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# On the saturation of the refractive index structure function – I. Enhanced hopes for long baseline optical interferometry

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**Summary.** The Kolmogorov–Obukhov law for the refractive index structure function is shown to be only an approximation valid in the inertial subrange of turbulence. The inclusion of saturation effects for larger separations leads to an asymptotically constant variance for the relative phase fluctuations for large baselines. This result enhances the possibilities for very long baseline optical interferometry.

## 1 Introduction

Atmospheric turbulence introduces random variations in temperature, humidity and therefore refractive index. The statistical description of these variations can be conveniently given in terms of structure functions. Tatarski's monograph (1961) on the nature of wave propagation through turbulent media contains all the details of such structure functions. In the inertial subrange of turbulence, the velocity structure function is proportional to the two-thirds power of the separation between the two points for which the function is evaluated. The corresponding law for the refractive index structure function, known in the literature as the Kolmogorov–Obukhov law, led to a theory of image formation through turbulent media (Fried 1966) which is only valid, however, in the inertial subrange of turbulence. A direct consequence of Fried's theory is a stringent requirement of narrow bandwidth for long baseline optical interferometry (Tango & Twiss 1980). In this paper, we look at an alternative formulation of the refractive index structure function which incorporates the effects of saturation at large separations. We then go on to describe the implications of this saturation for long baseline optical interferometry.

## 2 The refractive index structure function

The structure function for any random variable $X$ is defined as

$$D_X(\xi)=\langle|X(\mathbf{r}+\xi)-X(\mathbf{r})|^2\rangle \tag{1}$$

where the angular brackets denote an ensemble average. Resorting to the ergodic theorem, we can also consider equation (1) as a time average. In this case, we need to be careful about defining the time-scales over which the averaging is performed. If the process varies on time-scales longer than that over which the averaging is done, then we can consider $D_X$ to be a slowly varying function of time as well as of the separation $\xi$. The right-hand side of equation (1) can be expanded to yield

$$D_X(\xi)=\langle|X(\mathbf{r}+\xi)|^2\rangle+\langle|X(\mathbf{r})|^2\rangle-2\,\mathrm{Re}\langle X^*(\mathbf{r}+\xi)\,X(\mathbf{r})\rangle. \tag{2}$$

For stationary stochastic processes, the first two terms on the right-hand side of equation (2) are independent of position while the last term depends only on the separation $\xi$. Writing $\langle|X(\mathbf{r}+\xi)|^2\rangle=\langle|X(\mathbf{r})|^2\rangle=\sigma_X^2$ (for zero mean) and writing $\mathrm{Re}\langle X^*(\mathbf{r}+\xi)\,X(\mathbf{r})\rangle=\sigma_X^2\varrho_X(\xi)$, we have

$$D_X(\xi)=2\sigma_X^2[1-\varrho_X(\xi)] \tag{3}$$

where $\varrho_X$ is known as the normalized correlation function. Tatarski (1961, p. 34) has given an expression for the velocity structure function as follows:

$$D_V(\xi)=\tfrac{2}{3}\sigma_V^2\left[1-\frac{2^{2/3}}{\Gamma(1/3)}\left(\frac{\xi}{L_0}\right)^{1/3}K_{1/3}\left(\frac{\xi}{L_0}\right)\right] \tag{4}$$

where $L_0$ is the outer scale of turbulence and $K$ is the modified Bessel function. The function given by (4) asymptotically approaches $\tfrac{2}{3}\sigma_V^2$ for $\xi \gg L_0$, while for $\xi \ll L_0$ it follows a two-thirds power law. Matveev (1958) has given the following mathematically simpler expression of essentially similar behaviour for the temperature structure function:

$$D_T(\xi)=\langle\Delta T^2\rangle\{1-\exp[-(\xi/L_0)^{2/3}]\}. \tag{5}$$

In what follows, we adopt expression (5) as the basis for further calculations. The corresponding expression for the refractive index structure function can now be written as

$$D_N(\xi)=2\sigma_N^2\left\{1-\exp\left[-\left(\frac{\xi}{L_0}\right)^{2/3}\right]\right\} \tag{6}$$

with the identification

$$2\sigma_N^2\equiv C_N^2L_0^{2/3} \tag{7}$$

where $C_N$ is the conventionally used refractive index structure constant of Obukhov's law.

