{(i)} & \theta = \pi/2, \psi = \pi/2, 3\pi/2, n_i = \pm\dfrac{x - x_i}{r_i} = \pm 1; \\ \text{(ii)} & \theta = 0, \psi = 0, \pi, n_i = 0; \\ \text{(iii)} & \theta = \pi/2, \psi = 0, \pi, n_i = 0; \\ \text{(iv)} & \theta = 0, \psi = \pi/2, 3\pi/2, n_i = 0. \end{array}\right\} \quad ...(4.2)$$

It may be seen that under the above conditions $\frac{\partial\Omega}{\partial\theta} = \theta = \frac{\partial\Omega}{\partial\psi}$ (Choudhry 1977).

Hence we can conclude that under all the four conditions collinear libration points will exist provided $\partial\Omega/\partial x = 0$ gives real roots. These conditions are classified to represent the motion of the types 'spoke', 'level', 'arrow' and 'float' respectively.

Now it remains to investigate the existence of real roots for $\partial\Omega/\partial x = 0$ which will confirm the existence of collinear libration points in the various cases. According to the values of n_i, we shall have two cases $n_i = \pm 1$ or $n_i = 0$. Subsequently we may write $\dfrac{\partial\Omega}{\partial x} = 0$ as

$$x - \sum_{i=1}^{2} \frac{\alpha_i m_i(x - x_i)}{[(x - x_i)^2]^{3/2}} + k\sigma \sum_{i=1}^{2} m_i \frac{x - x_i}{[(x - x_i)^2]^{5/2}} = 0,$$

where $k = -3$ when $n_i = \pm 1$; and $k = 1.5$ when $n_i = 0$. Or,

$$f(x) + k\sigma \sum_{i=1}^{2} m_i \frac{x - x_i}{[(x - x_i)^2]^{5/2}} = 0, \qquad \ldots(4.3)$$

where $f(x) = x - \sum_{i=1}^{2} \alpha_i m_i \dfrac{x - x_i}{[(x - x_i)^2]^{3/2}}$. It may be noted that for $\sigma = 0$ $n_i = \pm 1$ and $n_i = 0$ the resulting equation coincides with the equation for collinear libration points for the classical case (Simmons, McDonald & Brown 1985). Thus there exist three collinear libration points which are discussed below for different values of α_1 and α_2.

(a) It can be shown that for small values of σ, equation (4.3) has distinct real roots, each of which tends to coincide with the libration points when $\sigma = 0$ (*cf.* Choudhry 1977).

Thus we may represent the collinear libration points as

$$L_j^{(s)} : x_0 = \alpha_{js},\ y_0 = 0,\ z_0 = 0\ (j = 1, 2, 3;\ s = 1, 2, 3, 4).$$

We shall assume that $s = 1, 2, 3, 4$ correspond to spoke, level, arrow, float, respectively.

(b) Let us now consider the case when the collinear libration points exist for all σ. For such libration points the conditions can be written as

$$x - \sum_{i=1}^{2} \frac{\alpha_i m_i(x - x_i)}{[(x - x_i)^2]^{3/2}} = 0,$$

$$\sum_{i=1}^{2} \frac{m_i(x - x_i)}{[(x - x_i)^2]^{3/2}} = 0. \qquad \ldots(4.4)$$

Let us call the libration points $L_j (j = 1, 2, 3)$ for all s according as it lies on the right of $M_2 (j = 1)$ or between M_1 and $M_2 (j = 2)$ on the left of $M_1 (j = 3)$.

If we take up the case of L_3, then $x - x_1 < 0$ and $x - x_2 < 0$. In this case we can write equations (4.4) as

$$x + \frac{\alpha_1 m_1}{(x_1 - x)^2} + \frac{\alpha_2 m_2}{(x_2 - x)^2} = 0, \quad \frac{m_1}{(x_1 - x)^4} + \frac{m_2}{(x_2 - x)^4} = 0. \quad \ldots(4.5)$$

The second equation cannot be satisfied since it is the sum of two positive terms and so L_3 will not exist for all σ.

