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Atmospheric Thermal Emission Effect on Chandrasekhar's Finite Atmosphere Problem

Soumya Sengupta[1,2]
[1] Indian Institute of Astrophysics, Koramangala 2nd Block, Sarjapura Road, Bangalore, 560034, India; soumya.s@iiap.res.in
[2] Pondicherry University, R.V. Nagar, Kalapet, 605014, Puducherry, India
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Abstract

The solutions of the diffuse reflection finite atmosphere problem are very useful in the astrophysical context. Chandrasekhar was the first to solve this problem analytically, by considering atmospheric scattering. These results have wide applications in the modeling of planetary atmospheres. However, they cannot be used to model an atmosphere with emission. We solved this problem by including the thermal emission effect along with scattering. Here, our aim is to provide a complete picture of the generalized finite atmosphere problem in the presence of scattering and thermal emission, and to give a physical account of the same. For that, we take an analytical approach using the invariance principle method to solve the diffuse reflection finite atmosphere problem in the presence of atmospheric thermal emission. We established general integral equations of the modified scattering function $S(\tau; \mu, \phi; \mu_0, \phi_0)$, transmission function $T(\tau; \mu, \phi; \mu_0, \phi_0)$, and their derivatives with respect to τ for a thermally emitting atmosphere. We customize these equations for the case of isotropic scattering and introduce two new functions $V(\mu)$, and $W(\mu)$, analogous to Chandrasekhar's $X(\mu)$, and $Y(\mu)$ functions, respectively. We also derive a transformation relation between the modified S and T functions and give a physical account of the $V(\mu)$ and $W(\mu)$ functions. Our final results are consistent with those of Chandrasekhar's at the low emission limit (i.e., only scattering). From the consistency of our results, we conclude that the consideration of the thermal emission effect in the diffuse reflection finite atmosphere problem gives more general and accurate results than considering only scattering.

Unified Astronomy Thesaurus concepts: Radiative transfer (1335); Radiative transfer equation (1336); Atmospheric effects (113); Diffuse radiation (383)

1. Introduction

Chandrasekhar did pioneering work on the process of radiative transfer, which is at the heart of observations as well as modeling in astrophysical contexts (Chandrasekhar 1960). One of his most interesting and useful methods is the invariance principle technique, which has a great deal of applications in atmospheric modeling. Although this principle was first introduced by Ambartsumian (1943, 1944), Chandrasekhar (1960) used this theory to solve the semi-infinite and finite atmosphere problems in its most elegant way by introducing the scattering function $S(\tau; \mu, \phi; \mu_0, \phi_0)$ and transmission function $T(\tau; \mu, \phi; \mu_0, \phi_0)$. The final results of those treatments can be represented in terms of the H function (semi-infinite case) Chandrasekhar (1947a) and X and Y functions (finite case) Chandrasekhar (1948). The values of the H function (Chandrasekhar & Breen 1947) and X and Y functions (Chandrasekhar et al. 1952; Chandrasekhar & Elbert 1952) in the case of isotropic scattering are directly used in atmosphere modeling. Even a simple transformation rule between S and T was established by Coakley (1973).

Although the results provided by Chandrasekhar (1960) have direct applications in stellar and planetary problems, the treatment is not complete in some sense as it does not consider atmospheric emission and scattering simultaneously. Bellman et al. (1967) included thermal emission in the planetary atmosphere problem and started a new technique called invariant embedding (Bellman & Wing 1992). In the context of exoplanetary transmission spectra modeling, Sengupta et al. (2020) and Chakrabarty & Sengupta (2020) showed the crucial effect of scattering and atmospheric reemission, respectively. Recently, Sengupta (2021) considered scattering and atmospheric emission simultaneously to study modifications of Chandrasekhar's semi-infinite atmosphere problem. However, the effect of emission on the finite atmosphere problem, which is more general than the semi-infinite one, remains unsolved.

In this work we solve the finite atmosphere problem in the case of isotropic scattering and emission by the same analytical procedure as shown by Sengupta (2021). For that we consider the local thermodynamic equilibrium condition in vertical atmospheric layers, which ensures the fact that each layer contributes to blackbody emission according to Kirchoff's law (Chandrasekhar 1960; Seager 2010). We used the invariance principle method (Ambartsumian 1944; Chandrasekhar 1960) to derive the modified scattering and transmission functions and the final radiation to show that our results are more general than Chandrasekhar's results. This treatment is also free from the isothermal atmosphere condition, which was a limitation of the work of Sengupta (2021).

In Section 2 we state the mathematical formulas of the invariance principles for a finite atmosphere following Chandrasekhar (1960). Section 3 is devoted to deriving the general integral equations of the scattering function (S) and transmission function (T) in case of thermal emission with scattering. The modified form of these functions specifically for the isotropic scattering case is shown in Section 4. Then we establish a simple transformation rule between $S(\mu)$ and $T(\mu)$ in Section 5 and give their physical interpretations in Section 6. The consistency of our new results with the literature is discussed in Section 7 and we conclude with an elaborated discussion in Section 8.

2. Invariance Principle for a Finite Atmosphere

The radiative transfer equation in case of plane-parallel approximation can be written as,

$$\mu\frac{dI_\nu(\tau_\nu, \mu, \phi)}{d\tau_\nu} = I_\nu(\tau_\nu, \mu, \phi) - \xi_\nu(\tau, \mu, \phi). \tag{1}$$

Here, $I_\nu(\tau_\nu, \mu, \phi)$ is the specific intensity at a particular frequency ν, direction cosine μ, angle of azimuth ϕ and optical depth range between τ_ν to $\tau_\nu + d\tau_\nu$. With these same parameters the source function is written as, $\xi_\nu(\tau_\nu, \mu, \phi)$.

For an atmosphere with simultaneous scattering and absorption, the optical depth can be defined as (Domanus & Cogley 1974; Sengupta et al. 2020; Sengupta 2021),

$$d\tau_\nu = -[\kappa_\nu(z) + \sigma_\nu(z)]dz = -\chi_\nu(z)dz \tag{2}$$

Here, $\kappa_\nu(z)$, $\sigma_\nu(z)$ and $\chi_\nu(z)$ are the volumetric absorption coefficient, scattering coefficient, and extinction coefficient at a particular frequency ν and depth z, respectively.

Note that, for the sake of simplicity in further calculations we suppress the subscript ν by considering all the calculations at a particular frequency. It should not be confused with the gray atmosphere approximation as there is no such assumption in the present work.

A finite atmosphere is bounded by optical depth $\tau = 0$ to $\tau = \tau_1$ (Chandrasekhar 1960). To provide a solution of the problem of only diffuse reflection from such an atmosphere, Chandrasekhar (1947b) used the invariance principle method. We will use the same methodology following Chandrasekhar (1960) to get a solution of the more general problem where atmospheric thermal emission is also included with diffuse scattering.

