Kodaikanal Obs. Bull. (1984) 4, 43-46.

MODELLING OF PLASMA STRUCTURES IN THE SOLAR ATMOSPHERE

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Abstract

A maximal helicity principle is proposed to study the stability of plasma structures in the solar atmosphere. This principle derives from the statistical treatment of the magnetohydrodynamical turbulence. The steady state of the plasma system results from subjecting the system to the invariance of total energy, the magnetic helicity and toroidal and poloidal magnetic fluxes. The statistical distributions of the velocity and magnetic fields are formulated.

1. Introduction

High resolution optical, UV, EUV and X-ray observations have established the spatial complexity of the solar corona. Active regions appear in the form of loops or arches. The spatial structure of these loops outlines the magnetic field geometry. From the observations of lines of Ca II, HI, CII, CIII, CIV, Ne VII, Mg VIII, Mg IX and Mg X, it is concluded that the loops consist of a cool core surrounded by a hot sheath which merges with the hot corona, Vaiana and Rosner (1978).

2. Maximal Helicity Principle for Stability

We assume plasma to be in the form of a cylinder of radius R and length L with rigid perfectly conducting walls at r = R. The MHD equations for an incompressible medium are:

$$\rho\left[\frac{\delta \vec{v}}{\delta t} + (\vec{V}.\vec{\nabla})\,\vec{V}\right] = \frac{1}{4\pi}(\vec{\nabla}\times\vec{B})\times\vec{B} - \vec{\nabla}p + \rho\vec{g} \quad (1)$$

$$\frac{\delta \vec{B}}{\delta t} = \vec{\nabla}\times(\vec{V}\times\vec{B}), \quad \vec{\nabla}.\vec{V} = \vec{\nabla}.\vec{B} = 0 \quad (2)$$

where, the symbols have their usual meaning. The boundary conditions on B and V are:

$$V_r \ (r=R) = 0 = B_r \ (r=R) \quad (3)$$

The geometry is indicated in Fig.1.
Using the incompressibility condition in the divergence of equation (1), one gets a relation between p, V and B as:

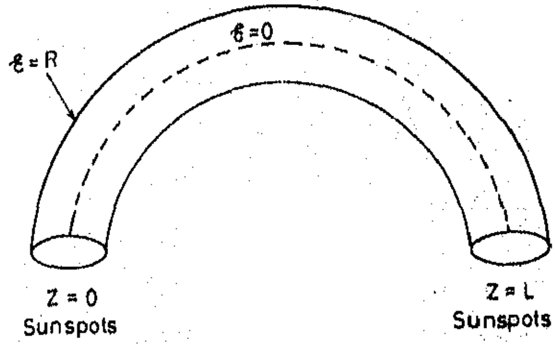

Fig.1. The geometry of a solar corona loop.

$$\nabla^2 p = 1/4\pi\ \nabla.[(\nabla\times B)\times B] - \nabla.[\rho(\vec{V}.\vec{\nabla})V] + \nabla.(\rho\vec{g}) \quad (4)$$

which determines the density and temperature profile of the plasma. The three invariants of the plasma system are:

Total energy $E = \int d^3 X\ (V^2 + B^2)$ (5)

Magnetic helicity $H_M = \int d^3 X\ \vec{A}.\vec{B}$ (6)

Cross helicity $H_C = \int d^3 X\ \vec{V}.\vec{B}$ (7)

The poloidal flux $\psi_p = \int_0^L dz \int_0^{2\pi} d\theta\ A_z$

$= \text{constant at } r = R$ (8)

where $\vec{A}$ is the vector potential.

The velocity and magnetic fields are expanded in terms of Chadrasekhar-Kendall functions given as:

$$\vec{a}(n,m,q) = e_r\left[\frac{im}{r} + \frac{ik_n}{\lambda(n,m,q)}\frac{\delta}{\delta r}\right]\psi(n,m,q)$$

$$+ e_\theta\left[\frac{\delta}{\delta r} - \frac{mk_n}{r\lambda(n,m,q)}\right]\psi(n,m,q) \quad (9)$$

$$+ e_z\left[\frac{\lambda^2(n,m,q) - k_n^2}{\lambda(n,m,q)}\right]\psi(n,m,q)$$

where

$$\psi(n,m,q) = J_m(\gamma_{nmq}\ r)\ e^{im\theta + ik_n z}$$

$$\lambda(n,m,q) = \pm[\gamma_{nmq}^2 + k_n^2]^{1/2}$$

$$K_n = \frac{2\pi n}{L}\ ;\ n=0, \pm 1, \ \ 2, \ldots\ldots\ldots;$$

$$m=0, \pm 1, \pm 2, \ldots\ldots\ldots$$

γ_{nmq} corresponds to that solution which makes $\vec{a}$ satisfy the boundary condition $a_r = 0$ at $r = R$. This gives for $m^2 + n^2 > 0$.