Next, let us take up the case of L_2, the libration point lying between M_1 and M_2. In this case $x - x_1 > 0$ and $x - x_2 < 0$ and equations (4.4) become

$$x - \frac{\alpha_1(1-\mu)}{(x+\mu)^2} + \frac{\alpha_2 \mu}{(x-1+\mu)^2} = 0,$$

$$\frac{1-\mu}{(x+\mu)^4} - \frac{\mu}{(x-1+\mu)^4} = 0. \quad \ldots(4.6)$$

From the second equation, we have $\{(x+\mu)/(x-1+\mu)\}^4 = (1-\mu)/\mu$. Now putting $(x+\mu)^4/(1-\mu) = (x-1+\mu)^4/\mu = k'^2$ (say), we get from the first of the above equations

$$k'^2(1-\mu)^{1/2} - \mu^{1/2} + k'(1-2\mu) - \alpha_1(1-\mu)^{1/2} + (\alpha_2\mu)^{1/2} = 0. \quad \ldots(4.7)$$

The equation (4.7) shows that two roots in k' will always exist and be of opposite signs. If $(x+\mu)^2/(1-\mu)^{1/2} = (x-1+\mu)^2/\mu^{1/2} = -$ ve then x will be imaginary and L_2 will not exist. Thus L_2 will exist for all σ and it will have only one position.

Lastly, let us consider the case of L_1, the libration points lying on the right of M_1. In this case $x - x_1 > 0$ and $x - x_2 > 0$, so we may write equations (4.4) as

$$x - \frac{\alpha_1(1-\mu)}{(x+\mu)^2} - \frac{\alpha_2\mu}{(x-1+\mu)^2} = 0, \quad \frac{1-\mu}{(x+\mu)^4} + \frac{\mu}{(x-1+\mu)^4} = 0. \quad \ldots(4.8)$$

By virtue of the second equation it follows that L_1 does not exist. However for small σ, L_1 does exist (Simmons, McDonald & Brown 1985)

5. Condition for the existence of coplanar libration points

For a libration point, we have

$$\frac{\partial\Omega}{\partial x} = 0 = \frac{\partial\Omega}{\partial y} = \frac{\partial\Omega}{\partial z} = \frac{\partial\Omega}{\partial\psi} = \frac{\partial\Omega}{\partial\theta}. \quad \ldots(5.1)$$

We find that

$$\frac{\partial\Omega}{\partial y} = y - \sum_{i=1}^{2} m_i \left\{\frac{\alpha_i y}{r_i^3} + \tfrac{3}{2}A\sigma\frac{y}{r_i^5}(1-3n_i^2) + \frac{3A\sigma n_i}{r_i^3}\left(-\frac{n_i y}{r_i^2} + \frac{a_{23}}{r_i}\right)\right\},$$

where $n_i = [(x - x_i)\sin\psi \sin\theta - y\cos\psi\sin\theta + z\cos\theta]/r_i$.

It follows that $\partial\Omega/\partial y = 0$ is satisfied by $y = 0$ either when $n_1 = 0$ or $\theta = 0$. Corresponding to the latter case we have for the case $\theta = 0$, $n_1 = z/r_1 = 0$,

$$\frac{\partial\Omega}{\partial\psi} = -\sum_{i=1}^{2} m_i \frac{3\sigma n_i}{r_i^4}\left[-a_{23}(x - x_i) + ya_{13}\right].$$

Thus $\frac{\partial\Omega}{\partial\psi} = 0$ is satisfied in either case, i.e., when $n_1 = 0$ or when $\theta = 0$. Similarly, $\frac{\partial\Omega}{\partial\theta} = -\sum_{i=1}^{2} m_i \frac{3A\sigma n_i}{r_i^4}\left[(x - x_i)\sin\psi\cos\theta - y\cos\psi\cos\theta - z\sin\theta\right]$

can be satisfied for $n_1 = 0$ as well as for $\theta = 0$ if in the latter case $\psi = 0$ or π. We find that

$$\frac{\partial\Omega}{\partial x} = x - \sum_{i=1}^{2} m_i \left\{\frac{\alpha(x - x_i)}{r_i^3} + \tfrac{3}{2}A\sigma\frac{x - x_i}{r_i^5}(1 - 3n_i^2) + \frac{3A\sigma n_i}{r_i^3}\left[-\frac{n_i(x - x_i)}{r_i^2} + \frac{a_{13}}{r_i}\right]\right\}$$

and
$$\frac{\partial\Omega}{\partial z} = -\sum_{i=1}^{2} m_i \left\{\frac{\alpha_i z}{r_i^3} + \tfrac{3}{2}A\sigma\frac{z}{r_i^5}(1 - 3n_i^2) + \frac{3A\sigma n_i}{r_i^3}\left(\frac{n_i z}{r_i^2} + \frac{a_{23}}{r_i}\right)\right\}$$

