COMPUTER EXPERIMENTS IN STELLAR DYNAMICS

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INTRODUCTION :

In studying the distribution of matter in the universe, one comes across many instances of collection of stars, where the internal dynamics is basically governed by the gravitational interaction of stars in that collection. The most obvious examples of these collections are different types of galaxies. Within the galaxies, stars are sometime found in clusters. Here again one can consider the system in the first approximation as self-gravitating. The subject of stellar dynamics is concerned with the dynamical behavior of these self-gravitating stellar systems. In what follows, we will describe some new methods of study of stellar systems made possible by availability of fast electronic computers.

In contrast to the investigators in most other scientific fields, the astrophysicist is at a certain disadvantage. The nature of the objects of interest in astrophysics precludes any experimentation. Even from the observational point of view, the situation is far from satisfactory, because the time scale of most of the astrophysical phenomena are much longer than the time for which astronomical records have been kept. This latter difficulty is sometimes overcome by taking a statistical approach. In the study of stellar evolution, for example, one can observe stars with different ages at a certain instant of time and use their observed physical properties to build possible evolutionary sequences. The situation in the study of stellar systems is generally more difficult because dynamical and morphological effects of evolution are not clearly understood. An additional complication in building evolutionary sequence for stellar systems is that all members of a particular type of stellar system appear to have been formed at approximately the same time, a possible exception to this are galactic clusters. This means that any dissimilarities within a type arise from different initial conditions and not from difference in ages. The advent of fast electronic computer has opened a new line of attack in the study of stellar systems. The methods of modelling and simulation have been applied to the problem, allowing experimentation with various initial conditions, thus partly compensating for the lack of real experiments.

THE BASIC PROBLEM:

We are interested in the evolution of a collection of stars moving under their own mutual gravitational attraction. Starting with certain initial positions and velocities, our problem is to find these quantities at a future time. If the stars are approximated by mass points, we write down the equations of motion for a system with n stars,

$$\ddot{r}_i = -\sum_{j=1}^{j=n}{}' \, Gm_j \frac{r_i - r_j}{|r_i - r_j|^3}, \quad i=1, \cdots n, \quad (1)$$

where r_i is the position and m_i the mass of i th star. This is the gravitational n-body problem and it can be made to represent a wide variety of stellar systems. In actual cases, we might have to consider other complications such as the presence of gas, external gravitational fields, but the basic problem can still be studied first.

THE EXACT METHOD:

If the number of stars in the system is not too large, the most straight-forward approach is to integrate these equations numerically; this would in principle solve our problem by providing the positions and velocities of stars at any desired time. Before considering possible ways in which this can be accomplished, let us carefully examine the nature of system of equations (1). We notice that each of equations has (n-1) singularities corresponding to possible collisions. Although actual collisions are unlikely due to the point nature of the masses, very close hyperbolic encounters and tight binaries are possible. Such close formations slow down the progress of computation drastically, and in some cases even force it to a halt due to the accumulation of truncation errors. The exact method cannot be applied to systems with very large number of stars. This is basically due to the fact that to calculate the force on all stars once, one must perform n^2 operations and therefore the computer time needed for evolving a system for a dynamical period also goes as n^2 for most integration schemes. The above difficulties limit the usefulness of standard methods of numerical solution of ordinary differential equation, to the gravitational n-body problem and have forced the investigators in the field to look for new techniques.

In the rest of this section, we will briefly describe the development of special techniques for the n-body problem.