## 3 Derivation of the phase structure function

As light propagates vertically down the atmosphere, the random refractive index fluctuations deflect the rays in a random fashion. The longer the ray travels through turbulence, the more deflections it suffers. This phenomenon is exactly analogous to a random walk problem where the variance of the walker's position from the initial origin increases as the square root of the number of steps taken. Thus the variance in the angular deflection of the ray from an initial direction will be proportional to the square root of the total number of small deflections suffered in its path. Using a phase screen approach, this can be translated into a variance of the phase fluctuations. For a single homogeneous phase screen of length $L$, equation (6.28) of Tatarski (1961) gives the following expression for the phase structure function $D_\phi(\boldsymbol{\rho})$ evaluated for two points separated by a distance $\varrho$ on a plane normal to the direction of propagation:

$$D_\phi(\boldsymbol{\rho})=2k^2L\int_0^\infty\{D_N[(x^2+\varrho^2)^{1/2}]\,D_N(x)\}\,dx \tag{8}$$

where $k$ is the wavenumber of the light. Substituting for $D_N$ from equation (6) we obtain

$$D_\varphi(\boldsymbol{\rho}) = 2k^2 L\, L_0 \sigma_N^2 F\left(\frac{\varrho}{L_0}\right) \tag{9}$$

where

$$F\left(\frac{\varrho}{L_0}\right) = 2L_0^{-1}\int_0^\infty dy \left\{\exp\left[\left(\frac{y}{L_0}\right)^{2/3} - \exp\left[\frac{(\varrho^2+y^2)^{1/3}}{L_0^{2/3}}\right]\right]\right\}. \tag{10}$$

Equation (10) can be integrated and the result is shown in Fig. 1. The slope of the curve plotted in a logarithmic scale for small values of $\varrho/L_0$ has a value 5/3. However, it should be noted that the five-thirds power dependence has limited validity, being true over a restricted range $\varrho/L_0 \lesssim 10^{-3}$. Above this range the slope diminishes and the function saturates beyond $\varrho/L_0 \gtrsim 1$. We now proceed to follow Roddier (1981) and sum up expressions similar to (9) for a continuous distribution of screens having different values of $\sigma_N^2$ and $L_0$ at different heights. Thus we can write the phase structure function on the ground as

$$D_\varphi(\varrho) = 2k^2 \int_0^H L_0(h)\, \sigma_N^2(h)\, F\left(\frac{\varrho}{L_0}\right) dh. \tag{11}$$

From Fig. 1 and equation (11) we can see that $D_\varphi$ is independent of $\varrho$ for $\varrho > \max[L_0(h)]$.

## 4 Implications for long baseline interferometry

To make further progress, it is essential to know the outer scale of turbulence $L_0$ and the variance of the refractive index fluctuations $\sigma_N$ as a function of height. In the absence of such data, we can

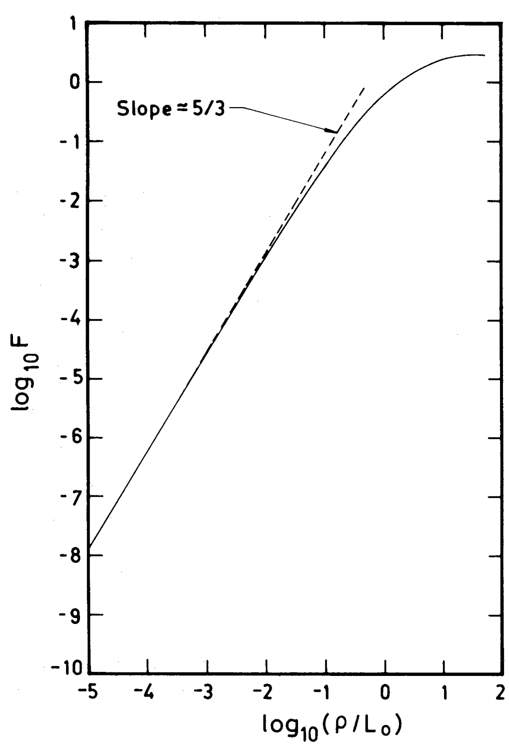


**Figure 1.** Plot of $\log F(\varrho/L_0)$ versus $\log(\varrho/L_0)$. The broken line is the extrapolation of the five-thirds behaviour at small separations.