Case A : *When* $n_1 = 0$

$$\left.\begin{aligned} \frac{\partial\Omega}{\partial x} &= x - \sum_{i=1}^{2} m_i \left\{\frac{\alpha_i(x - x_i)}{r_i^3} + \tfrac{3}{2}A\sigma\frac{(x - x_i)}{r_i^5}\right\}, \\ \text{and}\quad \frac{\partial\Omega}{\partial z} &= -\sum_{i=1}^{2} m_i \left\{\frac{\alpha_i z}{r_i^3} + \tfrac{3}{2}A\sigma\frac{z}{r_i^5}\right\}. \end{aligned}\right\} \qquad \ldots(5.2)$$

Case B: *When* $\theta = 0$

$$\frac{\partial\Omega}{\partial x} = x - \sum_{i=1}^{2} m_i \left\{\alpha_i\frac{(x - x_i)}{r_i^3} + \tfrac{3}{2}A\sigma\frac{x - x_i}{r_i^5}\left(1 - \frac{3z^2}{r_i^2}\right) - \frac{3A\sigma(x - x_i)z^2}{r_i^7}\right\},$$

$$\frac{\partial\Omega}{\partial z} = -\sum_{i=1}^{2} m_i \left\{\frac{\alpha_i z}{r_i^3} + \tfrac{3}{2}A\sigma\frac{z}{r_i^5}\left(1 - \frac{3z^2}{r_i^2}\right) + \frac{3A\sigma z}{r_i^4}\left(-\frac{z^2}{r_i^3} + \frac{1}{r_i}\right)\right\}.$$

In all the cases we find that when $\sigma = 0$, $\frac{\partial \Omega}{\partial z} = \Sigma\, \alpha_i m_i / r_i^3 = 0$ if $z \neq 0$. When α_1 and α_2 are of the same sign, there is no solution. When α_1 and α_2 are of opposite signs, for definiteness, let $\alpha_1 = \delta_1^3 \leqslant 0$ and $\alpha_2 = \delta_2^3 \geqslant 0$. Simmons, McDonald & Brown (1985) find that when $\mu \leqslant \frac{1}{2}$ there is one solution if $g(\delta_1, \delta_2) \leqslant 0$ and $k < 1$, otherwise there is no solution. When $\mu > \frac{1}{2}$

(i) there are two solutions iff $1 < k < \frac{1 + 3\mu}{4 - 3\mu}$ and $g(\delta_1, \delta_2) \leqslant 0$ and $h(\delta_1, \delta_2) < 0$

(ii) there is one solution iff either $k \leqslant 1$ and $g(\delta_1, \delta_2) \leqslant 0$ or $k > 1$ and $g(\delta_1, \delta_2) > 0$

or

$$1 < k \leqslant \frac{1 + 3\mu}{4 - 3\mu} \text{ and } g(\delta_1, \delta_2) = 0$$

or

$$1 < k \leqslant \frac{1 + 3\mu}{4 - 3\mu} \text{ and } h(\delta_1, \delta_2) = 0;$$

and otherwise there is no solution. When $\delta_1 = 0 = \delta_2$ an infinite number of solutions exist along the z-axis with z taking any value. When $\delta_1 \neq 0$, $\delta_2 = 0$, there is no solution.

Since the system $(m_1, m_2, \delta_1, \delta_2)$ is identical with $(m_2, m_1, \delta_2, \delta_1)$, we shall have symmetrically double solution in the corresponding cases. That is, either we shall have two solutions or four solutions or no solution, the conditions being the same when $1 - \mu$ is substituted for μ and δ_1 is substituted for δ_2. Here

$$g(\delta_1, \delta_2) = \delta_1(1 - \mu)^{4/3} + \delta_2\mu^{4/3} - \{-\delta_1(1 - \mu)^{1/3} + \delta_2\mu^{1/3}\}^4,$$

$$h(\delta_1, \delta_2) = \delta_1^2(1 - \mu)^{2/3} - \delta_2^2\mu^{2/3} - (2\mu - 1)^{5/3}(a/2)^{2/3},$$

$$r_1/r_2 = k,\ a = (\tfrac{3}{5})^{3/2} - (\tfrac{3}{5})^{5/2}.$$

If r_1 and r_2 be given in terms of $m_1, m_2, \delta_1, \delta_2$ for $\sigma = 0$ and $r_1 = r_1'$ and $r_2 = r_2'$ be the solution for $\sigma = 0$, then the expression

$$\frac{\partial \Omega}{\partial z} = \sum_{i=1}^{2} m_i \left(\frac{\alpha_i}{r_i^3} - \frac{6A\sigma}{r_i^5} \right)$$

can be expanded in powers of $r_1 - r_1'$ and $r_2 - r_2'$ and σ (when σ is small). By implicit function theorem for $\sigma \neq 0$ and for small values of σ also solutions will be available for r_1 and r_2 in terms of $m_1, m_2, \delta_1, \delta_2$. In other words for small values of σ we shall have solutions in both the cases A and B, provided the conditions stated above for $\sigma = 0$ are satisfied for the existence of roots.

