1979Ap&SS..66..505D

A TECHNIQUE FOR MEASUREMENT OF ROTATIONAL FREQUENCY

(Letter to the Editor)

G. C. DAS

Indian Institute of Astrophysics, Bangalore, India

(Received 6 July, 1979)

Abstract. On the basis of Chandrasekhar's proposition that the Coriolis force has an influence on the magnetohydrodynamic waves excited in cosmic phenomena, a simple approximate formula determining the rotational frequency from the effects of Coriolis force is derived. The method is expected to be of value for rotating medium studies and may be applicable in diagnosing the physical parameters of an ionized medium.

1. Introduction

Chandrasekhar (1953a) was first to show that the gaseous medium in the solar system is subjected to a Coriolis acceleration $2\Omega \times v$ and proposed, from his studies (Chandrasekhar, 1953b, c), that the Coriolis force – however small in magnitude – may play a significant role in magnetohydrodynamic waves in the Sun. Later, many workers have attempted to examine the role of Coriolis force on the wave propagation in cosmic phenomena. From the results of all observations, a conclusion can be made that the Coriolis force deriving from a uniform rotation plays no significant role in physical phenomena on a laboratory scale but exhibits a great influence in cosmic phenomena. Lehnert (1954), in the study of low-frequency waves, shows that the Coriolis force could modify the Alfvén's theory on sunspot. The numerical estimation made by Lehnert (1954) and Hide (1966), based on the ratio of the Coriolis force to the magnetic force for the gaseous in the interior of the Sun and for the conducting fluid in the Earth's core, respectively, showed that, with very long wavelengths, the Coriolis force plays a dominating role on the waves especially in the presence of the low magnetic field. Lehnert (1954) concluded from the calculation on the ratio of Coriolis force to the magnetic force estimated that

$$\frac{fc}{fm} = \frac{\Omega(\mu\varrho)^{1/2}}{B/2L},$$

where all the parameters have the usual meaning as defined by Lehnert (1954). For the typical value of rotation, $\Omega = 2 \times 10^{-6}\ \mathrm{s}^{-1}$, Alfvén velocity, $V = 2\ \mathrm{m\,s}^{-1}$ and $L = 100R_0$ (where R_0 is the solar radius), the ratio fc/fm is equal to 14. Taking all the references in Chandrasekhar and Lehnert as evidence that the Coriolis force does play a significant role in cosmic phenomena, we wish to show how to determine the

Astrophysics and Space Science **66** (1979) 505–510. 0004–640X/79/0662–0505$00.90
Copyright © 1979 by D. Reidel Publishing Co., Dordrecht, Holland, and Boston, U.S.A.

© Kluwer Academic Publishers • Provided by the NASA Astrophysics Data System

1979Ap&SS..66..505D

rotational frequency of a medium rotating with uniform angular rotation from a study of simple wave phenomena.

2. Basic Equations and Group Travel Time in a Rotating Medium

We consider an ionized medium consisting of electrons (subscript 'e') and ions (subscript 'i'). We assume, in the equilibrium state, the medium is pervaded by a uniform magnetic field and rotates with a uniform angular velocity around the magnetic field lines directed in a particular axis (say, the z-axis). The resulting centrifugal force can be neglected in studies of astrophysical problems. Such neglect is well-justified, as the magnitude of the rotational frequency is small. Moreover, we shall be interested mainly in studying the effects of Coriolis force in isolation. The basic governing equations, referred to a rotating frame of reference, are:

Equation of continuity:

$$\frac{\partial n_\alpha}{\partial t} + \operatorname{div}(n_\alpha v_\alpha) = 0, \tag{1}$$

Equation of motion:

$$\frac{\partial v_\alpha}{\partial t} + v_\alpha \cdot \nabla v_\alpha = \frac{q_\alpha}{m_\alpha}\left[E + \frac{v_\alpha \times H}{c}\right] + 2(v_\alpha \times \Omega), \tag{2}$$

together with Maxwell's equations:

$$\begin{aligned} \nabla \times H &= \frac{4\pi}{c}\sum_\alpha n_\alpha q_\alpha v_\alpha + \frac{1}{c}\frac{\partial E}{\partial t}, \\ \nabla \times E &= -\frac{1}{c}\frac{\partial H}{\partial t}, \\ \nabla H &= 0, \\ \nabla E &= 4\pi \sum q_\alpha n_\alpha, \end{aligned} \tag{3}$$

where $\alpha = i, e$ and v_α is the velocity of the α-type particle having the mass m_α and number density, n_α.