Consider a radiation of light πF incident on an atmosphere of optical thickness τ_1 along the direction $(-\mu_0, \phi_0)$. Then, the diffusely reflected and transmitted intensities can be represented as,

$$\begin{aligned} I(0, \mu, \phi) &= \frac{F}{4\mu}S(\tau_1, \mu, \phi; \mu_0, \phi_0), \\ I(\tau_1, -\mu, \phi) &= \frac{F}{4\mu}T(\tau_1, \mu, \phi; \mu_0, \phi_0), \end{aligned} \tag{3}$$

respectively, where $S(\tau_1, \mu, \phi; \mu_0, \phi_0)$ and $T(\tau_1, \mu, \phi; \mu_0, \phi_0)$ are the scattering and transmission functions. Note that these two intensities refer only to light that has suffered at least one scattering process and they do not include any direct transmission along the $(-\mu_0, \phi_0)$ direction. For a detailed discussion on this, we refer the reader to *Radiative Transfer* by Chandrasekhar (1960).

The four mathematical expressions of invariance principle in a finite atmosphere problem can be written as (Chandrasekhar 1960),

Principle I

$$\begin{aligned} I(\tau, +\mu, \phi) = {} & \frac{F}{4\mu}e^{-\tau/\mu_0}S(\tau_1 - \tau, \mu, \phi; \mu_0, \phi_0) \\ & + \frac{1}{4\pi\mu}\int_0^1\int_0^{2\pi} I(\tau, -\mu', \phi') \\ & \times S(\tau_1 - \tau, \mu, \phi; \mu', \phi')d\mu' d\phi'. \end{aligned} \tag{4}$$

Principle II

$$\begin{aligned} I(\tau, -\mu, \phi) = {} & \frac{F}{4\mu}T(\tau, \mu, \phi; \mu_0, \phi_0) \\ & + \frac{1}{4\pi\mu}\int_0^1\int_0^{2\pi} I(\tau, +\mu', \phi') \\ & \times S(\tau, \mu, \phi; \mu', \phi')d\mu' d\phi'. \end{aligned} \tag{5}$$

Principle III

$$\begin{aligned} \frac{F}{4\mu}S(\tau_1; \mu, \phi; \mu_0, \phi_0) = {} & \frac{F}{4\mu}S(\tau; \mu, \phi; \mu_0, \phi_0) \\ & + e^{-\tau/\mu}I(\tau, +\mu, \phi) \\ & + \frac{1}{4\pi\mu}\int_0^1\int_0^{2\pi} I(\tau, +\mu', \phi') \\ & \times T(\tau, \mu, \phi; \mu', \phi')d\mu' d\phi'. \end{aligned} \tag{6}$$

Principle IV

$$\begin{aligned} \frac{F}{4\mu}T(\tau_1; \mu, \phi; \mu_0, \phi_0) = {} & \frac{F}{4\mu}e^{-\tau/\mu_0}T(\tau_1 - \tau; \mu, \phi; \mu_0, \phi_0) \\ & + e^{-(\tau_1-\tau)/\mu}I(\tau, -\mu, \phi) \\ & + \frac{1}{4\pi\mu}\int_0^1\int_0^{2\pi} I(\tau, -\mu', \phi') \\ & \times T(\tau_1 - \tau, \mu, \phi; \mu', \phi')d\mu' d\phi'. \end{aligned} \tag{7}$$

These equations are derived and diagrammatically shown in Chandrasekhar (1947b, 1960) and Peraiah (2001). The boundary conditions used to calculate $S(\tau_1; \mu, \phi; \mu', \phi')$ and $T(\tau_1; \mu, \phi; \mu', \phi')$ are,

$$I(0, -\mu, \phi) = 0 \text{ and } I(\tau_1, +\mu, \phi) = 0. \tag{8}$$

Chandrasekhar (1960) used these boundary conditions in Equation (1) and derived the four invariance principles (4)–(7) in terms of the source functions $\xi(0, \mu, \phi)$ and $\xi(\tau_1, \mu, \phi)$. We directly use these relations in this paper.

3. The General Integral Equations for a Scattering and Thermally Emitting Atmosphere

When there is atmospheric emission as well as scattering, the source function ξ can be written as (Sengupta 2021),

$$\begin{aligned} \xi(\tau, \mu, \phi) = {} & \beta(\tau, \mu, \phi) + \frac{1}{4}Fe^{-\tau/\mu_0}p(\mu, \phi; -\mu_0, \phi_0) \\ & + \frac{1}{4\pi}\int_{-1}^1\int_0^{2\pi} p(\mu, \phi; \mu''; \phi'') \\ & \times I(\tau, \mu'', \phi'')d\phi'' d\mu''. \end{aligned} \tag{9}$$

Here, $p(\mu, \phi; \mu''; \phi'')$ and $\beta(\tau, \mu, \phi)$ are the phase function and atmospheric emission, respectively. The atmospheric emission β can be expanded (Bellman et al. 1967; Sengupta 2021) as follows,

$$\beta(\tau; \mu; \phi) = \sum_{m=0}^{N} \beta^m(\tau, \mu)\cos m(\phi - \phi_0). \tag{10}$$

For a planetary atmosphere, emission can be caused by different mechanisms (for example, see, Bellman et al. 1967;

Chakrabarty & Sengupta 2020; Malkevich 1963; Sengupta 2021; Seager 2010). In the current study, we consider an atmosphere where each horizontal layer is in local thermodynamic equilibrium and emits only in terms of Planck Emission (Seager 2010), as shown in Figure 1. Hence, considering $m = 0$ with no μ dependencies, Equation (10) will reduce into,

$$\beta(\tau, \mu, \phi) \approx B(T_\tau). \tag{11}$$

Here T_τ represents the absolute temperature of that particular atmospheric layer, which has optical depth τ. It is worth noting that in the case of thermal emission the exact expression of β is $\frac{\kappa}{\chi}B(T_\tau)$. But in the case of the low scattering limit (i.e., $\kappa \gg \sigma$), $\kappa \approx \chi$ and Equation (11) is valid (Sengupta 2021).

Assuming the low scattering approximation, Equation (9) will become,

$$\begin{aligned}\xi(0, \mu, \phi) = B(T_0) + \frac{1}{4}F\Bigg[& p(\mu, \phi; -\mu_0, \phi_0) \\ &+ \frac{1}{4\pi}\int_0^1\int_0^{2\pi} p(\mu, \phi; \mu''; \phi'') \\ &\times S(\tau_1; \mu'', \phi''; \mu_0, \phi_0)d\phi''\frac{d\mu''}{\mu''}\Bigg].\end{aligned} \tag{12}$$

$$\begin{aligned}\xi(\tau_1, \mu, \phi) = B(T_{\tau_1}) + \frac{1}{4}F\Bigg[& e^{-\tau_1/\mu_0}p(\mu, \phi; -\mu_0, \phi_0) \\ &+ \frac{1}{4\pi}\int_0^1\int_0^{2\pi} p(\mu, \phi; -\mu''; \phi'') \\ &\times T(\tau_1; \mu'', \phi''; \mu_0, \phi_0)d\phi''\frac{d\mu''}{\mu''}\Bigg].\end{aligned} \tag{13}$$