The first attempt at the numerial solution of the problem was made by von Horner (1960), who evolved clusters with small number of stars ($n < 16$). The time-step for integration was taken to be the same for all stars and was chosen so as to be small enough to correctly integrate the pair undergoing strongest interaction. The difficulty of this method is that the strongest interaction dictates the evolutionary progress of integration and slows down the whole program. To overcome this difficulty, Aarseth (1963) introduced individual time steps for each star allowing separate updating. In this method, the stars are no longer synchronized and therefore each time one must update the star which is left furthest behind. The recalculation of the force on a star requires synchronization of the positions of all other stars to the time for which the position of this star is known. Even with the extra time needed for synchronization and the search for the star left furthest behind, the method of individual time steps is much more efficient than von Horner's method. A further extension of this concept is the method of double individual time steps (Ahmad and Cohen 1973a). In this method, the force on a star arising from its neighbours (irregular force) is treated separately from the force due to the rest of the system (regular force). The advantage of this separation is that the regular force changes very slowly and therefore does not require frequent recalculation, resulting in considerable saving of computer time. The irregular force, which changes rapidly and therefore has to be frequently calculated, does not take too much time, because it is caused by a few stars. Another scheme along the similar lines is the category scheme of Hayli (1967), where the stars are divided into categories according to the strength of the gravitational interaction which they are undergoing, stars in different categories are then treated differently in force calculation.

The problem of close encounters and binaries is, to a certain extent, eased by the modifications discussed above. Complete solution of this problem of course requires elimination of the singularities in the differential equation. For the two body problem, it is now possible to eliminate the singularity by a change in variables, a technique known as regularization. In application to gravitational n-body problem, regularization can be used to eliminate the singularity in the closest interacting pair, while treating the rest of the system in conventional manner. The regularization technique is particularly useful for clusters where binaries play an important role (Aarseth 1971).

Even with the improvement in the integration methods, the largest number of stars that have been evolved for dynamically significant times is a thousand (Ahmad and Cohen 1973a). The exact n-body calculation can therefore realistically simulate only the small stellar systems among which the most important are open clusters. The evolution of open clusters has been studied by a number of investigators using this method. Among the many aspects of their evolution, which have been studied, we might mention, escape rate of stars from clusters (Wielen 1968), effect of galactic field on the evolution (Hayli 1971) and the role played by the binaries (Aarseth 1968). Another type of gravitational system that can be simulated is cluster of galaxies ($n < 1000$). Peebles (1972) and Aarseth (1963) have used the exact method for the study of cluster of galaxies. The exact schemes have been also found to be useful in studying different types of dynamical processes and some interesting insights have been obtained. Wielen (1967) for example, found that a star acquires escape energy in a single encounter rather than by a slow diffusion process envisaged by Chandrasekhar. Similarly, van Albada has done a detailed study of mechanism of formation of binaries and multiple systems. Recently, some of the statistcal theories of gravitational systems, first proposed by Chandrasekhar and von Neumann (1943), have been tested (Ahmad and Cohen 1972, 1973b, 1974).

We should point out at this stage an instability of the exact n-body integrations, first noticed by Miller (1964). He found that two systems which were very similar initially, exponentially diverged in 6-n dimensional phase space, as they were evolved on a computer. Standish (1968) has shown that this instability is caused by the singularity in the force law. Another manifestation of this instability is the fact that inspite of the time reversible nature of the equations of motion, actual computer integrations are not exactly reversible, even for small integration times. The introduction of regularization technique would tend to reduce this instability but not eliminate it.

The basic usefulness of the exact n-body calculations, apart from the simplicity of the direct approach, is that it is free of any questionable simplifying assumptions. This fact is particularly advantageous in testing predictions of a theory and in studying detailed dynamical processes. It is for these reasons that exact integrations have become very popular recently, inspite of the difficulties discussed above.

The limited number of stars that can be evolved by exact integration method precludes its usefulness for the simulation of most real stellar systems. For large system, one must search for statistical methods. Indeed, most of the quantities of interest in the evolutionary study of stellar systems such as the density, velocity distribution and escape rates are statistical quantities and one does not have to know exact particle trajectories to find these. Before we describe how the statistical methods are introduced, we should have a look at the nature of the force acting on a star in a stellar system.

