Black holes in nonflat backgrounds: The Schwarzschild black hole in the Einstein universe

K. Rajesh Nayak*
Indian Institute of Astrophysics, Bangalore 560 034, India

M. A. H. MacCallum†
School of Mathematical Sciences, Queen Mary and Westfield College, London, United Kingdom

C. V. Vishveshwara‡
Indian Institute of Astrophysics, Bangalore 560 034, India
and Bangalore Association for Science Education (BASE), Bangalore 560 001, India
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As an example of a black hole in a non-flat background a composite static spacetime is constructed. It comprises a vacuum Schwarzschild spacetime for the interior of the black hole across whose horizon it is matched onto the spacetime of Vaidya representing a black hole in the background of the Einstein universe. The scale length of the exterior sets a maximum to the black hole mass. To obtain a non-singular exterior, the Vaidya metric is matched to an Einstein universe. The behavior of scalar waves is studied in this composite model.

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I. INTRODUCTION

For more than three decades now, black holes have been investigated in great depth and detail. However, almost all these studies have focused on isolated black holes possessing two basic properties: namely, time independence characterized by the existence of a timelike Killing vector field and asymptotic flatness. On the other hand, one cannot rule out the important and, perhaps, realistic situation in which the black hole is associated with a non-flat background. This would be the case if one takes into account the fact that the black hole may actually be embedded in the cosmological spacetime or surrounded by local mass distributions. In such situations one or both of the two basic properties may have to be given up. If so, the properties of isolated black holes may be modified, completely changed or retained unaltered. Black holes in non-flat backgrounds form, therefore, an important topic. Very little has been done in this direction. Some of the issues involved here have been outlined in a recent article by Vishveshwara [1]. As has been mentioned in that article, there may be fundamental questions of concepts and definitions involved here. Nevertheless, considerable insight may be gained by studying specific examples even if they are not entirely realistic. In this regard the family of spacetimes derived by Vaidya [2], which is a special case of Whittaker's solutions [4], representing in a way black holes in cosmological backgrounds have been found to be helpful. Nayak and Vishveshwara [3] have studied these spacetimes, concentrating on the geometry of the Kerr black hole in the background of the Einstein universe, which dispenses with asymptotic flatness while preserving time symmetry. In the present paper, we specialize to the simpler case of the Schwarzschild black hole in the background of the Einstein universe, which we may call the Vaidya-Einstein-Schwarzschild (VES) spacetime. This allows us to study this spacetime in considerable detail as well as investigate a typical physical phenomenon, namely the behavior of scalar waves in this spacetime as the background.

The rest of this paper is organized as follows. In Sec. II, we consider the line element of the VES spacetime and the energy-momentum tensor. In Sec. III, we match the metric of the VES spacetime to the Schwarzschild vacuum metric across the black hole surface. Similarly we match the VES spacetime to the Einstein universe at large distances. In Sec. IV, we investigate the behavior of scalar waves propagating in this spacetime. Section V comprises the concluding remarks.

II. LINE ELEMENT AND THE ENERGY-MOMENTUM TENSOR

As mentioned earlier, an account of Vaidya's black hole spacetimes in cosmological spacetimes may be found in Refs. [2] and [3]. By setting the angular momentum to zero in the Kerr metric we obtain the line element of the Schwarzschild spacetime in the background of the Einstein universe:

$$ds^2=\left(1-\frac{2m}{R\tan\left(\frac{r}{R}\right)}\right)dt^2-\left(1-\frac{2m}{R\tan\left(\frac{r}{R}\right)}\right)^{-1}dr^2 - R^2\sin^2\left(\frac{r}{R}\right)[d\theta^2+\sin^2\theta\, d\phi^2] \tag{2.1}$$

where m is the mass and the coordinates range from $0 \leqslant r/R \leqslant \pi$, $0 \leqslant \theta \leqslant \pi$ and $0 \leqslant \phi \leqslant 2\pi$. In the limits $m=0$ and $R=\infty$, we recover respectively the Einstein universe and the Schwarzschild spacetime. The parameter R is a measure of the cosmological influence on the spacetime. As the spacetime is static, the black hole is identified as the surface on