Further, we consider a plane wave propagating along the direction of magnetic field lines in such a way that all the perturbed parameters vary as $\exp[i(k \cdot r - \omega t)]$. Following Uberoi and Das (1970), the dispersion relation can be easily written in the form

$$\begin{vmatrix} A & iB \\ -iB & A \end{vmatrix} = 0, \tag{4}$$

© Kluwer Academic Publishers • Provided by the NASA Astrophysics Data System

1979Ap&SS..66..505D

where

$$A = \omega^2 - c^2k^2 - \frac{\omega^2\omega_{pi}^2}{\omega^2 - \pi_i^2} - \frac{\omega^2\omega_{pe}^2}{\omega^2 - \pi_e^2},$$

$$B = \omega\left[\frac{\pi_i\omega_{pi}^2}{\omega^2 - \pi_e^2} - \frac{\pi_e\omega_{pe}^2}{\omega^2 - \pi_e^2}\right],$$

where $\pi_i = \omega_{ci} + 2\Omega$, $\pi_e = \omega_{ce} - 2\Omega$ and all the conventional symbols have their usual meanings together with another mode of waves

$$\omega^2 = \omega_p^2 : \omega_p^2 = \omega_{pi}^2 + \omega_{pe}^2.$$

The plane-wave solution represents two circularly polarized waves propagating with different phase velocities along the magnetic lines. Now, in the case of a medium free from the applied magnetic field, the dispersion relation (4) becomes

$$n^2 = \frac{c^2k^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega(\omega - 2\Omega)}. \tag{5}$$

The simplification on the medium free from applied magnetic field is made, not because of mathematical simplification, but in order to study the effects of the Coriolis force separately. The inherent reason for this will be made clear later.

The refractive indices show that the Coriolis force splits the wave into two circularly-polarized waves propagating with the different phase velocities. The polarization occurs because of a creation of an equivalent magnetic field $2\Omega(cm_\alpha/q_\alpha)$, produced by the Coriolis force resulting from the uniform angular rotation. As the wave progresses along the axis of rotation, the plane of polarization rotates through an angle (Faraday rotation) which is proportional to the line integral of the product of the electron density and the rotational frequency – i.e., $\int_0^h \Omega n_e \, dz$. The measurements of Faraday rotation suffer from an inherent disadvantage, since both electron density and rotational frequency are functions of z. The consideration of the uniform rotational frequency gives rise to a straightforward calculation of Faraday rotation and can be employed to estimate the electron density. In the case of the variation in rotation, we assume the variation of plasma density is very small; and we develop here another method in determining the rotational frequency. The method is based on the measurement of group travel-time of a low frequency wave over the path from the source of the wave to an observer, and is given by the line integral

$$t(\omega) = \int_0^h \frac{ds}{v_g}, \tag{6}$$

where v_g is the group velocity of the wave along the ray path. The refractive indices show that, for the frequency near to twice that of rotational frequency, the integral receives the major contribution from the left circularly polarized wave as compared to that from the right-circular wave. This leads us to restrict our attention to the

© Kluwer Academic Publishers • Provided by the NASA Astrophysics Data System

1979Ap&SS..66..505D

wave propagating with the frequency nearer to twice that of rotational frequency and thus, consequently, the major contribution on the travel time is from the left-circular wave. During the analysis, one should note the peculiar phenomenon of the polarization and the intermode coupling of the waves. The Coriolis force plays an equivalent role of multiple ions in introducing a characteristic frequency – called crossover frequency – at which the reversal of wave polarization occurs (Uberoi and Das, 1972); and, as a result of which, there is an abrupt change of group-velocity due to the polarization changes at the crossover frequency. These phenomena must be taken into account together with the restriction on wave frequency which is nearly equal to the rotational frequency; and, consequently, the contribution of the left-circularly polarized wave is always considered. Thus, the integral (1) should then be considered along the path where the wave is completely left circularly-polarized. The assumptions modify the group velocity u_g which obtains as

$$u_g = \frac{c}{n + \omega(\partial n/\partial\omega)}, \tag{7}$$

where n is the refractive indices for left circularly-polarized wave. Using (5), (7) can be written in the form

$$u_g = \frac{c(2\Omega)^{1/2}(2\Omega - \omega)^{3/2}}{\omega_p(2\Omega - \omega/2)}. \tag{8}$$

Now then, the group travel-time integral (6) becomes

$$t(\omega) = \int_0^h \frac{\omega_p(2\Omega - \omega/2)\,\mathrm{d}s}{(2\Omega)^{1/2}(2\Omega - \omega)^{3/2}}. \tag{9}$$

Equation (9) exhibits the inherent disadvantage in the group travel-time in which $t(\omega)$ is proportional to the integral over the product of electron density and the rotational frequency. The disadvantage arises when both the density and the rotational frequency are functions of path along which the wave progresses. In order to evaluate the integral (9), we modify the expression behind the integral sign by a justifiable assumption that the density variation along the axis of rotation remains almost the same, and thus can be taken out in front of the integral. Equation (9) then becomes a simple function of rotational frequency and the wave frequency. So modified Equation (9) is then required for the determination of rotation and the density of the ionized medium.

