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On gravitating stellar systems I. formulation using distribution function method

U.S. Pandey
Physics Department, D.D.U. Gorakhpur University, Gorakhpur-273009, India

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Abstract. An interesting work on self-gravitating gaseous disks has concerned the behaviour of the precursor of massive central dark object that might be the energy source existing at centers of galaxies and powering quasars. With this view, we present a model of self gravitating gas flattened in a disk. We make the following assumptions: (i) the interaction force between individual pairs of particles is negligibly small, (ii) the distribution function f in phase space satisfies collision-free Boltzmann equation and (iii) the system is conservative. We set up a third order linear differential equation satisfied by the disk for axisymmetric case. We obtain a particular solution for the disk in steady state. We derive the expression for gravitational potential W and surface mass density σ of the disk. We also calculate the velocity profiles.

Keywords : galaxies : nuclei – galaxies : active

1. Introduction

A galaxy is regarded as a compact self-contained system bound together through gravitation. Individual galaxies are sharply defined and well separated having empty space between them. A typical spiral galaxy has a mass of 10^{11} solar mass and a radius of 10 kpc. Steady state of galaxies is maintained with inertial forces balancing the force of gravity that is responsible for the collapse of the system. Virial equilibrium demands velocities of the order 200 $km\ s^{-1}$ for the kinetic energy of motion. Disks of spiral galaxies reflect that these velocities primarily take the form of an ordered circular motion in the plane of the disk about its centre. Less coherent and more random velocities are also possible and pertain to other type of galaxies. In several of these systems, there are motions responsible for the rotation of the disk plane. Thus, more general type of disks are possible and that disks in circular motion happen to be some special configuration.

The 21 cm line observations with 91 m telescope at NRAO (Roberts 1967) show strikingly (Oort 1965) that central peak contour in M31 and M33 depict ring-like distribution. Distribution of hydrogen displays ring-pattern in the observation of many galaxies (NGC 2403, NGC 5055, NGC 5457, IC 342 and our Galaxy). One may refer to Kerr and Westerhout (1965) for hydrogen distribution in our Galaxy. It is thus tempting to think that a balance primarily between gravitational force and centrifugal force (if pressure is negligible) persuades neutral hydrogen distribution to assume a ring pattern.

The physical picture that we have in mind is presented below. The inner central part of an AGN presumably consists of a massive dark object. We, however adopt the general feature of gaseous disk discussed by Kundt (1996). Accordingly, a homogeneous (hydrogen) disk is prone to star formation above $\sigma_{Jeans} = 2\Omega c_s/\pi G$ and gets hot and / or degenerate near its mid-plane above σ_{deg}, where Ω and c_s denote the rotation speed and the speed of sound.

The gas around the massive dark object spirals towards it with average rate $M_{in} \leq 1 M_{\odot}/yr$ (Kundt 1990). During this process, a small fluctuation results into enormous oversupply of matter in the inner regions. Consequently, σ will grow beyond σ_{Jeans} - resulting into a spike in the inner rotation curve evidenced by the starburst. After a few 10^8 years or so, oversupply may raise surface mass density σ to stellar values, i.e. $\sigma \sim 10^{11.2} gm\ cm^{-2}$ and nuclear burning front will appear out to solar system distances. This is reflected as the activity of galactic central part of an AGN.

For understanding the physical properties of the distribution described above, we present a model. We list the main assumptions.

(1) We assume that interaction forces between individual pairs of particles (collisions) are negligibly small as compared with regular force. Let the relaxation time of the gaseous system be short enough to facilitate rapid energy exchange to help maintain the gas under statistical distribution. One requires that number of particles is significantly large compared to unity however pair interaction is weaker than collective interaction.

(2) We define a function f in phase space to describe the system. Thus, $f(t, x, y, z, u_x, u_y, u_z)\ dxdydzdu_xdu_ydu_z$ gives the number of gas particles at time t in a volume element $dxdydz$ around a point (x, y, z) with velocities lying between u_x and u_x+du_x, u_y and u_x+du_x, u_z and u_z+du_z. The function satisfies collision-free Boltzmann equation (Camm 1950), i.e.,

$$\frac{\partial f}{\partial t} + u_x \frac{\partial f}{\partial x} + u_y \frac{\partial f}{\partial y} + u_z \frac{\partial f}{\partial z} + \frac{\partial f}{\partial u_x} \frac{du_x}{dt} + \frac{\partial f}{\partial u_y} \frac{du_y}{dt} + \frac{\partial f}{\partial u_z} \frac{du_z}{dt} = 0. \quad (1)$$