*Email address: nayak@iiap.ernet.in
†Email address: m.a.h.maccallum@qmw.ac.uk
‡Email address: vishu@iiap.ernet.in

which the time-like Killing vector becomes null, i.e., $g_{00}=0$ above, which is the static limit and the Killing event horizon. The black hole is therefore given by

$$2m=R\tan\left(\frac{r}{R}\right). \tag{2.2}$$

We shall now work out the energy-momentum tensor for this metric. The components of the Einstein tensor are given by

$$G_1^1=G_2^2=G_3^3=\frac{1}{3}G_0^0=\frac{1}{R^2}\left(1-\frac{2m}{R\tan(r/R)}\right). \tag{2.3}$$

The Einstein field equations, including the cosmological term Λ for generality, although it could equally well be considered to be included in ρ and p below, are given by

$$R_{ab}-\frac{1}{2}g_{ab}R=\kappa T_{ab}+\Lambda g_{ab}, \tag{2.4}$$

where $\kappa=8\pi G/c^2$ and the Latin indices a,b range from 0 to 3 (Greek indices $\mu,\nu=1$–3).

The energy-momentum tensor is taken to be that of a perfect fluid,

$$T_{ab}=(\rho+p)u_a u_b-pg_{ab}, \tag{2.5}$$

u^a being the static four velocity:

$$u^a=\frac{1}{\sqrt{g_{00}}}\delta_0^a. \tag{2.6}$$

Then density ρ and pressure p are given by

$$\rho=\frac{3}{\kappa R^2}\left(1-\frac{2m}{R\tan(r/R)}\right)-\Lambda/\kappa, \tag{2.7}$$

$$p=\frac{-1}{\kappa R^2}\left(1-\frac{2m}{R\tan(r/R)}\right)+\Lambda/\kappa. \tag{2.8}$$

The behavior of ρ and p can be easily ascertained from the above equations.

$\Lambda>0$. We find that $\rho,p\leqslant 0$ in some region outside the black hole, violating the weak energy condition.

$\Lambda\leqslant 0$. In this case, $\rho>0$ but $p<0$ everywhere outside the black hole and tends to zero on it. However, $\rho+p\geqslant 0$ thereby satisfying the weak energy condition.

For convenience, we take $\Lambda=0$. Then

$$\rho+3p=0. \tag{2.9}$$

This suggests that the spacetime is a special case of the solutions obeying the condition $\rho+3p=$ *constant* discussed by Whittaker [4], and it is easy to check that this is so (it is the case $B=G=\lambda=0$, $c=-2m$, $\alpha=1/R$, with the time coordinate scaled so that $n=1$, in Whittaker's notation).

Thus the behavior of the energy-momentum tensor is reasonable, since in the Einstein universe itself we have $p<0$, while ρ and p satisfy the weak energy condition.

III. MATCHING TO THE SCHWARZSCHILD VACUUM AND THE EINSTEIN UNIVERSE

In this section we shall match the VES spacetime to the Schwarzschild vacuum spacetime on one side and to the Einstein universe on the other. The possibility of matching to the Schwarzschild vacuum at the black hole surface, without a surface layer or shell, is strongly indicated by the fact that the Einstein tensor of the VES spacetime goes to zero on the surface. We shall show that this is indeed possible. In order to do this, we will first write the line element in Kruskal-like coordinates, among which we will find admissible coordinates in which the matching can be carried out, so that the requirements [5] become simply the continuity of the metric and its first derivative.

The Kruskal form of the VES line element is arrived at by the following transformations:

$$r^*=\frac{R^2}{4m^2+R^2}\left\{r+2m\ln\left[-2m\cos\left(\frac{r}{R}\right)+R\sin\left(\frac{r}{R}\right)\right]\right\}, \tag{3.1}$$

$$u=t-r^*,\quad v=t+r^*, \tag{3.2}$$

$$\hat{U}=-\exp\left[-\frac{u}{4m}\frac{4m^2+R^2}{R^2}\right], \tag{3.3}$$

$$\hat{V}=\exp\left[\frac{v}{4m}\frac{4m^2+R^2}{R^2}\right]. \tag{3.4}$$

Then we obtain

$$\mathrm{d}s^2=\left(\frac{4mR^2}{4m^2+R^2}\right)^2\frac{1}{R\sin(r/R)}e^{-r/2m}\mathrm{d}\hat{U}\mathrm{d}\hat{V}-[R\sin(r/R)]^2\mathrm{d}\Omega^2. \tag{3.5}$$