Now using Equations (12) and (13) in Equations (23)–(26); p. 168 in Chandrasekhar (1960) we will get (for a detailed derivation see Appendix),

$$\begin{aligned}&\left[\left(\frac{1}{\mu} + \frac{1}{\mu_0}\right)S(\tau_1; \mu, \phi; \mu_0, \phi_0) + \frac{\partial S(\tau_1; \mu, \phi; \mu_0, \phi_0)}{\partial \tau_1}\right] \\ &= 4U(T_0)\left[1 + \frac{1}{4\pi}\int_0^1\int_0^{2\pi} S(\tau_1; \mu, \phi; \mu', \phi')\frac{d\mu'}{\mu'}d\phi'\right] \\ &+ p(\mu, \phi; -\mu_0, \phi_0) + \frac{1}{4\pi}\int_0^1\int_0^{2\pi} p(\mu, \phi; \mu''; \phi'') \\ &\times S(\tau_1; \mu'', \phi''; \mu_0, \phi_0)d\phi''\frac{d\mu''}{\mu''} \\ &+ \frac{1}{4\pi}\int_0^1\int_0^{2\pi} S(\tau_1; \mu, \phi; \mu', \phi')\Bigg[p(-\mu', \phi'; -\mu_0, \phi_0) \\ &+ \frac{1}{4\pi}\int_0^1\int_0^{2\pi} p(-\mu', \phi'; \mu''; \phi'') \\ &\times S(\tau_1; \mu'', \phi''; \mu_0, \phi_0)d\phi''\frac{d\mu''}{\mu''}\Bigg]\frac{d\mu'}{\mu'}d\phi',\end{aligned} \tag{14}$$

$$\begin{aligned}&\left[\frac{\partial S(\tau_1; \mu, \phi; \mu_0, \phi_0)}{\partial \tau_1}\right] = 4U(T_{\tau_1}) \\ &\times\left[e^{-\tau_1/\mu} + \frac{1}{4\pi}\int_0^1\int_0^{2\pi} T(\tau_1; \mu, \phi; \mu', \phi')\frac{d\mu'}{\mu'}d\phi'\right] \\ &+ \left[\exp\left\{-\tau_1\left(\frac{1}{\mu_0} + \frac{1}{\mu}\right)\right\}p(\mu, \phi; -\mu_0, \phi_0)\right. \\ &+ \frac{1}{4\pi}e^{-\tau_1/\mu}\int_0^1\int_0^{2\pi} p(\mu, \phi; -\mu''; \phi'') \\ &\left.\times T(\tau_1; \mu'', \phi''; \mu_0, \phi_0)d\phi''\frac{d\mu''}{\mu''}\right] \\ &+ \frac{1}{4\pi}\int_0^1\int_0^{2\pi} T(\tau_1; \mu, \phi; \mu', \phi')\Bigg[e^{-\tau_1/\mu_0}p(\mu', \phi'; -\mu_0, \phi_0) \\ &+ \frac{1}{4\pi}\int_0^1\int_0^{2\pi} p(\mu', \phi'; -\mu''; \phi'') \\ &\times T(\tau_1; \mu'', \phi''; \mu_0, \phi_0)d\phi''\frac{d\mu''}{\mu''}\Bigg]\frac{d\mu'}{\mu'}d\phi'.\end{aligned} \tag{15}$$

$$\begin{aligned}&\left[\frac{1}{\mu}T(\tau_1; \mu, \phi; \mu_0, \phi_0) + \frac{\partial T(\tau_1; \mu, \phi; \mu_0, \phi_0)}{\partial \tau_1}\right] \\ &= 4U(T_{\tau_1})\left[1 + \frac{1}{4\pi}\int_0^1\int_0^{2\pi} S(\tau_1; \mu, \phi; \mu', \phi')\frac{d\mu'}{\mu'}d\phi'\right] \\ &+ \left[e^{-\tau_1/\mu_0}p(-\mu, \phi; -\mu_0, \phi_0)\right. \\ &+ \frac{1}{4\pi}\int_0^1\int_0^{2\pi} p(-\mu, \phi; -\mu''; \phi'') \\ &\left.\times T(\tau_1; \mu'', \phi''; \mu_0, \phi_0)d\phi''\frac{d\mu''}{\mu''}\right] \\ &+ \frac{1}{4\pi}\int_0^1\int_0^{2\pi} S(\tau_1; \mu, \phi; \mu', \phi')\Bigg[e^{-\tau_1/\mu_0}p(\mu', \phi'; -\mu_0, \phi_0) \\ &+ \frac{1}{4\pi}\int_0^1\int_0^{2\pi} p(\mu', \phi'; -\mu''; \phi'') \\ &\times T(\tau_1; \mu'', \phi''; \mu_0, \phi_0)d\phi''\frac{d\mu''}{\mu''}\Bigg]\frac{d\mu'}{\mu'}d\phi'\end{aligned} \tag{16}$$

$$\begin{aligned}&\left[\frac{1}{\mu_0}T(\tau_1; \mu, \phi; \mu_0, \phi_0) + \frac{\partial T(\tau_1; \mu, \phi; \mu_0, \phi_0)}{\partial \tau_1}\right] \\ &= 4U(T_0)\left[e^{-\tau_1/\mu} + \frac{1}{4\pi}\int_0^1\int_0^{2\pi} T(\tau_1; \mu, \phi; \mu', \phi')\frac{d\mu'}{\mu'}d\phi'\right] \\ &+ \Bigg[p(-\mu, \phi; -\mu_0, \phi_0) + \frac{1}{4\pi}\int_0^1\int_0^{2\pi} p(-\mu, \phi; \mu''; \phi'') \\ &\times S(\tau_1; \mu'', \phi''; \mu_0, \phi_0)d\phi''\frac{d\mu''}{\mu''}\Bigg]e^{-\tau_1/\mu} \\ &+ \frac{1}{4\pi}\int_0^1\int_0^{2\pi} T(\tau_1; \mu, \phi; \mu', \phi')\Bigg[p(-\mu', \phi'; -\mu_0, \phi_0) \\ &+ \frac{1}{4\pi}\int_0^1\int_0^{2\pi} p(-\mu', \phi'; \mu'', \phi'') \\ &\times S(\tau_1; \mu'', \phi''; \mu_0, \phi_0)d\phi''\frac{d\mu''}{\mu''}\Bigg]\frac{d\mu'}{\mu'}d\phi',\end{aligned} \tag{17}$$

where $U(T_0) = \frac{B(T_0)}{F}$, $U(T_{\tau_1}) = \frac{B(T_{\tau_1})}{F}$.

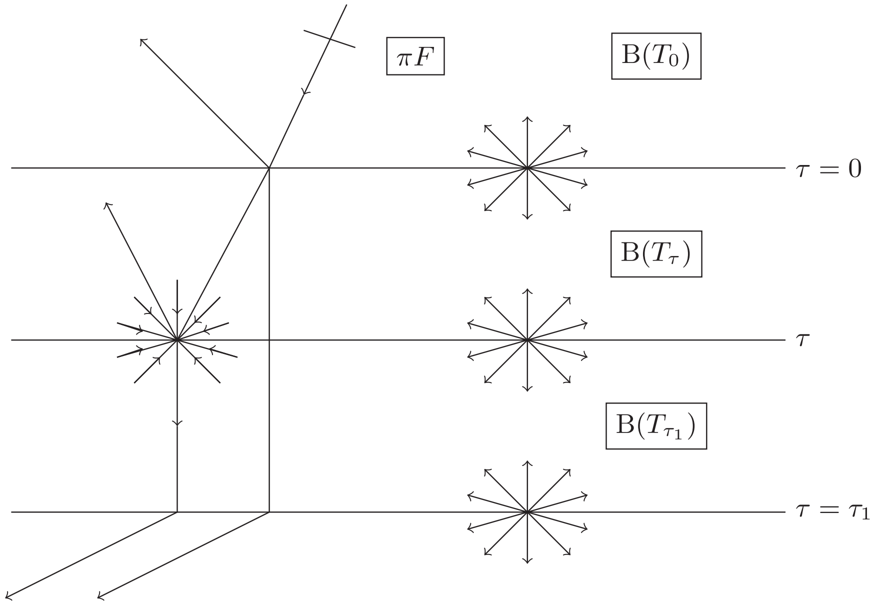

Figure 1. This figure depicts the total effect of diffuse scattering and thermal emission B for a finite atmosphere. Considering an arbitrary layer with optical depth τ $(0 < \tau < \tau_1)$ we show the effect of scattering and emission, while B is isotropic in nature and depends only on the temperature of the emitting layer.