(3) We assume that the system is conservative and therefore the force is derivable

from the potential function W. We may now put eq. (1) as

$$\frac{\partial f}{\partial t} + u_x \frac{\partial f}{\partial x} + u_y \frac{\partial f}{\partial y} + u_z \frac{\partial f}{\partial z} + \frac{\partial W}{\partial x}\frac{\partial f}{\partial u_x} + \frac{\partial W}{\partial y}\frac{\partial f}{\partial u_y} + \frac{\partial W}{\partial z}\frac{\partial f}{\partial u_z} = 0. \tag{2}$$

In a more compact form eq. (2) appears as

$$\frac{\partial f}{\partial t} + \mathbf{u}.\nabla f + \nabla W.\nabla \mathbf{u} f = 0. \tag{3}$$

Let us impose geometrical constraint on eq. (3). For a stratified plane disk the function f as well as gravitational potential W are restricted. In a transition from spheroidal form to stratified plane form of the disk, θ is squeezed enough to lose its meaning. We assume axial symmetry. The function f and W depend only on ϕ and u_ϕ. Eq. (3) for steady state now becomes

$$\frac{u_\phi}{r}\frac{\partial f}{\partial \phi} + \frac{\partial W}{r\partial \phi}\frac{\partial f}{\partial u_\phi} = 0. \tag{4}$$

A solution of eq. (4) will provide the distribution function f, the mass density ρ and the gravitational acceleration g.

2. General formulation

2.1 Differential equation for the system

Let us define number density as

$$n(\phi) = \int f(\phi, u_\phi) du_\phi \tag{5}$$

and mass density as

$$\rho(\phi) = m \int f(\phi, u_\phi) du_\phi, \tag{6}$$

where m denotes mass of a particle. Further, we define two new functions

$$P(\phi, s) = m \int f(\phi, u_\phi) e^{iu_\phi s} du_\phi \tag{7}$$

and

$$Q(\phi, s) = 2\pi G \int_0 P(\phi, s) r d\phi. \tag{8}$$

It is easy to see that $P(\phi, 0)$ = mass density and $Q(\phi, 0) = 2\pi G\sigma$ = gravitational acceleration and σ being the surface mass density defined as

$$\sigma = \int P(\phi, 0) r d\phi. \tag{9}$$

Now, multiply eq, (4) by $me^{iu_\phi s}$ and integrate with respect to u_ϕ

$$\left(\frac{m}{r}\right)\frac{\partial}{\partial\phi}\left[\int u_\phi f(\phi,u_\phi)e^{iu_\phi s}du_\phi\right]+Q(\phi,0)m\int\frac{\partial f}{\partial u_\phi}e^{iu_\phi s}du_\phi=0$$

which goes over to

$$-i\frac{\partial^2 P(\phi,s)}{r\partial s\partial\phi}-isQ(\phi,0)P(\phi,s)=0. \tag{10}$$

In terms of $Q(\phi,s)$, eq. (10) appears as

$$\frac{\partial^3 Q(\phi,s)}{r\partial s\partial\phi^2}+sQ(\phi,0)\frac{\partial Q(\phi,s)}{r\partial\phi}=0. \tag{11}$$

Eq. (11) is a third order linear differential equation which must be satisfied by a stratified, self gravitating gaseous disk around galactic nuclei. In this formulation, the only restriction on the function $Q(\phi,s)$ is that it must yield a non-negative $f(\phi,u_\phi)$. For non-steady state (the term $\frac{\partial f}{\partial t}$ survives) the differential equation becomes

$$\frac{\partial^2 Q(t,\phi,s)}{r\partial t\partial\phi}-i\frac{\partial^3 Q(t,\phi,s)}{r\partial s\partial\phi^2}-isQ(t,\phi,0)\frac{\partial Q(t,\phi,s)}{r\partial\phi}=0. \tag{12}$$

2.2 A particular solution

In order to solve eq. (11), we assume that the number of particles in any velocity range at a particular ϕ do not depend upon ϕ. Let

$$Q(\phi,s)=\Phi(\phi)S(s). \tag{13}$$

In view of the above, eq. (11) becomes

$$\frac{d^2\Phi}{r^2d\phi^2}\frac{dS}{ds}+s\Phi S\frac{d\Phi}{rd\phi}=0. \tag{14}$$

