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A MODEL OF THE "STEADY" AND "FLUCTUATING" PARTS OF THE SUN'S INTERNAL MAGNETIC FIELD

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ABSTRACT

We model the structure of the "steady" (slowest varying) part of the Sun's internal poloidal magnetic field assuming it to be (for given reasons and in the relevant domain) a current-free field whose field lines "isorotate" according to the Sun's internal rotation given by helioseismology. This part of the field can be described as that of a central dipole and a central hexapole both parallel to the rotation axis and embedded in a uniform external field. The field structure contains a critical surface (running along the base of the convection zone in the low latitudes), where a discontinuity of ~ 7 nHz per unit flux in the gradient of rotation (with respect to magnetic flux) may be winding a poloidal field (of $\sim 10^{-4}$ to 1 G) into a toroidal field ~ 2 MG in $\sim 10^5$–10^9 yr. Small deviations from isorotation indicate presence of MHD perturbations whose latitude structure and time scales may be similar to those dominant in the solar cycle.

Subject headings: Sun: magnetic fields — Sun: rotation

1. INTRODUCTION

Helioseismology can be used in two ways to study the internal magnetic field of the Sun. In one approach, Dziembowski & Goode (1991) analyzed Libbrecht's (1989) data for the observed frequency splittings of the intermediate degree acoustic modes and derived the presence of a quadrupole toroidal field of $\sim 2 \pm 1$ MG just below the base of the convection zone. A purely toroidal field, however, is unlikely to be stable (Taylor 1980), or even in equilibrium (Spruit 1987), unless it is connected to, and replenished from a poloidal field (Kuhn 1988). The presence of a directly unobservable *steady* poloidal field of primordial origin is possible on theoretical grounds (Cowling 1953) and is also indicated indirectly by some observations (Piddington 1976). In this paper we first determine, theoretically, a likely configuration of the internal poloidal field that can remain in a "steady" state with the helioseismologically determined internal rotation of the Sun.

At any epoch the "steady" and the "fluctuating" parts of rotation and magnetic field must considered simultaneously. The fluctuations may be contributed, for example, by superposition of a large number of the Sun's global oscillations (for example, in this context, hydromagnetic oscillations), on a hierarchy of "dynamical" time scales. The "steady" parts of the rotation and the magnetic field may themselves vary on very long time scales (e.g., diffusion time scales), much longer than the period of the slowest mode of global oscillation. The latter, for example, may be of the order of decades. However, for determining the "steady" part of the magnetic field, in this paper we approximate it as if it is absolutely steady.

Analysis of sunspot data for the period 1874–1976 suggests that the underlying steady part of the Sun's magnetic field must be predominantly axisymmetric (Gokhale & Javaraiah 1990). Hence, according to Cowling's theorem, the resistive and the inductive terms in the induction equation for the steady part of the field must vanish separately. Thus the steady part of the internal field must be current free (except near the "boundaries") and must be in isorotation (Ferraro 1937) with the steady part of the rotation of the plasma. In concluding so, other plasma flows like meridional circulation, etc. are neglected if they exist. If the time-dependent part of the rotation is small, then the isorotation of the field with the steady part of the plasma rotation implies approximate isorotation with the helioseismologically "observed" rotation of the plasma. This requires existence of a functional relation between the "observed" rotation rate $\Omega(\boldsymbol{r})$ at a point $\boldsymbol{r}$ and the flux function $\phi(\boldsymbol{r})$ of the "steady" poloidal field linking through the circle of revolution of the point.

One may compute many models of an "isorotating" field at the epoch of observation by assuming $\phi(\boldsymbol{r})$ to be different arbitrary functions of the observed $\Omega(\boldsymbol{r})$. However, the models so computed will be only numerical, since $\Omega(\boldsymbol{r})$ is known only numerically. Moreover, none of the models so computed will represent the *real* steady field, since the *real* steady field must be isorotating only with the unknown "steady" part of helioseismologically "observed" internal rotation. Therefore it is necessary to model the "steady" part of the real field by determining the best combination of terms, and the best set of values for the coefficients in an appropriately chosen *analytical* expression for the magnetic field that isorotates with $\Omega(\boldsymbol{r})$.

In such a model, that part of the helioseismologically determined $\Omega(\boldsymbol{r})$, which can be best fitted to the isorotation law, represents the *steady* part of the rotation field. The residual part of $\Omega(\boldsymbol{r})$ then represents the fluctuating part of rotation.

In the present paper we choose, as a starting form, the simplest expression, viz., the multipole expansion of a potential field with odd north-south parity. The physical justification for this choice and the method of analysis are given in § 2.