only make qualitative statements regarding $D_\phi(\varrho)$. However, descriptions exist in the literature of several experiments which have been conducted for horizontal propagation of light. Since it is well known that the maximum degradation of images occurs very close to the ground, the results of such experiments could give the most stringent limits on the capabilities of interferometers looking up into the sky. To enumerate only a few such results, we can start from Tatarski's (1961, p. 212) experiments which yielded a correlation length for amplitude fluctuations of approximately $(\lambda L)^{1/2}$ where $\lambda$ is the wavelength and $L$ the length of turbulence over which the ray is propagated before reaching the detector. Bertolotti *et al.* (1968) demonstrated that the phase structure function has a knee around a separation of a few centimetres for horizontal propagation near the ground. Gaskill (1969) showed that his experiments for the atmospheric degradation of holographic images are consistent with the theory of locally stationary Gaussian processes and found saturation near 10 cm. Bouricius & Clifford (1970) found the knee of the phase structure function to be around 100 cm. Using this knee as an indicator of the outer scale of turbulence, Clifford *et al.* (1971) monitored the outer scale of turbulence and found it to be $1.3\,\mathrm{m}\pm20$ per cent. For vertical propagation of light, the experiment performed by Danjon on Sirius and Rigel at Strasbourg on 1926 February 13 (Danjon 1955, fig. 8) shows a clear saturation of the semi-amplitude of the movement of the fringes at separations of a few decimetres between the interfering beams. More recent correlation measurements on star light also show a sharp drop in the correlation at a separation of a few centimetres (Roddier & Roddier 1973, fig. 3). All these results are consistent with our contention that $D_\phi$ must saturate owing to the saturation of $D_N$ at separations comparable with the outer scale of turbulence.

This can only mean that the sensitivity of an interferometer will not decrease any further from its capabilities at a baseline of a metre or so. It is interesting to note that an essentially similar statement was made 30 years ago (Danjon 1955) but has been generally ignored by the builders of modern interferometers. To quote Danjon: '. . . the fluctuations for elements several decimetres apart are uncorrelated. It may be remarked in passing that this fact favours the use of interference methods, since there is no reason to fear that the atmospheric disturbance will increase indefinitely as the distance $d$ between the beams is made larger.'

## 5 Discussion

By including the effects of saturation in the structure functions describing the fluctuations of atmospheric parameters, we have shown that the phase structure function at the ground level does not increase without bound but saturates to a constant value. As mentioned earlier, the precise value of the separation at which saturation sets in, as well as the asymptotic constant value of the phase fluctuations at saturation, can only be known from a knowledge of $L_0$ and $\sigma_N$ as a function of height. Danjon (1955) obtains a semi-amplitude of one wavelength for the fringe fluctuations, but obviously this will differ from location to location as well as from time to time.

A few comments in the five-thirds dependence of the structure function at small separation seem to be in order. The departure from the above law begins at around $\varrho/L_0\approx10^{-3}$ so that for $L_0\approx50$–$100$ m the five-thirds law must be strictly valid for $\varrho\lesssim0.5$–$1$ m. Furthermore, some of the published experimental results, e.g. Bouricius & Clifford (1970, fig. 4), in fact show a smaller slope, although these workers claim a five-thirds behaviour. It would be worthwhile to investigate whether the data can give signatures of saturation even at small separations. There are other effects which must be considered, namely the large-scale gradients in phase which could probably appear at large separations. This would require a continuous monitoring of the position of the 'white light fringe'. However, these large-scale changes occur also on large time-scales and the hope would be that enough time will be available to track the position of zero phase difference. The consequences of saturation for the estimates of 'isoplanatic patch' as also for temporal correlations of the phase fluctuations are not considered here. These are certainly worth detailed investigation and will form the topic of another paper.

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