RANDOM FORCE AND MEAN FIELD FORCE :

Chandrasekhar (1960) has pointed out that the total force F_T on a star can be divided up into two parts :

$$F_T = F_M + F_R,$$

where F_M, the mean field force, is due to the smoothed out distribution of matter, and F_R, the random force, is due to local fluctuations in the density. The mean field force is a deterministic quantity and can be found by either a direct application of Gauss's law (in systems with symmetry) or via Poisson equation. The random force, on the other hand, is a stochastic quantity and its effect can be treated statistically. By this separation of force, it is in fact possible to bypass the particle description altogether and write down a kinetic equation (generally the Fokker-Planck equation) for the gravitational system (Chandrasekhar 1960). Corresponding to these two types of forces, there are two different relaxation mechanisms operating in a stellar system which have quite different time scales. The time scale for the mean field relaxation is the crossing time T_C given by

$$T_C = \frac{R}{V},$$

where V is the rms velocity of stars and R is a measure of the size of the system. The relaxation due to the random force, called collisional relaxation, operates on the time scale given by the relaxation time. The relaxation time is the time taken by random encounters to change the kinetic energy of a star by an amount equal to its initial value. The ratio of the average value of the relaxation time, T_R, to the crossing time, is given by

$$\frac{T_R}{T_C} \sim \frac{n}{\log n}.$$

COLLISION-LESS SYSTEMS :

From the above expression, it is clear that for systems with large number of stars, collisional relaxation is unimportant on the time scale of a few crossing times and one could neglect the random force treating the system as essentially collision-less. A good example of such a system is our galaxy, where the relaxation time turns out to be many orders of magnitude larger than its age. In collision-less systems, the problem is reduced to that of solving the Fokker-Planck equation without encounter term (called the Vlasov equation), coupled with the Poisson equation. Although in principle this can be done directly, in practice it is found more convenient not to abandon the particle approach completely. The solution of Vlasov equation is then obtained indirectly by solving the equation of motion of groups of stars or stars in the same cell of phase space. A particularly simple situation occurs when the system has certain spatial symmetries, because the force is then directly obtained by Gauss's law. The z-motion of the stars in the disk of the Galaxy has been simulated by infinite parallel sheets (Lecar and Cohen 1968). Similarly, a model of concentric spherical shells has been applied to the study of collision-less stellar systems with spherical symmetry (Henon 1973). In absence of symmetry, the force can be obtained via the potential by solving Poisson equation. A number of methods of solving Poisson equation have been used to study the evolution of the galactic disk (Hohl and Hockney 1969; Miller and Prendergast 1968).

MONTE CARLO METHODS :

The simulation of globular cluster presents a different problem. The number of stars here ($n \simeq 10^5$) is too large to be handled by exact method but not large enough to make collisional relaxation negligible. The effect of random force is important and somehow must be taken into account. One way to do this is to solve the Fokker-Planck equation directly, but again an indirect approach is found more convenient. This is the essence of the Monte Carlo methods used to study the evolution of spherical stellar system. Here the stars are evolved by calculating the mean field from Gauss's law. The effect of random encounters could be superimposed on this mean field motion in a variety of ways. Henon (1971) introduces this effect by making a randomly selected pair to undergo an "average" binary collision, while Spitzer and Hart (1971) incorporate this effect via the distribution for the net momentum transfer.

The Monte Carlo methods offer a very efficient and fast means of dynamical study of the globular clusters. The basic assumption of this technique is of course that the Fokker-Planck equation is the correct kinetic equation for the gravitational systems. Much theoretical work needs to be done either (a) to firmly establish the validity of this assumption or (b) to find the correct kinetic equation for the gravitational system. Work of Lee (1968) and that of Gilbert (1970) could be mentioned here as examples of efforts along these two different directions, respectively.

CONCLUSION :

We would like to emphasize once again the similarity of these simulation techniques to the real laboratory experiments. The numerical experiments are very useful in a field where actual experimentation is not possible and where analytical results are very hard to come by. Such is indeed the case with stellar dynamics.

As the field is still relatively new, there are a number of unsolved problems. The question of the instability of the exact method has not been fully investigated. In particular, the effect of this instability on the macroscopic quantities has to be better understood. The validity of the Monte Carlo approach has to be firmly established before its results can be taken as strictly correct. A comparision of an identical system evolved exactly with one evolved by Monte Carlo scheme would be very useful, particularly if the end systems are highly evolved.

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