The Kruskal line element for the Schwarzschild vacuum spacetime,

$$\mathrm{d}s^2=16m_s^2\frac{1}{r_s}e^{-r_s/2m_s}\mathrm{d}\hat{U}\mathrm{d}\hat{V}-r_s^2\mathrm{d}\Omega^2, \tag{3.6}$$

may be recovered from Eq. (3.5) by the limit $R=\infty$. As usual the Kruskal coordinates for the Schwarzschild space cover the whole maximally extended spacetime and not only the region where the coordinates t, r are valid. Now we proceed to carry out the matching at the horizons.

The horizon of the VES metric is at $r=r_0$ where $2m=R\tan(r_0/R)$. To match to the Schwarzschild metric at the horizon the angular variable part requires $2m_s=R\sin(r_0/R)$. Let us use $r'=R\sin(r/R)$ as the radial variable in the VES region. We can rescale both the $\hat{U}$ and $\hat{V}$ of each of the metrics by constant factors $4m_s/\sqrt{e}$ and $4mR^2e^{-r_0/4m}/(4m^2+R^2)$ respectively, giving new coordinates U, V, to reduce the metrics to the forms

$$\mathrm{d}s^2=\frac{1}{r'}e^{(r_0-r)/2m}\mathrm{d}U\mathrm{d}V-(r')^2\mathrm{d}\Omega^2, \tag{3.7}$$

$$\mathrm{d}s^2=\frac{1}{r_s}e^{(2m_s-r_s)/2m_s}\mathrm{d}U\mathrm{d}V-r_s^2\mathrm{d}\Omega^2. \tag{3.8}$$

Then we see the metric is continuous if we identify r_s and r' at the future horizon $U=0$, $r'=r_s=2m_s=R\sin(r_0/R)$, $r=r_0$. To deal with derivatives, start with

$$UV=\left(\frac{4mR^2}{4m^2+R^2}\right)^2 e^{-r_0/2m}\exp\left(2\frac{4m^2+R^2}{4mR^2}r^*\right) \tag{3.9}$$

on the VES side, so that, there,

$$V=2\left(\frac{4mR^2}{4m^2+R^2}\right)e^{-r_0/2m}\exp\left(2\frac{4m^2+R^2}{4mR^2}r^*\right)\frac{\mathrm{d}r^*}{\mathrm{d}U} \tag{3.10}$$

and therefore

$$\frac{\mathrm{d}r'}{\mathrm{d}U}=\frac{\mathrm{d}r'}{\mathrm{d}r}\qquad\frac{\mathrm{d}r}{\mathrm{d}r^*}\qquad\frac{\mathrm{d}r^*}{\mathrm{d}U}$$

$$=\cos(r/R)\quad\left(1-\frac{2m}{R\tan(r/R)}\right)\quad Ve^{r_0/2m}\frac{4m^2+R^2}{8mR^2}$$

$$\times\exp\left(-2\frac{4m^2+R^2}{4mR^2}r^*\right).$$

As $r\to r_0$ the product

$$\left(1-\frac{2m}{R\tan(r/R)}\right)\exp\left(-2\frac{4m^2+R^2}{4mR^2}r^*\right)$$

approaches $e^{-r_0/2m}/2m_s$ and we get

$$\frac{\mathrm{d}r'}{\mathrm{d}U}=\cos(r_0/R)V\frac{4m^2+R^2}{16mm_sR^2}$$

$$=V\frac{R}{\sqrt{4m^2+R^2}}\frac{4m^2+R^2}{16mm_sR^2}$$

$$=V\frac{\sqrt{4m^2+R^2}}{2mR}\frac{1}{8m_s}=\frac{V}{16m_s^2}$$

which is obviously the same as for $\mathrm{d}r_s/\mathrm{d}U$ in the Schwarzschild metric. Now the derivatives of the metric coefficients will match if