In order to employ Equation (9) to this end, we assume that the variation of the rotational frequency is approximated as $\Omega(s) = \Omega(0) + s\Omega'(0)$, where $\Omega(0)$ is the rotation at the source of the wave, $\Omega'(0)$ is the gradient along the axis of rotation. The integral (9) takes the form

$$t(\omega) = \frac{\omega_p(0)}{c}\int_0^h \frac{[2\Omega(0) + 2s\Omega'(0)]\,\mathrm{d}s}{[2\Omega(0) + 2s\Omega'(0)]^{1/2}[2\Omega(0) + 2s\Omega'(0) - \omega]^{3/2}}. \tag{10}$$

© Kluwer Academic Publishers • Provided by the NASA Astrophysics Data System

1979Ap&SS..66..505D

After some straightforward mathematical manipulation, the group travel-time, for the frequency nearer to $2\Omega(0)$ is, approximately, given by

$$t(\omega) = \frac{\omega_p(0)[2\Omega(0)]^{1/2}}{c2\Omega'(0)[2\Omega(0) - \omega]^{1/2}}. \tag{11}$$

In the derivation of the expression (11), we impose a suitable condition on the wave frequency compared to the rotational frequency. The derived expression shows that the group travel-time changes with the rotational frequency, and can become very large as 2Ω approaches the wave frequency.

The expression (11) is the required equation in determining the rotational frequency on the basis of the condition on the wave frequency near twice that of the rotational frequency. Otherwise, the method will be subject to some error; but the latter is not very difficult to estimate for the assumed models. The analysis of employing the method through expression (11) can be described as follows: First, an arbitrary rotational frequency is chosen for a set of observation of the group travel-times for different wave frequencies. The observed travel-time for this fixed rotation but with the different frequencies should show a linear relation with $[2\Omega(0) - \omega]^{-1/2}$, and the plot of the observed time against $[2\Omega(0) - \omega]^{-1/2}$ will indicate whether or not the chosen rotational frequency is appropriate, depending on the nature of the curve. This plot is to be made for several assumed rotational frequencies until a best straight line is recovered. The rotation for which the straight line is obtained is the appropriate rotational frequency of the system and the slope of the line θ is given by

$$\theta = \frac{\omega_p(0)2\Omega(0)^{1/2}}{c[2\Omega'(0)]}, \tag{12}$$

yielding the density of the medium as

$$n_e(0) = \frac{c\theta}{4\pi\, e^2}\left\{\frac{2\Omega'(0)}{[2\Omega(0)]^{1/2}}\right\}, \tag{13}$$

Thus, expression (11) is the working equation for the rotational frequency and expression (13) is to be used in determination of the density of the medium. It is mentioned earlier that the Coriolis force always introduces an equivalent magnetic field causing the dispersiveness of the wave. However, the well-known fact is that the magnetic field in the ionosphere causes the dispersion of the wave, by virtue of which an ion-cyclotron whistler deriving from the left circularly-polarized wave is produced. If the conclusion that the equivalent magnetic field could produce the whistler phenomena in the rotating ionized medium is confirmed, then the method described here could be important in diagnosing the physical parameters of a rotating medium. The effect of the magnetic field is, so far, not taken into account; as our main aim has been to show that the effect of the Coriolis force could be useful as a diagnosis technique. But, in the presence of the magnetic field, the rotation of the charged particles due to the magnetic field will be superimposed on the rotational

© Kluwer Academic Publishers • Provided by the NASA Astrophysics Data System

1979Ap&SS..66..505D

frequency. In the present case, the rotational frequency is replaced by an equivalent rotation $\bar{\Omega} \equiv \omega_{c_\alpha} + 2\Omega$ where ω_{c_α} is the cyclotron frequency of α-type particles and the mathematical development goes parallel as described before. In the case of non-uniform magnetic field along the axis of rotation or the fluctuating magnetic field, we continue the same type of expansion in magnetic field or in ω_{c_α} as similar to those made in rotational frequency and the technique needs some mathematical manipulation in determining the group travel-time and the expression for the density of the medium.

3. Conclusion

Though the adopted model is an idealized one (because the solid body rotation is an idealistic assumption in an ionized medium) yet the present study should give an insight of a diagnostic technique in determining the physical parameters. Moreover, a major disadvantage is absent due to the valid assumption made on the density variation. In order to employ the method as a diagnostic technique for a more realistic model, further investigation is needed before the method could be applied to problems arising in space physics.

Acknowledgement

The author gratefully acknowledges the kind hospitality of Dr M. K. V. Bappu, Director, Indian Institute of Astrophysics, Bangalore. The work has been supported by CSIR-Government of India.

References

Chandrasekhar, S.: 1953a, George Darwin Lecture, *Monthly Notices Roy. Astron. Soc.* **113**, 667.
Chandrasekhar, S.: 1953b, *Proc. Roy. Soc. A.* **217**, 306.
Chandrasekhar, S.: 1953c, *Astrophys. J.* **118**, 7.
Hide, R.: 1966, *Phil. Trans. Roy. Soc.* (*Lond.*) **A259**, 615.
Lehnert, B.: 1954, *Astrophys. J.* **119**, 647.
Uberoi, C. and Das, G. C.: 1970, *Plasma Phys.* **12**, 661.
Uberoi, C. and Das, G. C.: 1972, *Plasma Phys.* **14**, 201.

© Kluwer Academic Publishers • Provided by the NASA Astrophysics Data System