$$Rk_n\ \ \gamma_{nmq}\ J'_m(\gamma_{nmq}R) + m\lambda(n,m,q)\ Jm(\gamma_{nmq}R) = 0 \quad (10)$$

The eigen values for $m = n = 0$ are determined from the ratio of toroidal to poloidal flux:

$$\frac{\psi_t}{\psi_p} = \frac{-R}{L}\ \frac{\gamma_{ooq}\ J'_o\ (\gamma_{ooq}R)}{\lambda(o,o,q)\ Jo(\gamma_{ooq}R)} \quad (11)$$

Now, all the fields and the invariants can be expressed in terms of $\vec{a}$ by using

$$\vec{B} = \sum_{n,m,q} Cnmq\ \ \lambda(n,m,q)\ \ \xi(n,m,q)\ \ \vec{a}(n,m,q) \quad (12)$$

$$\vec{V} = \sum_{n,m,q} Cnmq\ \ \lambda(n,m,q)\ \ \eta(n,m,q)\ \ \vec{a}(n,m,q) \quad (13)$$

The statistical mechanics of the V and B fields is then formulated in a phase space whose coordinates are the real and imaginary parts of ξ'S and η'S. The most probable state of the system is then constructed subject to the constraints of conservation laws. This leads to a canonical ensemble which has an exponential term having a lagrange multiplier for every conserved quantity. Lagrange multipliers are determined by requiring a given value of the expectation value of the conserved quantity, Montgomery et al. (1978). One chooses the canonical distribution subject to the invariance of E, H_M, H_C, ψ_t and ψ_p as:

$$D = \text{Constant}\ \exp\ [-\alpha E - \beta H_M - \gamma' H_C - \delta\psi_t] \quad (14)$$

where α, β, γ' and δ are the Lagrange multipliers. For initially quiescent conditions $H_C = 0$ and, therefore $\gamma' = 0$. One can factor out the probability distribution for each expansion coefficient. The probability distribution for η(n,m,q) is:

$$Pn(n,m,q) = Kn,m,q\ \exp\ [-\alpha\lambda^2(n,m,q)\ |\eta(n,m,q)|^2] \quad (15)$$

From this we find

$$<|\eta(n.m.q)|^2> = \alpha^{-1}\ \lambda^{-2}(n,m,q) \quad (16)$$

Similarly, one can carry out the procedure for other coordinates ξ(n,m,q). One notices from equation (16) that even for initially quiescent conditions there is a finite mean square velocity for every mode in the system. Four scalar quantities, $<E>$, $<H_M>$, $<\psi_t>$ and $<\psi_p>$ are sufficient to determine the level of excitation of every mode in the system.

3. Application to Coronal Loops

3.1. Radial Temperature Variation

We propose that the steady state of the loop can be described in terms of a single Chandrasekhar-Kendall function characterized by m = n = 0. Equation (4) for this choice can be solved to find the radial variation of the plasma temperature as:

$$T(r) = \frac{m_i}{k_B} \lambda^4 (0,0,1)|\eta^2(0,0,1)| \; C^2(0,0,1) \; X$$

$$[\, x \{ J_0^2(x) + J_1^2(x) \} - J_1(x)\, Jo(x)] + To \quad \text{and}$$

$$\frac{\delta T}{\delta r} = \frac{m_i}{k_B} \lambda^5 (0,0,1) \; |\eta^2 (0,0,1)| \; C^2(0,0,1) \; X \; \frac{J_1^2(x)}{x} \qquad (17)$$

where $T = T_0$ at $r = 0$; $x = \lambda r$, λ is determined from equation (11). One can substitute for T(r) from eq. (17) in the line flux calculations and determine the spatial widths of the lines CII, CIII, OIV, OVI, Ne VIII and Mg X which have mean temperatures of formation 5×10^4K, 9×10^4K, 2×10^5K, 3×10^5K, 6×10^5K and 1.4×10^6K respectively. Now $\lambda^2\eta^2$ (0,0,1) is a measure of the turbulent velocity in the fluid and can be estimated from the observed Doppler widths of these lines. For $\lambda^2\eta^2 \sim 10^{14}(\text{cm/sec})^2$ and $T_0 = 6\times10^3$K, $\lambda = 2/R$, $R = 10^9$cm, B ~1.5 Gauss, we find the spatial widths to be $0.02\lambda^{-1}$, $0.025\lambda^{-1}$, $0.12\lambda^{-1}$, $0,1215\lambda^{-1}$, $0.3\lambda^{-1}$ and $> 0.5\lambda^{-1}$ for CII, CIII, OIV, OVI, Ne VII and Mg X respectively. This behaviour certainly conforms to the cool core and hot sheath type of loops.