So in general we shall have two libration points, but for the case $1 < k < (1 + 3\mu)/(4 - 3\mu)$, $g(\delta_1, \delta_2) \leqslant 0$, $h(\delta_1, \delta_2) < 0$ we shall have four solutions.

For general values for σ we shall have in Case A

$$\frac{\partial\Omega}{\partial z} = - z \sum_{i=1}^{2} m_i \left\{\frac{\alpha_i}{r_i^3} + \tfrac{3}{2}\frac{A\sigma}{r_i^5}\right\} = 0,$$

$$\frac{\partial\Omega}{\partial x} = x \sum_{i=1}^{2} m_i \left\{\frac{\alpha_i(x - x_i)}{r_i^3} + \tfrac{3}{2}A\sigma\frac{(x - x_i)}{r_i^5}\right\} = 0.$$

So we shall have four equations given as

$$\sum_{i=1}^{2} \frac{\alpha_i m_i}{r_i^3} = 0, \quad \sum_{i=1}^{2} \frac{m_i}{r_i^5} = 0,$$

$$x - \sum_{i=1}^{2} \frac{\alpha_i m_i(x - x_i)}{r_i^3} = 0, \quad \sum_{i=1}^{2} \frac{m_i(x - x_i)}{r_i^5} = 0$$

to determine two unknowns x and z. It is easily seen that the second equation has no solution, since it is a sum of two positive terms; and so a general solution for all σ is not possible. However as we have seen above that for small σ the solution will exist as an analytic continuation of the solution corresponding to $\sigma = 0$.

Similarly in Case B the solution for the general values of σ in general will not exist, but for small σ, the solution will be available.

6. Triangular libration points

We know that the libration points are given by the equations (4.1). From the last three equations of (4.1) it is seen that triangular libration points exist under the following cases :

(a) $n_1 = n_2 = 0, \quad z = 0, \quad y \neq 0.$

From the first two of the equations (4.1) we see that x-coordinates of the libration points are given by

$$x - \sum_{i=1}^{2} m_i \left\{\frac{\alpha_i(x - x_i)}{r_i^3} + \tfrac{3}{2}A\sigma\frac{(x - x_i)}{r_i^5}\right\} = 0, \qquad \text{...(6.1)}$$

$$1 - \sum_{i=1}^{2} m_i \left(\frac{\alpha_i}{r_i^3} + \tfrac{3}{2}\frac{A\sigma}{r_i^5}\right) = 0, \qquad \text{...(6.2)}$$

whence we get

$$\frac{\alpha_1}{r_1^3} - \frac{\alpha_2}{r_2^3} + \tfrac{3}{2}A\sigma\left(\frac{1}{r_1^5} - \frac{1}{r_2^5}\right) = 0. \qquad \text{...(6.3)}$$

By putting $\alpha_1 = \delta_1^3$, $\alpha_2 = \delta_2^3$ and letting $r_1/r_2 = k$, equation (6.3) reduces to

$$k^5(r_2^2\delta_2^3 + \tfrac{3}{2}A\sigma) - r_2^2k^2\delta_1^3 - \tfrac{3}{2}A\sigma = 0.$$

If $\sigma = 0$

$$r_2^2k^2(\delta_2^3k^3 - \delta_1^3) = 0,$$

and equation

$$k = \frac{\delta_1}{\delta_2} = \frac{r_1}{r_2}. \qquad \text{...(6.4)}$$

Similarly from equation (6.2), we get

$$1 - \frac{\delta_1^3(1-\mu)}{r_1^3} - \frac{\delta_2^3\mu}{r_2^3} = 0,$$

which by virtue of equation (6.4) gives $r_1 = \delta_1$ and $r_2 = \delta_2$. In order that for all σ a solution may exist, we should have $\delta_1^3/r_1^3 = \delta_2^3/r_2^3$ and $1/r_1^5 = 1/r_2^5$. Hence

$$\left(\frac{r_1}{r_2}\right)^3 = \left(\frac{\delta_1}{\delta_2}\right)^3 = 1 = k^3,$$

$$r_1 = r_2, \delta_1 = \delta_2.$$

Thus solutions will exist for all σ, if $k = 1$. In this case we shall get $r_1 = r_2$ and $\delta_1 = \delta_2$. Putting these values in equation (6.2), we have

$$r_1^5 - r_1^2\delta_1^3 - \tfrac{3}{2}A\sigma = 0. \qquad \text{...(6.5)}$$

Thus a positive real root always exists for r_1.