$$\frac{1}{2m}\frac{\mathrm{d}r}{\mathrm{d}r'}=\frac{1}{2m_s}=\frac{1}{R\sin(r_0/R)} \tag{3.11}$$

at the horizon, but $\mathrm{d}r/\mathrm{d}r'=1/\cos(r/R)$ and consequently

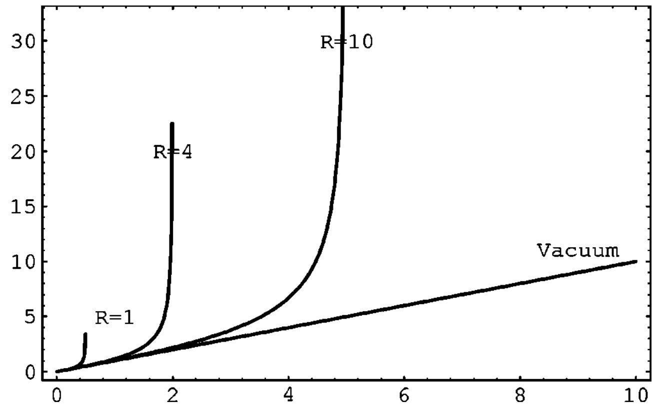

FIG. 1. Plot of m as a function of m_s for different values of R.

$$\frac{1}{2m}\frac{\mathrm{d}r}{\mathrm{d}r'}=\frac{1}{2m\cos(r_0/R)}=\frac{1}{R\sin(r_0/R)} \tag{3.12}$$

because at the horizon $2m=R\tan(r_0/R)$. This completes the matching at the future horizon. Clearly a similar matching with the roles of U and V reversed applies at the past horizon in a Kruskal picture.

We may note that matching the metric component g_{33} yields the relation between the Schwarzschild vacuum mass m_s and the VES mass m:

$$m=m_s\left[1-\left(\frac{2m_s}{R}\right)^2\right]^{-1/2}. \tag{3.13}$$

This clearly exhibits the influence of the cosmological matter distribution on the bare black hole mass. Figure 1 shows plots of m as a function of m_s for different values of R. We note that $2m_s\leqslant R$, so the length scale in the exterior puts a bound on the black hole mass, in a way which may be analogous with the bound found in [6] for the mass of a black hole in an Einstein space.

It is also worth emphasizing that a consequence of this matching is that all the horizon properties, such as the surface gravity, are necessarily the same as those of the usual Schwarzschild black hole. Whether this is reassuring or disappointing is a matter of opinion. It does not imply, however, that properties which depend on the behavior in the exterior region, such as the behavior of waves, will be the same.

To investigate such behaviors we need a well-behaved non-vacuum exterior. Unfortunately, formulas (2.7) and (2.8) show that the energy density and pressure of the VES spacetime blow up as $r/R\to\pi$ and this is in fact a naked singularity. To remove it we try to match to the Einstein universe [which, remember, is a limit of Eq. (2.1)]. It is easy to see that the best hope of doing so without a surface layer is at $r/R=\pi/2$, where we could match both the angular part of the metric and its derivative. In fact, at this radius the VES line element reduces to

$$\mathrm{d}s^2=\mathrm{d}t^2-\mathrm{d}r^2-R^2\sin^2\left(\frac{r}{R}\right)\mathrm{d}\Omega^2 \tag{3.14}$$

which is the line element of the Einstein universe. The metric components of the two spacetimes automatically match, without any change of coordinates, and the first derivative of the angular parts on both sides vanishes. But the first derivative of the tt parts is discontinuous, thereby giving rise to a surface distribution of matter. The components of the corresponding energy-momentum tensor may be computed following Mars and Senovilla [5]. We find that this leads to a trace free tensor.

More specifically, the jump in the fundamental form of the $r=$const surfaces is

$$[K_{tt}]=-m/R^2 \tag{3.15}$$

and the non-zero components of the δ-function parts of the curvature and Ricci tensor are given by

$$Q^r_{ttr}=-m/R^2, \quad R_{tt}=R_{rr}=m/R^2. \tag{3.16}$$

Such a layer might be interpreted as a domain wall.