3.2. Statistical Mechanics of V and B Fields

In order to formulate the statistical mechanics, we include two values of the parameter q which labels the various roots of equation (11) as λ_0, λ_1, etc. One can express the average values of E, H_M and ψ_t in terms of λ'S and the Lagrange multipliers. The Lagrange multipliers can be solved for by assigning some given values to $<E>$ $<H_M>$, $<\psi_t>$ and $<\psi_p>$. As a numerical example, say $L = 5\times10^9$cm, $\psi_t/\psi_p = 1$ then equation (11) gives $\lambda_0 = 2.2/R$ and $\lambda_1 = -2.6/R$. The values of α, β and δ are determined from the following relationships:

$$\alpha = [2(\bar{E}-\lambda_1\bar{H}_m)\;(\bar{E}-\lambda_1\bar{H}_m)]^{-1} \; X$$

$$[\, - \{ -3\bar{E} + \tfrac{3}{2}(\lambda_0+\lambda_1)\;\bar{H}_m \} \pm$$

$$\{ \bar{E}^2 - \tfrac{9}{4}(\lambda_0+\lambda_1)^2 \;\bar{H}_m^{-2} - 8\lambda_1\lambda_0\,\bar{H}_m^{-2}$$

$$- \bar{E}(\lambda_0+\lambda_1)\;\bar{H}_m \}^{1/2} \,] \qquad (18)$$

$$\beta = \frac{\lambda_1}{2(\bar{E} - 1/\alpha - \lambda_1\bar{H}_m)} - \frac{\lambda_0}{2(\bar{E} - 1/\alpha - \lambda_0\bar{H}_m)} \qquad (19)$$

and

$$\delta = \bar{\psi}_t \left[\frac{\delta_0^2\,(\bar{E} - 1/\alpha - \lambda_1 H_m)}{\lambda_0(\lambda_0 - \lambda_1)} + \frac{\delta_1^2\,(\bar{E} - 1/\alpha - \lambda_0\bar{H}_m)}{\lambda_1(\lambda_1 - \lambda_0)} \right]^{-1} \qquad (20)$$

where $\delta_0 = 2\pi RC(0,0,1)\;\gamma_{001}\;J_1(\gamma_{001}R)$

$\delta_1 = 2\,RC(0,0,2)\;\gamma_{002}\;J_1\,\gamma_{002}R)$

$\bar{E}$, $\bar{H}_M$ and $\bar{\psi}_t$ are the observables. Since for the case of coronal loops, there are no direct measurements of these quantities, we shall have to fix their values from other considerations. For example $\bar{E}$ can be obtained from energy balance arguments which give $\bar{E} \sim 10^{28}$ ergs (Levine and Withbroe 1977). We choose $H_M \sim 0.05\times10^{37}$ ergs cm. The two values of α are $\sim 2.04\times10^{-28}$ $(ergs)^{-1}$ and 0.98×10^{-28} $(ergs)^{-1}$. We find $\alpha \sim 2.04\times10^{28}$ $(ergs)^{-1}$ to be appropriate for the approximations involved in arriving at (18). Eq. (19) gives $\beta = 0.2\times10^{-37}$ $(ergs\ cm)^{-1}$, and $\delta = 10^{-36}\ \bar{\psi}_t$ $(Gauss\ cm^2)^{-1}$. A representative choice of $\bar{\psi}_t \sim 10^{17}$ Maxwell gives $\delta = 10^{-19}$.

Thus the distribution function for this particular indicative example is :

$$D = Const\ e^{-2.04\bar{E}/E}\ e^{0.01\bar{H}m/Hm} \times e^{-0.01\psi_t/\bar{\psi}_t} \qquad (21)$$

We observe that total energy has a rather narrow distribution compared to the distributions of the magnetic helicity and toroidal flux. More details about the application to coronal loops are presented in the paper by Krishan (1983a,b).

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