By virtue of equation (6.4) we shall assume that for small σ, the solutions are given by

$$r_1 = \delta_1 + A_0(\sigma), \quad r_2 = \delta_2 + A_1(\sigma), \qquad \text{...(6.6)}$$

where $A_0(\sigma)$ and $A_1(\sigma)$ are some continuous functions of σ such that

$$\lim_{\sigma\to 0} A_0(\sigma) = \lim_{\sigma\to 0} A_1(\sigma) = 0.$$

Let us restrict to first order terms alone in σ and write

$$r_1 = \delta_1 + \lambda_0\sigma, \quad r_2 = \delta_2 + \lambda_1\sigma.$$

Since equation (6.5) and a similar equation for r_2 and δ_2 hold for all σ, putting $r_1 = \delta_1 + \lambda_0\sigma$ in equation (6.5) and restricting to first order terms in σ we get $\lambda_0 = A/2\delta_1^4$ and $\lambda_1 = A/2\delta_2^4$. Now, we can write the solutions (6.6) as

$$r_1 = \delta_1 + \frac{A}{2\delta_1^4}, r_2 = \delta_2 + \frac{A}{2\delta_2^4}. \qquad \text{...(6.7)}$$

It may be noted that triangular libration point will exist only when $\delta_1 > 0$ and $\delta_2 > 0$.

(b) Next let us consider the problem of the existence of the triangular points of libration for the case when $n_1 = n_2 \neq 0$.

In order that $z = 0$ may be a solution of $\partial\Omega/\partial z = 0$, $\theta = \pi/2$. Here it follows that $n_1 = n_2$ when $r_1 = r_2 = r$, $\psi = 0, \pi$

$$n = n_1 = n_2 = \pm\, y/r.$$

Then from $\partial\Omega/\partial\psi = 0$ we have

$$-\,3A\sigma n \cos\psi \,\frac{[m_1(x - x_1) + m_2(x - x_2)]}{r^4} = 0,$$

whence we find that

$$x = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = 0. \qquad \ldots(6.8)$$

From $\partial\Omega/\partial x = 0$ we get $x = 0$ for $n_1 = n_2 = n$, $r_1 = r_2 = r$, $\theta = \pi/2$, $\psi = 0, \pi$ whence $(\delta_1^3 - \delta_2^3)\,\mu(1 - \mu) = 0$ and so $\delta_1 = \delta_2$. The equation $\partial\Omega/\partial\theta = 0$ is identically satisfied. From $\partial\Omega/\partial y = 0$ we get on simplification

$$y\left[1 - \left\{\frac{\alpha_1 m_1 + \alpha_2 m_2}{r^3} + \tfrac{3}{2}A\sigma\,\frac{1}{r^5}\left(1 - \frac{3y^2}{r^2}\right) + \frac{3A\sigma}{r^3}\left(-\frac{y^2}{r^4} - \frac{1}{r}\right)\right\}\right] = 0.$$

Setting $r_1 = r_2$, gives $\mu = \tfrac{1}{2}$ and $y^2 = r^2 - \tfrac{1}{4}$. Therefore, r is given by

$$1 - \frac{\delta_1}{r^3} - A\sigma\,\frac{\delta_1}{r^5} - 15A\sigma\,\frac{\delta_1}{8r^7} = 0. \qquad \ldots(6.9)$$

Thus in this case the libration points are given by

$$x = 0,\ y = \sqrt{r^2 - \tfrac{1}{4}},\ z = 0,$$

where r is given by the equation (6.9)

$$r_1 = r_2 = r,\quad n_1 = n_2 = n = \pm\,\frac{y}{r},\ \theta = \frac{\pi}{2},\quad \psi = 0, \pi,\quad \delta_1 = \delta_2.$$

7. Conclusions

We have shown that the oblateness of the infinitesimal mass does not contribute to the radiation pressure due to the two radiating, as well as attracting, bodies of finite masses. Secondly, we have found the possibility of the existence of nine libration points of which three are collinear, four coplanar and two triangular. In general there exist seven libration points of which three are collinear, two coplanar and two triangular.

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