In eq. (14) variables are separable, we write

$$\frac{dS}{ds}=-A^2sS \tag{15}$$

and get

$$A^2\frac{d^2\Phi}{r^2d\phi^2}=\Phi\frac{d\Phi}{rd\phi}. \tag{16}$$

Eq. (15) yields a solution $S=exp(-\frac{1}{2}s^2A^2)$ so as to make $S(0)=1$. Differentiate eq. (8) with respect to ϕ to get

$$\left(\frac{\partial Q}{r\partial\phi}\right)=2\pi GP(\phi,s).$$

Rewriting above we obtain

$$P(\phi, s) = \frac{1}{2\pi G}\frac{d\Phi}{rd\phi}S. \tag{17}$$

Using Fourier transform, we write from eq. (7) as

$$f(\phi, u_\phi) = \frac{1}{2\pi m}\int_{-\infty}^{+\infty} P(\phi, s)e^{-iu_\phi s}ds. \tag{18}$$

Making use of eq. (17), we determine the function f as

$$f(\phi, u_\phi) = \frac{1}{2\sqrt{2}\pi^{\frac{3}{2}}Gm}\left(\frac{d\Phi}{rd\phi}\right) exp\left(-\frac{1}{2}\frac{u_\phi^2}{A^2}\right). \tag{19}$$

Integrate eq .(16) with respect to $rd\phi$, to get

$$2A^2\frac{d\Phi}{rd\phi} = \Phi^2 + constant. \tag{20}$$

Writing the constant as B^2 and $\Phi(\phi) = B\tanh\xi$,

$$\frac{d\Phi}{rd\phi} = \frac{d\Phi}{d\xi}\frac{d\xi}{rd\phi} = \frac{B}{r}sech^2\xi\left(\frac{d\xi}{d\phi}\right). \tag{21}$$

From eq. (21) & (20) one gets

$$\left(\frac{d\xi}{d\phi}\right) = \frac{rB}{2A^2}. \tag{22}$$

If we measure ϕ from the point where $\Phi(\phi) = 0$, we get

$$\left(\frac{\xi}{\phi}\right) = \frac{rB}{2A^2}. \tag{23}$$

It thus follows that $\Phi = B\tanh(\frac{rB}{2A^2}\phi)$. Hence

$$\frac{d\Phi}{rd\phi} = \frac{B^2}{2A^2}sech^2\left(\frac{rB}{2A^2}\phi\right). \tag{24}$$

Using Poisson equation we get

$$\frac{d}{rd\phi}\left(\frac{dW}{rd\phi}\right) = 4\pi G\rho(\phi) = \frac{d\Phi}{rd\phi}.$$

Since,

$$Q(\phi, 0) = \Phi(\phi) = \frac{dW}{rd\phi},$$

this implies that

$$\rho(\phi) = \rho_0 sech^2\left(\frac{rB}{2A^2}\phi\right) \tag{25}$$

where $\rho_0 \equiv \frac{B^2}{8\pi G A^2}$ = mass density in the plane $\phi = 0$. The distribution function is now expressed as

$$f(\phi, u_\phi) = \sqrt{\frac{2}{\pi}} \frac{\rho_0}{mA} sech^2 \left\{ (2\pi G\rho_0)^{\frac{1}{2}} \frac{r\phi}{A} \right\} \exp\left(-\frac{1}{2} \frac{u_\phi^2}{A^2} \right). \tag{26}$$

Mass per unit area of the disk may be obtained as

$$\sigma = \int \rho(\phi) r d\phi = \int \rho_0 sech^2 \left\{ (2\pi G\rho_0)^{\frac{1}{2}} \frac{r\phi}{A} \right\} r d\phi$$

or

$$\sigma = \sigma_0 \tanh \left\{ (2\pi G\rho_0)^{\frac{1}{2}} \frac{r\phi}{A} \right\}, \tag{27}$$

where

$$\sigma_0 \equiv \frac{\rho_0 A}{(2\pi G\rho_0)^{\frac{1}{2}}} = \left(\frac{\rho_0 A^2}{2\pi G} \right)^{\frac{1}{2}}. \tag{28}$$

Thus, surface mass density σ at different phases ϕ can be calculated using eq. (27).