We now have a composite model consisting of a vacuum Schwarzschild black hole matched onto the VES spacetime which is itself matched to the Einstein universe.

IV. SCALAR WAVES

In the last section we constructed a model for a black hole in a non-flat background. The interior of the black hole consists of the Schwarzschild vacuum. The exterior is the VES spacetime matched onto the Einstein universe. One can explore black hole physics in the exterior and compare it with the effects one encounters in the case of the usual isolated black holes. As an example of such possible studies, we shall consider some properties of scalar waves propagating in this spacetime. Other phenomena occurring in this spacetime, such as the classical tests of general relativity and the geodesics, have been investigated by Ramachandra and Vishveshwara [7].

Because of the time and spherical symmetries of the spacetime, the scalar wave function may be decomposed as

$$\psi=e^{i\omega t}\mathcal{R}(r)Y_l{}^m(\theta,\phi). \tag{4.1}$$

The limits of the radial coordinate are given by $R\tan(r/R)=2m$ to $(r/R)=\pi$ with the VES spacetime extending from $R\tan(r/R)=2m$ to $r/R=\pi/2$ and the Einstein universe from $r/R=\pi/2$ to π. We set the radial function

$$\mathcal{R}(r)=\frac{u(r)}{R\sin(r/R)} \tag{4.2}$$

and define

$$\mathrm{d}r^*=\frac{dr}{1-\dfrac{2m}{R\tan(r/R)}}. \tag{4.3}$$

Then we obtain the Schrödinger equation governing the radial function

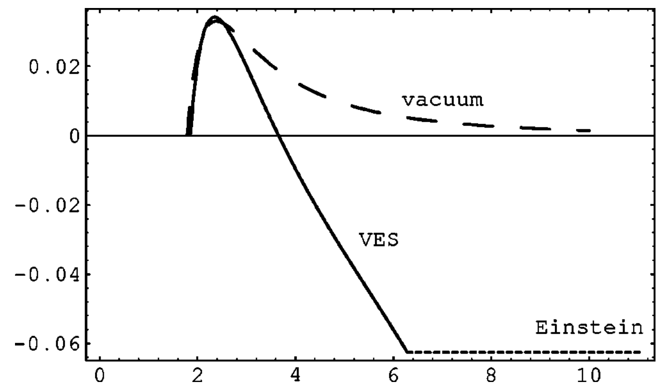

FIG. 2. Plot of effective potential $V(r)$ of VES spacetime for $R/m=4$ and $l=0$. For comparison the effective potentials of the vacuum Schwarzschild spacetime (dashed line) and the Einstein universe (dotted line) are also shown.

$$\frac{d^2u}{dr^{*2}}+[\omega^2-V(r)]u=0. \tag{4.4}$$

The effective potential that controls the propagation of the scalar waves is given by

$$V(r)=\left(1-\frac{2m}{R\tan(r/R)}\right)\left[\frac{l(l+1)}{R^2\sin^2(r/R)} + \frac{2m}{R^3\sin^2(r/R)\tan(r/R)}-\frac{1}{R^2}\left(1-\frac{2m}{R\tan(r/R)}\right)\right]. \tag{4.5}$$

We shall now discuss a few aspects of the behavior of the scalar waves as reflected by the nature of the effective potential.

We have drawn $V(r)$ in Fig. 2 for $l=0$. The figure shows the corresponding effective potential for the vacuum Schwarzschild exterior also which can be obtained by setting $R=\infty$. Both curves start from zero at the black hole and go through a maximum. Thus both potentials possess potential barriers. As in the case of the Schwarzschild vacuum, now too waves can be reflected at the barrier while the transmitted part is absorbed by the black hole. On the other hand, whereas the vacuum potential goes asymptotically to zero, in the present case the potential becomes negative at $r/R=\pi/2$ and continues as a constant, i.e., $-1/R^2$, in the Einstein universe up to $r/R=\pi$. The fact that the effective potential is negative as above raises the possibility of ω^2 being negative as well. This would be equivalent to ω being imaginary, thereby giving rise to exponential growth with time of the scalar wave function. This would mean instability of the model spacetime against scalar perturbations. However, one can see that negative values of ω^2 are ruled out by the boundary condition at $r/R=\pi$. In the Einstein universe sector the Schrödinger equation reduces to

$$\frac{\mathrm{d}^2u}{\mathrm{d}r^{*2}}+\left(\omega^2+\frac{1}{R^2}\right)u=0. \tag{4.6}$$