Equations (14)–(17) represent the integral equations governing the problem of diffuse reflection and transmission in the presence of atmospheric thermal emission of a plane-parallel atmosphere with a finite optical depth.

4. The Integral Equations in Isotropic Scattering

It is evident that these four integral equations have an explicit dependency on the phase function $p(\mu, \phi; \mu', \phi')$. The different types of phase functions are discussed in Chandrasekhar (1960) and Sengupta (2021). Here, we specifically study the effect of thermal emission in the isotropic scattering case only. It can be treated in terms of single scattering albedo $\tilde{\omega}_0$ (Sengupta 2021) as,

$$p(\mu, \phi; \mu_0, \phi_0) = \varpi_0.$$

This axial symmetry in the phase function is also a property of the scattering and transmission functions, and they can be expressed in axisymmetric terms as, $S(\tau_1; \mu; \mu')$ and $T(\tau_1; \mu; \mu')$.

Then, Equation (15) will become,

$$\begin{aligned}\left[\frac{\partial S(\tau_1; \mu, \mu_0)}{\partial \tau_1}\right] = {} & 4U(T_{\tau_1})\left[e^{-\tau_1/\mu} + \frac{1}{2}\int_0^1 T(\tau_1; \mu; \mu')\frac{d\mu'}{\mu'}\right] \\ & + \left[\exp\left\{-\tau_1\left(\frac{1}{\mu_0} + \frac{1}{\mu}\right)\right\}\tilde{\omega}_0\right. \\ & \left(+\frac{1}{2}e^{-\tau_1/\mu}\int_0^1 \tilde{\omega}_0 T(\tau_1; \mu'', \mu_0)\frac{d\mu''}{\mu''}\right] \\ & + \frac{1}{2}\int_0^1 T(\tau_1; \mu, \mu')\left[e^{-\tau_1/\mu_0}\tilde{\omega}_0\right. \\ & \left. + \frac{1}{2}\int_0^1 \tilde{\omega}_0 T(\tau_1; \mu'', \mu_0)\frac{d\mu''}{\mu''}\frac{d\mu'}{\mu'}\right]\end{aligned}$$

$$\boxed{\left[\frac{\partial S(\tau_1; \mu, \mu_0)}{\partial \tau_1}\right] = 4U(T_{\tau_1})W(\mu) + \varpi_0 W(\mu_0)W(\mu).} \quad (18)$$

Here we define two new functions as,

$$V(\mu) = 1 + \frac{1}{2}\int_0^1 S(\tau_1; \mu, \mu')\frac{d\mu'}{\mu'}, \quad (19)$$

and

$$W(\mu) = e^{-\tau_1/\mu} + \frac{1}{2}\int_0^1 T(\tau_1; \mu, \mu')\frac{d\mu'}{\mu'}. \quad (20)$$

Similarly, Equations (14), (16), and (17) can be expressed in terms of the V and W functions as follows,

$$\boxed{\begin{aligned}&\left(\frac{1}{\mu} + \frac{1}{\mu_0}\right)S(\tau_1; \mu, \mu_0) = 4[U(T_0)V(\mu) - U(T_{\tau_1})W(\mu)] \\ &+\varpi_0[V(\mu)V(\mu_0) - W(\mu)W(\mu_0)],\end{aligned}} \quad (21)$$

$$\left[\frac{1}{\mu_0}T(\tau_1; \mu, \mu_0) + \frac{\partial T(\tau_1; \mu, \mu_0)}{\partial \tau_1}\right] = 4U(T_0)W(\mu) + \varpi_0 V(\mu_0)W(\mu), \quad (22)$$

$$\left[\frac{1}{\mu}T(\tau_1; \mu, \mu_0) + \frac{\partial T(\tau_1; \mu, \mu_0)}{\partial \tau_1}\right] = 4U(T_{\tau_1})V(\mu) + \varpi_0 W(\mu_0)V(\mu). \quad (23)$$

Subtracting Equation (23) from Equation (22) gives,

$$\boxed{\begin{aligned}\left(\frac{1}{\mu_0}-\frac{1}{\mu}\right)T(\tau_1;\mu,\mu_0) &= 4[U(T_0)W(\mu)-U(T_{\tau_1})V(\mu)]\\ &\quad+\widetilde{\omega}_0[V(\mu_0)W(\mu)-W(\mu_0)V(\mu)].\end{aligned}} \tag{24}$$

Subtracting Equation $(22)^{*}\frac{1}{\mu}$ from Equation $(23)^{*}\frac{1}{\mu_0}$ gives,

$$\boxed{\begin{aligned}&\left(\frac{1}{\mu_0}-\frac{1}{\mu}\right)\frac{\partial T(\tau_1;\mu,\mu_0)}{\partial\tau_1}\\ &=4\left[\frac{1}{\mu_0}U(T_{\tau_1})V(\mu)-\frac{1}{\mu}U(T_0)W(\mu)\right]\\ &+\widetilde{\omega}_0\left[\frac{1}{\mu_0}W(\mu_0)V(\mu)-\frac{1}{\mu}V(\mu_0)W(\mu)\right].\end{aligned}} \tag{25}$$

Thus, the functional form of V and W functions (Equations (19) and (20)) can be modified as,

$$V(\mu)=1+\frac{1}{2}\mu\int_0^1\{4[U(T_0)V(\mu)-U(T_{\tau_1})W(\mu)] + \widetilde{\omega_0}[V(\mu)V(\mu_0)-W(\mu)W(\mu_0)]\}\frac{d\mu'}{\mu'+\mu}, \tag{26}$$

and

$$W(\mu)=e^{-\tau_1/\mu}+\frac{1}{2}\mu\int_0^1 4[U(T_0)W(\mu)-4U(T_{\tau_1})V(\mu)] + \widetilde{\omega}_0[V(\mu_0)W(\mu)-W(\mu_0)V(\mu)]\frac{d\mu'}{\mu-\mu'}. \tag{27}$$

The final emitted radiation from $\tau=0$ and $\tau=\tau_1$ can be expressed from Equation (3) as,

$$I(0,\mu)=\frac{\mu_0}{\mu+\mu_0}[B(T_0)V(\mu)-B(T_{\tau_1})W(\mu)] + \frac{F}{4}\frac{\mu_0}{\mu+\mu_0}\widetilde{\omega}_0[V(\mu)V(\mu_0)-W(\mu)W(\mu_0)], \tag{28}$$

and

$$I(\tau_1,-\mu)=\frac{\mu_0}{\mu-\mu_0}[B(T_0)W(\mu)-B(T_{\tau_1})V(\mu)] + \frac{F}{4}\frac{\mu_0}{\mu-\mu_0}\widetilde{\omega}_0[V(\mu_0)W(\mu)-W(\mu_0)V(\mu)]. \tag{29}$$