2.3 Gravitational potential

The gravitational potential W at any phase ϕ can be obtained as

$$\left(\frac{dW}{r d\phi} \right) = \Phi(\phi) = 2A(2\pi G\rho_0)^{\frac{1}{2}} \tanh \left\{ (2\pi G\rho_0)^{\frac{1}{2}} \frac{r\phi}{A} \right\}.$$

Integrate the above equation to get

$$W = \frac{2A^2 (2\pi G\rho_0)^{\frac{1}{2}}}{2\pi G \frac{\rho}{A}} \ln sech \left\{ (2\pi G\rho_0)^{\frac{1}{2}} \frac{r\phi}{A} \right\} + constant. \tag{29}$$

The requirement that $W = 0$ at $\phi = 0$ yields

$$W = 2A^2 \ln sech \left\{ (2\pi G\rho_0)^{\frac{1}{2}} \frac{r\phi}{A} \right\}. \tag{30}$$

Rearranging eq. (30) we get

$$\exp\left\{ \frac{W}{A^2} \right\} = sech^2 \left\{ (2\pi G\rho_0)^{\frac{1}{2}} \frac{r\phi}{A} \right\}. \tag{31}$$

Substituting eq. (31) into eq. (26),

$$f(\phi, u_\phi) = \sqrt{\frac{2}{\pi}} \frac{\rho_0}{mA} \exp\left[-\frac{\left(\frac{1}{2}u_\phi^2 - W\right)}{A^2} \right]. \tag{32}$$

It is interesting to note that the distribution function $f(\phi, u_\phi)$ depends upon total energy $(= 1/2u_\phi^2 - W)$ per unit mass.

For a non-steady case, when one assumes that the function $f(t, \phi, u_\phi) = \Phi(t, \phi)U(\phi)$, one may obtain

$$f(t, \phi, u_\phi) = \sqrt{\frac{2}{\pi}} \frac{\rho_0}{mA} sech^2 \left\{ \frac{(r\phi - ut)}{A} (2\pi G\rho_0)^{\frac{1}{2}} \right\} \times \exp\left[- \frac{\frac{1}{2}(u_\phi - u)^2}{A^2} \right]. \tag{33}$$

Eq. (33) represents the distribution of gas moving with uniform and constant velocity u in ϕ -direction. We have thus obtained the distribution of an isothermal, gravitating stratified gas around galactic nuclei.

2.4 Surface mass density σ

The surface mass density σ when hyperbolic tangent term in eq. (27) becomes unity is given by

$$\sigma_0 = \left(\frac{\rho_0 A^2}{2\pi G} \right)^{\frac{1}{2}} = \frac{B}{4\pi G} = \frac{g_r}{4\pi G}. \tag{34}$$

where $B \equiv g_r$ = acceleration due to gravity at the disk surface at radius r. Thus, $g_r = \frac{GM(r)}{r^2}$; M(r) = mass of the gaseous disk at r. Hence,

$$\sigma_0 = \frac{1}{4\pi} \frac{GM(r)}{Gr^2} = \frac{M(r)}{4\pi r^2}. \tag{35}$$

It is to be noted that variation of σ with phase ϕ indicates that several of these distributions communicate changes to each other in a manner so as to maintain equilibrium in the disk. Thus, σ dependence of the spatial position of the disk provides clue to its evolutionary status.

3. Calculation of velocity profiles

In the following, we present the velocity profiles for the disk understudy. Since,

$$\frac{dW}{rd\phi} = \Phi(\phi) = \frac{u_\phi^2}{r} = \Omega^2 r,$$

this gives

$$\Omega^2 = \frac{B}{r} \tanh \xi = \frac{B}{r} \tanh \left(\frac{rB\phi}{2A^2} \right)$$

or

$$\Omega^2 = \frac{2A(2\pi G\rho_0)^{\frac{1}{2}}}{r} \tanh \left\{ (2\pi G\rho_0)^{\frac{1}{2}} \frac{r\phi}{A} \right\}. \tag{36}$$

From equations (27) and (36) one gets

$$\sigma = \frac{\Omega^2 r}{2\pi G}. \tag{37}$$

Therefore surface mass density depends upon how Ω varies. Velocity profiles are obtained as

$$V^2 = 2Ar(2\pi G\rho_0)^{\frac{1}{2}} \tanh\left\{(2\pi G\rho_0)^{\frac{1}{2}} \frac{r\phi}{A}\right\}, \tag{38}$$

where $u_\phi = V$.