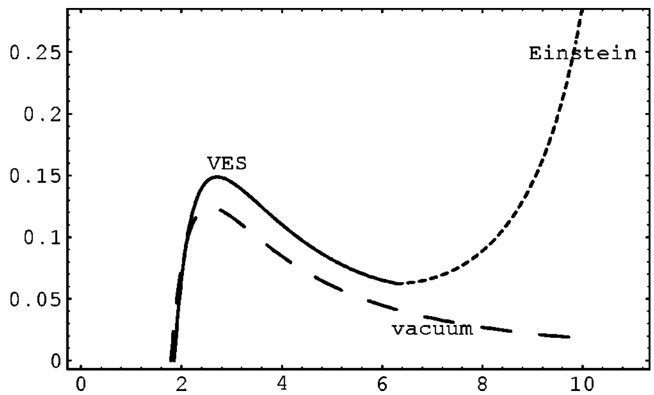

FIG. 3. Plot of effective potential $V(r)$ for $R/m=4$ and $l=1$.

We note that in the Einstein universe, we have $r^*=r$. Furthermore, since $\mathcal{R}=u(r)/R\,\sin(r/R)$, the function $u \sim \sin[(\omega^2+1/R^2)^{1/2}r]$ has to go to zero faster than $\sin(r/R)$ at $r/R=\pi$. This boundary condition requires that $|(R^2\omega^2+1)^{1/2}|$ be an integer greater than 1, and thence that ω^2 be positive. Therefore the spacetime is stable against scalar perturbations.

This is true in the case of vacuum Schwarzschild spacetime as well as the Einstein universe. However, the stability against gravitational perturbations is a different matter altogether. Whereas the Schwarzschild vacuum exterior is stable, the Einstein universe is not [8]. Whether the combination of the two spacetimes is stable, unstable or conditionally stable is an intriguing open question.

For $l>0$, the equivalent potential has the additional term $l(l+1)/R^2\sin^2(r/R)$.

We sketch $V(r)$ for $l=1$ in Fig. 3. Once again the potential goes to zero at the black hole and possesses a barrier region. The additional term goes to infinity at $(r/R)=\pi$, thereby behaving like a centrifugal barrier commonly encountered in the scattering phenomenon. The radial function is exponential wherever the value of $V(r)$ is greater than ω^2 and is a running wave when $\omega^2>V(r)$. Details of such solutions can be studied easily.

V. CONCLUDING REMARKS

The motivation for the present work stems from the need for detailed study of black holes in non-flat backgrounds in comparison and contrast to isolated black holes. A comprehensive investigation of this problem would be a formidable task indeed. We have confined ourselves in this paper to a specific example that relaxes the condition of asymptotic flatness while preserving time-symmetry. The starting point here is the static black hole in the Einstein universe which belongs to the family of solutions presented by Vaidya. In this spacetime the black hole is well defined as the Killing horizon. However, the nature of the interior of the black hole is not entirely clear. Furthermore, it is not obvious *a priori* whether the exterior can be matched smoothly to the Schwarzschild vacuum across the black hole surface. We have shown that this is possible by carrying out this matching using Kruskal coordinates in the two regions. Similarly we have matched the spacetime to the Einstein universe at the other end. This provides a composite model of a black hole in a non-flat background.

In the spacetime considered above, different phenomena may be studied and compared to their counterparts in the gravitational field of an isolated Schwarzschild black hole. As an example, we have briefly discussed the behavior of scalar waves. The spacetime being considered proves to be stable against scalar perturbations as is the Schwarzschild vacuum exterior. This is true of the Einstein universe as well. However, whereas the Schwarzschild spacetime is stable against gravitational perturbations, the Einstein universe is not. It would be quite interesting to see whether the spacetime we have considered, which involves both of the above ones, is gravitationally stable or not. Even if the model presented here is unrealistic, it should provide a testing ground for investigating external influences on the otherwise isolated black holes.

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