Equations (28) and (29) can be expressed as,

$$\begin{bmatrix}(\mu+\mu_0)I(0,+\mu)\\(\mu-\mu_0)I(\tau_1,-\mu)\end{bmatrix}=\begin{bmatrix}V(\mu) & W(\mu)\\ W(\mu) & V(\mu)\end{bmatrix} \times\begin{bmatrix}\frac{F}{4}\widetilde{\omega}_0\mu_0V(\mu_0)+\mu_0B(T_0)\\ -\frac{F}{4}\widetilde{\omega}_0\mu_0W(\mu_0)-\mu_0B(T_{\tau_1})\end{bmatrix}. \tag{30}$$

5. A Simple Transformation Rule

Chandrasekhar (1960) introduced two crucial functions, $S(\tau_1;\mu,\phi;\mu_0,\phi_0)$ and $T(\tau_1;\mu,\phi;\mu_0,\phi_0)$, while considering diffuse scattering in a finite atmosphere. The transformation rule between these two functions was established by Coakley (1973) as,

$$\begin{aligned}S(\tau_1,\mu,\phi;-\mu_0,\phi_0)e^{-\tau_1/\mu_0}&=T(\tau_1,\mu,\phi;\mu_0,\phi_0)\\ T(\tau_1,\mu,\phi;-\mu_0,\phi_0)e^{-\tau_1/\mu_0}&=S(\tau_1,\mu,\phi;\mu_0,\phi_0).\end{aligned} \tag{31}$$

Here we will show that these rules are indeed true while the thermal emission is included in this problem under some circumstances. We replace μ_0 by $-\mu_0$ in Equation (14) and multiplying both sides by $e^{-\tau_1/\mu_0}$ and get,

$$\begin{aligned}&\therefore\left[\left(\frac{1}{\mu}-\frac{1}{\mu_0}\right)S(\tau_1;\mu,\phi;-\mu_0,\phi_0)e^{-\tau_1/\mu_0} + \frac{\partial S(\tau_1;\mu,\phi;-\mu_0,\phi_0)}{\partial\tau_1}e^{-\tau_1/\mu_0}\right]\\ &=4U(T_0)\left[1+\frac{1}{4\pi}\int_0^1\int_0^{2\pi}S(\tau_1;\mu,\phi;\mu',\phi')\frac{d\mu'}{\mu'}d\phi'\right]e^{-\tau_1/\mu_0}\\ &\quad+p(\mu,\phi;\mu_0,\phi_0)e^{-\tau_1/\mu_0}\\ &\quad+\frac{1}{4\pi}\int_0^1\int_0^{2\pi}p(\mu,\phi;\mu'';\phi'')\\ &\quad\times S(\tau_1;\mu'',\phi'';-\mu_0,\phi_0)e^{-\tau_1/\mu_0}d\phi''\frac{d\mu''}{\mu''}\\ &+\frac{1}{4\pi}\int_0^1\int_0^{2\pi}S(\tau_1;\mu,\phi;\mu',\phi')\left[p(-\mu',\phi';\mu_0,\phi_0)e^{-\tau_1/\mu_0}\right.\\ &\quad+\frac{1}{4\pi}\int_0^1\int_0^{2\pi}p(-\mu',\phi';\mu'';\phi'')\\ &\quad\left.\times S(\tau_1;\mu'',\phi'';-\mu_0,\phi_0)e^{-\tau_1/\mu_0}d\phi''\frac{d\mu''}{\mu''}\right]\frac{d\mu'}{\mu'}d\phi'.\end{aligned} \tag{32}$$

Now if we make use of Equation (31) and the symmetric properties of the phase function $p(\mu,\phi;-\mu_0,\phi_0)=p(-\mu,\phi;\mu_0,\phi_0)$ and $p(-\mu,\phi;-\mu_0,\phi_0)=p(\mu,\phi;\mu_0,\phi_0)$ then Equation (32) will be,

$$\begin{aligned}&\therefore\frac{1}{\mu}T(\tau_1;\mu,\phi;\mu_0,\phi_0)+\frac{\partial T(\tau_1;\mu,\phi;\mu_0,\phi_0)}{\partial\tau_1}\\ &=4U(T_0)e^{-\tau_1/\mu_0}\left[1+\frac{1}{4\pi}\int_0^1\int_0^{2\pi}S(\tau_1;\mu,\phi;\mu',\phi')\frac{d\mu'}{\mu'}d\phi'\right]\\ &\quad+p(-\mu,\phi;-\mu_0,\phi_0)e^{-\tau_1/\mu_0}\\ &+\frac{1}{4\pi}\int_0^1\int_0^{2\pi}p(\mu,\phi;\mu'';\phi'')T(\tau_1;\mu'',\phi'';\mu_0,\phi_0)d\phi''\frac{d\mu''}{\mu''}\\ &+\frac{1}{4\pi}\int_0^1\int_0^{2\pi}S(\tau_1;\mu,\phi;\mu',\phi')\left[p(\mu',\phi';-\mu_0,\phi_0)e^{-\tau_1/\mu_0}\right.\\ &\quad+\frac{1}{4\pi}\int_0^1\int_0^{2\pi}p(\mu',\phi';-\mu'';\phi'')\\ &\quad\left.\times T(\tau_1;\mu'',\phi'';\mu_0,\phi_0)d\phi''\frac{d\mu''}{\mu''}\right]\frac{d\mu'}{\mu'}d\phi'.\end{aligned} \tag{33}$$

Observing the similarity of this equation with Equation (16), we can write the condition on thermal emission,

$$U(T_{\tau_1}) = U(T_0)e^{-\tau_1/\mu_0} \qquad B(T_{\tau_1}) = B(T_0)e^{-\tau_1/\mu_0}. \tag{34}$$

So the transformation rule between the S function and T function (Equation (31)) will be valid when the thermal emission from different atmospheric layers are connected by Equation (34). This blackbody emission can be named as the reduced thermal emission.

For the isotropic scattering case, the transformation rules of V and W can be obtained using Equations (31), (19), and (20) as,

$$V(-\mu)e^{-\tau_1/\mu} = W(\mu) \qquad W(-\mu)e^{-\tau_1/\mu} = V(\mu). \tag{35}$$

6. The Physical Meaning of the V and W Functions

We introduced two new functions, $V(\mu)$ and $W(\mu)$, which resemble Chandrasekhar's $X(\mu)$ and $Y(\mu)$ functions, respectively, in the presence of thermal emission with the diffusely reflecting finite atmosphere problem. The physical meaning of the X function and Y function has been discussed by Chandrasekhar (1960), Van de Hulst (1948), and Peraiah (2001). Here we will discuss the additional effects introduced in the V and W functions.