4. The radial solution

For a straitified plane disk if one imposes the restriction that physical quantities of interest depend only on r and u_r and define the distribution function $f = f(r, u_r)drdu_r$ as the number of particles located at r having velocities lying between u_r and $u_r + du_r$, the number density may be written as

$$n(r) = \int f(r, u_r)du_r. \tag{39}$$

The mass density is defined as

$$\rho(r) = m \int f(r, u_r)du_r. \tag{40}$$

It is straightforward to obtain the radial dependence from eq.(3) for axisymmetric and steady state case as

$$u_r \frac{\partial f}{\partial r} + \frac{\partial W}{\partial r}\frac{\partial f}{\partial u_r} = 0. \tag{41}$$

a treatment analogous to one given in section (2.1) yields the differential equation for this system as

$$\frac{\partial^3 Q(r,s)}{\partial r \partial^2 r} + sQ(r,0)\frac{\partial Q(r,s)}{\partial r} = 0. \tag{42}$$

We obtain a particular solution of eq.(42) for mass density as

$$\rho(r) = \rho_0 sech^2\left(\frac{rB}{2A^2}\right), \tag{43}$$

where ρ_0 is given in section (2.2). Surface mass density σ is given by

$$\sigma(r) = \sigma_0 \left\{tanh(2\pi G\rho_0)^{1/2}\frac{r}{A}\right\}, \tag{44}$$

where σ_0 is expressed by eq.(28).

The distribution function f is expressed as

$$f(r,u_r)=\sqrt{\frac{2}{\pi}}\frac{\rho_0}{mA}exp.\Big[-(\frac{1}{2}u_r^2-W)/A^2\Big]. \tag{45}$$

Equations (43) - (45) describe the radial solution. Gravitational potential W is again written as

$$W=2A^2\ ln\ sech\left\{(2\pi G\rho_0)^{1/2}\frac{r}{A}\right\}. \tag{46}$$

5. The complete solution (r and ϕ dependence)

It is found that the r - solution and ϕ - solution are particular solutions of a linear differential equation of third order. Hence, the sum of these two solutions must also represent a solution. This is what we call as the complete solution. Thus, general expression for mass density, surface mass density, distribution function and gravitational potential are given by

$$\rho(r,\phi)=\rho_0\left[sech^2\left(\frac{rB}{2A^2}\right)+sech^2\left(\frac{rB\phi}{2A^2}\right)\right], \tag{47}$$

$$\sigma(r,\phi)=\sigma_0\left[tanh\left\{(2\pi G\rho_0)^{1/2}\frac{r}{A}\right\}+tanh(2\pi G\rho_0)^{1/2}\frac{r\phi}{A}\right], \tag{48}$$

$$f(r,\phi,u_r,u_\phi)=\sqrt{\frac{2}{\pi}}\frac{\rho_0}{mA}\left[exp.-\left\{\frac{1}{2}(u_r^2+u_\phi^2)-W\right\}/A^2\right], \tag{49}$$

$$W=2A^2\left[ln\ sech\left\{(2\pi G\rho_0)^{1/2}\frac{r}{A}\right\}+ln\ sech\left\{(2\pi G\rho_0)^{1/2}\frac{r\phi}{A}\right\}\right]. \tag{50}$$

Velocity profile for the complete solution is expressed by

$$V_c^2=2Ar(2\pi G\rho_0)^{1/2}\Big[tanh(2\pi G\rho_0)^{1/2}\frac{r}{A}+tanh(2\pi G\rho_0)^{1/2}\frac{r\phi}{A}\Big], \tag{51}$$

where $V_c^2=u_r^2+u_\phi^2$.

6. Discussion

We have derived the distribution function f for a particular configuration of the gaseous disk around galactic nuclei. We have obtained mass density ρ, surface mass density σ, gravitational potential W and velocity profile. This type of gas distribution was originally suggested by Oort (1965). One finds sufficiently high concentration of ionized hydrogen near R = 5 kpc (where neutral hydrogen is deficient). A large HI concentration appears between 5 kpc and 15 kpc. There is thus inhomogeneous distribution of gas primarily lying in a thin layer in a planar configuration.

In a subsequent analysis, we aim to present self-gravitating disk parameters, e.g. (i) σ and its variation with phase, (ii) radial dependence of physical parameters, (iii) dependence of parameters perpendicular to the disk plane, etc. In fact, the evolution of σ with time is expected to determine various disk phases, namely, σ_{Jeans}, σ_{MW} (for Milky Way), σ_{deg} (for degenerate disk) and σ_B (for burning disk).

It has been suggested (Pandey and Gupta 1998) that nuclear activity at the centres of galaxies might be due to instabilities in the self-gravitating gaseous disks. Active galactic nuclei might evolve subsequently through LINER - Starburst - Central nuclear burning phases. The surface mass density σ and the core temperature are two parameters which may decipher the evolution of AGNs. It is to be noted that non-thermal coronal activities and stellar flares through magnetic reconnections has been studied by Osterbrock and Mathews (1986) and Benz (1994).

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