Let there be a point source above the layer $\tau = 0$, which has unit brightness (with a total emitting flux of 4π from the point source). Now the flux will be scattered and transmitted multiple times by both the atmospheric layers at $\tau = 0$ and $\tau = \tau_1$ (see Figure 2). In addition to it there is a contribution of thermal emission $B(T_0)$ and $B(T_{\tau_1})$, respectively, from those layers. Now for an observer at a large distance, the combination of the same point source and the illuminated atmosphere will again appear as a point source and only the combined effect can be observed. If that distant observer is in the $(+\mu, \phi)$ direction from the atmosphere (i.e., above the atmospheric layer $\tau = 0$ in Figure 2), then $V(\mu)$ will be the total observed brightness. In the same way, if the observer is in the $(-\mu, \phi)$ direction from the atmosphere (i.e., below the atmospheric layer $\tau = \tau_1$ in Figure 2), then $W(\mu)$ will be the total observed brightness. In both the cases, the factor $\frac{1}{\mu}$ is positive.

In other words, $V(\mu)$ and $W(\mu)$ represent the relative change of the incident and transmitted flux along the $(+\mu)$ and $(-\mu)$ directions, respectively, due to the presence of the atmosphere. This relative change shows the combined effect of scattering, transmission, and thermal emission by the atmospheric layers.

Clearly in the absence of thermal emission, the contribution of thermal emission will be removed and the observed brightness will be a combination of atmospheric scattering and transmission of the point source flux only. In such circumstances, the $V(\mu)$ and $W(\mu)$ functions will reduce into Chandrasekhar's $X(\mu)$ and $Y(\mu)$ functions only (see Section 7 for more discussion). Also Figure 2 will reduce into the figure given in Van de Hulst (1948).

In case of a semi-infinite atmosphere, the bottom layer will be extended at $\tau_1 \to \infty$ as shown in Figure 3. In such circumstances, the distant observer can observe the combined effect from $(+\mu, \phi)$ direction only. Hence the $W(\mu)$ function will vanish and the $V(\mu)$ function will give the combined effect of scattering and thermal emission. In such a case *the* $V(\mu)$ function will reduce into the well known $M(\mu)$ function as introduced by Sengupta (2021) for the semi-infinite atmosphere case.

Hence the V and W functions represent the relative change of the flux from point source due to the presence of atmospheric scattering, transmission, and thermal emission.

7. Consistency Check

In this section we will show that how our results reduce into previous literature results at specific boundary conditions. It is expected that, when the atmospheric thermal emission is very much less than the incident flux (i.e., $B(T_0)$, $B(T_{\tau_1}) \ll F$), then our solutions should match with the results of only the scattering case as derived by Chandrasekhar (1960).

In the case of the no thermal emission limit, $U(T_0)$, $U(T_{\tau_1}) \to 0$ and thus Equations (26) and (27) will reduce into those of Chandrasekhar's X and Y functions, as shown by Chandrasekhar (1960) (p. 181; Equations (84)–(85))

$$\lim_{U\to 0} V(\mu) \to X(\mu),$$

and

$$\lim_{U\to 0} W(\mu) \to Y(\mu).$$

Hence in the limit of $U \to 0$, Equations (21) and (24) will become,

$$\left[\frac{1}{\mu} + \frac{1}{\mu_0}\right] S(\tau_1;\, \mu,\, \mu_0) = \varpi_0[X(\mu)X(\mu_0) - Y(\mu)Y(\mu_0)],$$

and

$$\left[\frac{1}{\mu_0} + \frac{1}{\mu}\right] T(\tau_1;\, \mu,\, \mu_0) = \varpi_0[X(\mu_0)Y(\mu) - Y(\mu_0)X(\mu)]. \tag{36}$$

These equations are the same as given by Chandrasekhar (1960) (p. 181; Equations (80)–(81)). The no thermal emission limit will also affect the final radiation coming out from both of the boundaries at $\tau = 0$ and $\tau = \tau_1$. Hence Equations (28) and (29) will reduce into the scattering-only case. For the no thermal emission case we can write the matrix Equation (30) as follows,

$$\begin{bmatrix} (\mu + \mu_0)I(0, +\mu) \\ (\mu - \mu_0)I(\tau_1, -\mu) \end{bmatrix} = \begin{bmatrix} X(\mu) & Y(\mu) \\ Y(\mu) & X(\mu) \end{bmatrix} \times \begin{bmatrix} \frac{F}{4}\varpi_0\mu_0 X(\mu_0) \\ -\frac{F}{4}\varpi_0\mu_0 Y(\mu_0) \end{bmatrix}. \tag{37}$$

This is the same form as derived by Chandrasekhar (1960) (p. 201; Equations (108) and (109)) in the case of diffuse scattering.

Now we will show the two limiting cases of optical depth.

1. Semi-infinite optical depth ($\tau_1 \to \infty$): In this condition the expression of the function $V(\mu)$ will reduced into

$$V(\mu) = 1 + 2U(T)M(\mu)\mu\log\left[1 + \frac{1}{\mu}\right] + \frac{\varpi_0}{2}\mu M(\mu)\int_0^1 \frac{M(\mu')}{\mu + \mu'}d\mu' = M(\mu). \tag{38}$$

This expression is same as the M function derived by

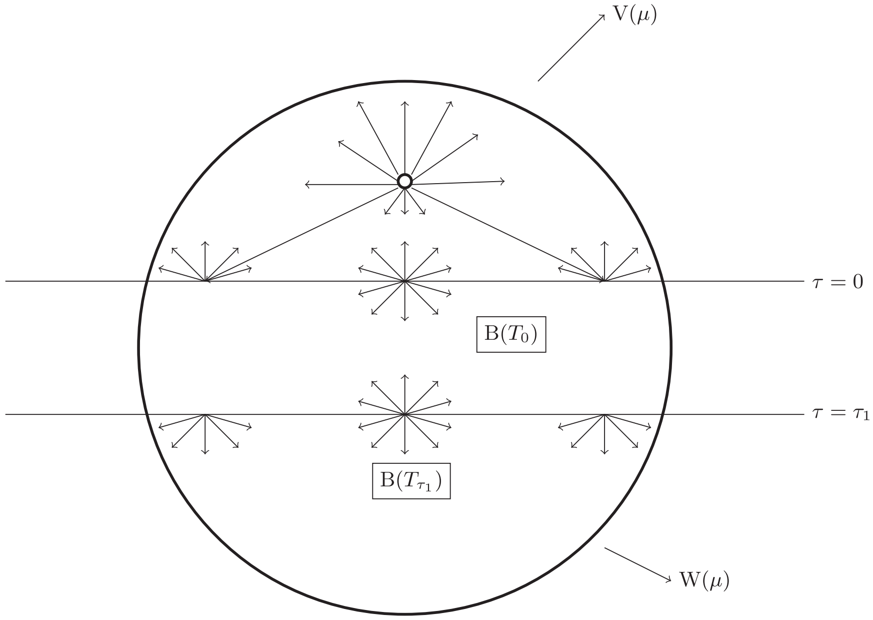

Figure 2. This figure explains the physical meaning of the $V(\mu)$ and $W(\mu)$ functions. The total brightness observed by a distant observer will be $V(\mu)$ (for the observer above $\tau = 0$) and $W(\mu)$ (for the observer below $\tau = \tau_1$) times the brightness of the point source alone. Hence all the scattering, transmission, and thermal emission from the atmosphere contributes in terms of the V and W functions.

Sengupta (2021) in the context of a semi-infinite atmosphere. Hence we can say that the finite atmosphere problem boils down to the semi-infinite atmosphere problem at this limiting value.

Now using the transformation rule (35), the W function can be represented as,

$$W(\mu) = \lim_{\tau_1 \to \infty} V(-\mu) e^{-\tau_1/\mu} \to 0. \tag{39}$$

2. Small optical depth ($\tau_1 \to 0$): In such a case the W function will be,

$$W(\mu) \to e^{-\tau_1/\mu} \tag{40}$$

and using the transformation rule we get,

$$V(\mu) = W(-\mu) e^{-\tau_1/\mu} \to 1. \tag{41}$$

These values are same as shown by Peraiah (2001) in the cases of the Y and X functions, respectively.

8. Discussion

The finite atmosphere diffuse reflection problem was first introduced by Chandrasekhar (1960) for a scattering-only atmosphere where no atmospheric emission was considered. Here for the first time we include the thermal emission effect simultaneously with the isotropic scattering from each atmospheric layers in the finite atmosphere diffuse reflection problem. The thermal emission modifies Chandrasekhar's results in terms of the factor $U(T)$, where U is the ratio of blackbody emission (B) and irradiation flux (F) (see Sections 3 and 4). Moreover, the modified scattering and transmission functions obey the same transformation rules as established by Coakley (1973) (as shown in Section 5). Then, we show that our results are consistent with those of the Chandrasekhar (1960) results in the limit of low atmospheric thermal emission (i.e., $B \ll F$). Hence, it can be said that our treatment of thermal emission and scattering for the finite atmosphere problem is more general than Chandrasekhar's one. In the exoplanetary context, Chandrasekhar's results are used to model the reflection, transmission, and emission spectra of highly irradiated low emitting planets (Madhusudhan & Burrows 2012). But as it is evident that when thermal emission and scattering occurs comparably in a planetary atmosphere (e.g., low irradiating ultrahot Jupiters), then our model will provide more accurate results than Chandrasekhar's results.

The thermal emission from each atmospheric layer will travel through other layers as well and will undergo scattering and transmission. For example, emission from the layer at $\tau = 0$ is $B(T_0)$, which is scattered along the direction (μ, ϕ) and contributes to the final radiation $I(0, \mu)$ in terms of $B(T_0)V(\mu)$ (see Equation (28)). In the same way it contributes along the direction ($-\mu$, ϕ) in the radiation in terms of $B(T_0)W(\mu)$. Thus, the flux F irradiated the atmosphere and the atmospheric thermal emission will follow the same rules of scattering and transmission. So this theory is applicable to those planetary atmospheres where (1) the atmospheric thermal emission is comparable to the irradiated stellar flux and (2) the atmosphere gives infrared scattering effects.

Here we have revisited the connection relation between the scattering and transmission functions (see Equation (31)) as established by Coakley (1973). This transformation rule describes the interchange between the S and T functions depending on the orientation of the incident beam in the case of only diffuse scattering in a finite atmosphere (Coakley 1973). In this work, we first show that this relation is indeed true in the case of a thermally emitting atmosphere. Second, a transformation rule for the V and W functions (see Equation (35)) as well as a connection relation between the thermal emission flux at different atmospheric layers, which can be named as reduced thermal emission, is established. It ensures that if a beam is incident from the upper side of the layer $\tau = 0$, then the V and W functions can be represented as shown in Figure 2. But if a light beam is incident from the lower side of $\tau = \tau_1$ layer then

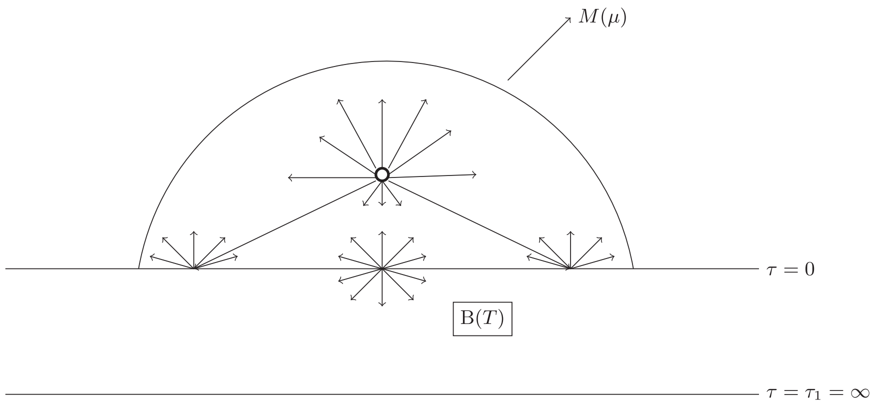

Figure 3. A pictorial view of the $M(\mu)$ function derived by Sengupta (2021) for a semi-infinite atmosphere. The total radiation observed from $(+\mu, \phi)$ direction will be the addition of the light from the source, scattered by the atmosphere and the thermally emitted radiation from the atmospheric layer. Here all the layers are at same temperature T.

the positions of V and W functions in Figure 2 will interchange. This shows the symmetry of the solutions provided here. Note that the transformation rules are applicable only if the atmospheric emission at different optical depths are connected by the relation given in Equation (34).

The applicability of the transformation rule in the case of thermal emission also ensures the fact that in case of comparable emission and scattering from an atmosphere, both of these effects are in equal footing. Hence, both effects should be considered for a full-proof modeling.

The $V(\mu)$ and $W(\mu)$ functions are analogous to Chandrasekhar's X and Y functions mentioned in Chandrasekhar (1960). They represent the relative changes of the radiation from the layer $\tau = 0$ along the direction (μ, ϕ) and from $\tau = \tau_1$ along $(-\mu, \phi)$, respectively, due to the presence of the atmosphere. Hence, the atmospheric presence can be realized in terms of diffuse reflection and atmospheric thermal emission from the corresponding layer. In other words it can be said that they act as the source function for the direction (μ, ϕ) at $\tau = 0$ and the direction $(-\mu, \phi)$ at $\tau = \tau_1$, respectively.

We showed (in Section 7) that at the semi-infinite limit (i.e.,$\tau_1 \rightarrow \infty$), our finite atmosphere results will reduce into that of the semi-infinite results obtained by Sengupta (2021). Hence we can say that the $M(\mu)$ function (see Sengupta 2021 for details) is a semi-infinite counterpart of the more general $V(\mu)$ function, as shown in Figure 3. It reveals the semi-infinite limiting case of the $V(\mu)$ function.

The work presented by Sengupta (2021) considered atmospheric thermal emission in the semi-infinite atmosphere case and was thus limited by the condition of translational invariant thermal emission in the atmosphere. For a planetary atmosphere it means that such a theory is applicable only for those planets that have an isothermal atmosphere. In this work we removed that limitation by considering the finite atmosphere problem, which does not need a translational invariant thermal emission and in such case the scattering function $S(\tau, \mu, \phi; \mu_0, \phi_0)$ varies with the optical depth of the atmosphere. This will provide the opportunity to model the atmospheric spectra for any type of atmospheric temperature structure with simultaneous emission and scattering.

In this work we considered only the isotropic scattering case, which is indeed the first step to modifying the finite atmosphere scattering problem in the presence of thermal emission. This work can be expanded for the general cases of scattering with the same recipe and the modifications will follow accordingly. To include the numerical approach, one should use the Heyney–Greenstein phase function (Henyey & Greenstein 1941),

$$p(\cos\Theta) = \frac{1 - g^2}{(1 + g^2 - 2g\cos\Theta)^{\frac{2}{3}}},$$

where $g \in [-1, 1]$ is the asymmetry parameter, as shown by Bellman et al. (1967) and Batalha et al. (2019).

Atmospheric thermal emission is the simplest possible emission process considered in our work. However, we assumed the low scattering limit $\sigma \ll \kappa$ to use the simple Planck function as the atmospheric emission term (see Equation (11)). It simplifies the mathematical derivations significantly. However, this restriction can be removed by replacing $B(T_\tau)$ with $\frac{\kappa}{\chi}B(T_\tau)$ and the results will follow accordingly. Although the physical interpretations remain unaltered.

In the case of exoplanetary atmosphere modeling, the atmospheric emission cannot always be simplified by the Planck emission. The upper atmospheres of exoplanets do not hold the local thermodynamic equilibrium condition (Seager 2010; Sengupta 2021). In such cases different types of atmospheric emission can be considered by the general atmospheric emission function β as shown in Equation (10). With an appropriate choice of the β parameter for thermal reemission, anisotropic emission can be considered as discussed by Sengupta (2021).

Finally, the polarization effect is not considered in this work. That can be included for a finite atmosphere in the same way as in the semi-infinite atmosphere case discussed by Sengupta (2021).

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Appendix
Derivation of the Scattering Function and its Derivative

In this section we will show the derivation of the scattering function $S(\tau_1, \mu, \phi; \mu_0, \phi_0)$ and its derivative $\frac{\partial S(\tau, \mu, \phi; \mu_0, \phi_0)}{\partial \tau}\Big|_{\tau=\tau_1}$, which has directly been written in Equations (14) and (15). For the simplicity of calculations we write $\frac{\partial S(\tau_1, \mu, \phi; \mu_0, \phi_0)}{\partial \tau_1}$ instead of $\frac{\partial S(\tau, \mu, \phi; \mu_0, \phi_0)}{\partial \tau}\Big|_{\tau=\tau_1}$ in the main text of Section 3. To start with we will use the scattering function relation with the source function equation given by Chandrasekhar (1960) (p. 168; Equations: (23) and (25)) as follows,

$$\frac{1}{4}\left[\left[\frac{1}{\mu}+\frac{1}{\mu_0}\right]S(\tau_1;\mu,\phi;\mu_0,\phi_0) + \frac{\partial S(\boldsymbol{\tau};\mu,\phi;\mu_0,\phi_0)}{\partial \boldsymbol{\tau}}\Big|_{\tau=\tau_1}\right] = \xi(0,+\mu,\phi) + \frac{1}{4\pi}\int_0^1\int_0^{2\pi} S(\tau_1;\mu,\phi;\mu',\phi') \times \xi(0,-\mu',\phi')\,\frac{d\mu'}{\mu'}d\phi', \tag{A1}$$

and

$$\frac{1}{4}F\frac{\partial S(\boldsymbol{\tau};\mu,\phi;\mu_0,\phi_0)}{\partial \boldsymbol{\tau}}\Big|_{\tau=\tau_1} = \exp(-\tau_1/\mu)\xi(\tau_1,+\mu,\phi) + \frac{1}{4\pi}\int_0^1\int_0^{2\pi} T(\tau_1;\mu,\phi;\mu',\phi')\xi(\tau_1,+\mu',\phi')\frac{d\mu'}{\mu'}d\phi'. \tag{A2}$$

Now the source functions $\xi(0, \mu, \phi)$ and $\xi(\tau_1, \mu, \phi)$ in thermal emission case are given in Equations (12) and (13), respectively. Making use of them with the boundary conditions (Equations (8)) the above two equations can be written as,

$$\frac{F}{4}\left[\left[\frac{1}{\mu}+\frac{1}{\mu_0}\right]S(\tau_1;\mu,\phi;\mu_0,\phi_0) + \frac{\partial S(\boldsymbol{\tau};\mu,\phi;\mu_0,\phi_0)}{\partial \boldsymbol{\tau}}\Big|_{\tau=\tau_1}\right] = B(T_0)\left[1+\frac{1}{4\pi}\int_0^1\int_0^{2\pi} S(\tau_1;\mu,\phi;\mu',\phi')\frac{d\mu'}{\mu'}d\phi'\right] + \frac{1}{4}F\left[p(\mu,\phi;-\mu_0,\phi_0) + \frac{1}{4\pi}\int_0^1\int_0^{2\pi} p(\mu,\phi;\mu'';\phi'')S(\tau_1;\mu'',\phi'';\mu_0,\phi_0)d\phi''\frac{d\mu''}{\mu''} + \frac{1}{4\pi}\int_0^1\int_0^{2\pi} S(\tau_1;\mu,\phi;\mu',\phi')\left\{p(-\mu',\phi';-\mu_0,\phi_0) + \frac{1}{4\pi}\int_0^1\int_0^{2\pi} p(-\mu',\phi';\mu'';\phi'') \times S(\tau_1;\mu'',\phi'';\mu_0,\phi_0)d\phi''\frac{d\mu''}{\mu''}\right\}\frac{d\mu'}{\mu'}d\phi'\right], \tag{A3}$$

and

$$\frac{F}{4}\left[\frac{\partial S(\boldsymbol{\tau};\mu,\phi;\mu_0,\phi_0)}{\partial \boldsymbol{\tau}}\Big|_{\tau=\tau_1}\right] = B(T_{\tau_1})\left[e^{-\tau_1/\mu} + \frac{1}{4\pi}\int_0^1\int_0^{2\pi} T(\tau_1;\mu,\phi;\mu',\phi')\frac{d\mu'}{\mu'}d\phi'\right] + \frac{1}{4}F\left[\exp\left\{-\tau_1\left(\frac{1}{\mu_0}+\frac{1}{\mu}\right)\right\}p(\mu,\phi;-\mu_0,\phi_0) + \frac{1}{4\pi}e^{-\tau_1/\mu}\int_0^1\int_0^{2\pi} p(\mu,\phi;-\mu'';\phi'') \times T(\tau_1;\mu'',\phi'';\mu_0,\phi_0)d\phi''\frac{d\mu''}{\mu''} + \frac{1}{4\pi}\int_0^1\int_0^{2\pi} T(\tau_1;\mu,\phi;\mu',\phi')\left\{e^{-\tau_1/\mu_0}p(\mu',\phi';-\mu_0,\phi_0) + \frac{1}{4\pi}\int_0^1\int_0^{2\pi} p(\mu',\phi';-\mu'';\phi'') \times T(\tau_1;\mu'',\phi'';\mu_0,\phi_0)d\phi''\frac{d\mu''}{\mu''}\right\}\frac{d\mu'}{\mu'}d\phi'\right]. \tag{A4}$$

Hence, by multiplying Equations (A3) and (A4) by $\frac{4}{F}$ and replacing the quantities $\frac{B(T_0)}{F}$, $\frac{B(T_{\tau_1})}{F}$ by $U(T_0)$ and $U(T_{\tau_1})$, respectively, we will get Equations (14) and (15), respectively. In similar fashion, the transmission function $T(\tau_1; \mu, \phi; \mu_0, \phi_0)$ and its derivative $\frac{\partial T(\tau;\mu,\phi;\mu_0,\phi_0)}{\partial \tau}\Big|_{\tau=\tau_1}$ can be derived.

ORCID iDs

Soumya Sengupta https://orcid.org/0000-0002-7006-9439

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