We find (in § 3) that the observed rotation in the convective envelope ("CE"), can be fitted very well to a linear relation $\Omega(\boldsymbol{r}) = \Omega_0 + A\phi(\boldsymbol{r})$, but the best fit requires that the flux function $\phi(\boldsymbol{r})$ contains a term $\phi_C(\boldsymbol{r})$ representing "central" sources as well as a term $\phi_E(\boldsymbol{r})$ representing field from "external" sources as defined in § 4.1. At the present accuracies of rotation data, $\phi_C(\boldsymbol{r})$ corresponds to a combination of a dipole and a linear hexapole (both located at the center, parallel to the rotation axis), and $\phi_E(\boldsymbol{r})$ corresponds to an asymptotically uniform field B_0. The fit gives the strengths of the dipole and the hexapole, in terms of B_0 and the solar radius R_0. The strength B_0

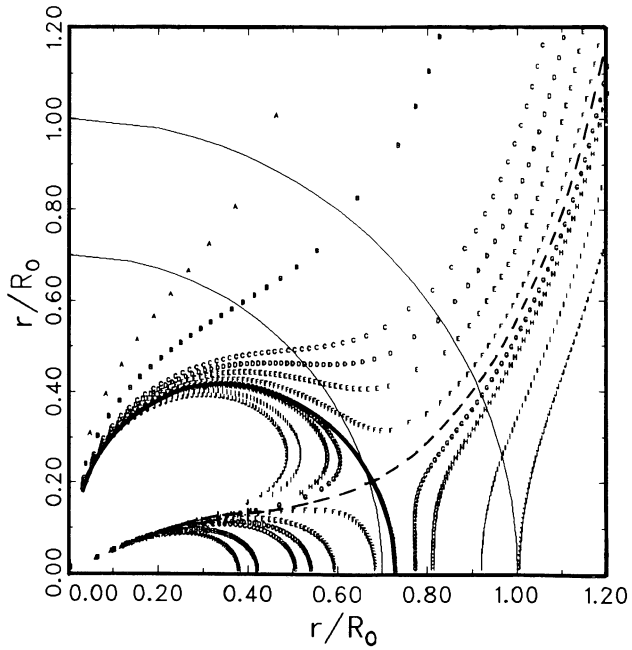

FIG. 1.—Structure of the "steady" part of the poloidal field in one quadrant of a meridional plane as defined by the combination of terms $l = 1$ and 3 in eq. (4) with the "best estimates": $\mu_1 = 0.624$ and $\mu_3 = 0.156$ (cf. eq. [7]). The field lines correspond to flux values A: 0.50, B: 1.00, C: 1.50, D: 1.60, E: 1.70, F: 1.80, G: 1.87, H: 1.90, I: 2.00, J: 2.10, in units of $\pi B_0 R_0^2$. The continuous and the broken lines represent branches of the separatrix S_* defined by $\phi = \phi_* =$ 1.84 units. The field directions can be marked relative to that of B_0, which is presently indeterminate. The field at the surface and that outside is subject to deformation by several processes into observationally unrecognizable patterns.

cannot be determined from this fit itself, but can be independently estimated to be in the range $\sim 10^{-4}$–1 G, as shown in § 4.2.

The resulting field structure (Fig. 1) (§ 3.2) has a closed "critical field line" $\phi = \phi_*$ running almost along the base of the convection zone in the "sunspot" latitudes.

The "observed" rotation in the outer radiative core "ORC" suggests that in the interior of the surface S_*, generated by the rotation of the critical field line at the rate Ω_* $(=\Omega_0 + A\phi_*)$, one may have $\Omega_0 \approx \Omega_*$ and $A \approx 0$.

Nevertheless, we show in § 4 that the observed rotation *just inside* S_* can also be expressed in the form of equation (9), where $(A_C - A_E)$ may be ~ 7 nHz per unit flux. In § 4 we show that this difference between A_C and A_E would correspond to a small deviation from isorotation near S_*. The resulting field winding might be adequate to yield the 2 MG toroidal field (varying with ϑ as $\sin^3 \vartheta \cos \vartheta$), near the base of the convection zone (cf. Dziembowski & Goode 1991), in $\sim 10^6$–10^9 yr, depending upon the value of B_0 (§ 4.2).

The fit over "ORC + CE" $(0.4 \leq r/R_0 \leq 1.0)$ also indicates the possibility of requiring, in the magnetic potential, higher order terms $(l \geq 5)$ with small strengths which are highly uncertain at present. We show in § 5 that if these terms are significant, then in spite of the large uncertainties in their strengths, they would imply the presence of time-dependent torsional MHD perturbations with time scales and main latitudinal structure similar to those of the solar magnetic cycle.

In § 6 we point out some interesting properties of the structure of the steady field and the possible role of the MHD fluctuations in (1) transport of energy away from the central regions and (2) transport of angular momentum and poloidal magnetic flux away from the rotation axis. We also discuss the difficulties in directly observing and determining the steady field, and suggest two possible methods to do so.

Finally we suggest that further study of the steady and the fluctuating internal magnetic fields will be simpler if (1) in the helioseismological determination of the frequency splittings, the rotation is assumed to have the same form as that of the flux function of an axisymmetric field with prescribed physical properties (e.g., a current-free field as in the present model, or, a field in the form of a solution of the magnetic diffusion equation), and (2) the observations are continued throughout the solar cycle.

2. DETERMINATION OF THE "STEADY" FIELD

2.1. *The Shapes of Isorotation Contours and the Sources for the "Steady" Poloidal Magnetic Field*

Christensen-Dalsgaard & Schou (1988), Dziembowski, Goode & Libbrecht (1989), and Sekii (1989) have analyzed the helioseismological data of Libbrecht (1989) in different ways. Except for the differences in the value and the gradient of rotation rate just below the base of the convection zone, the characteristics of internal rotation given by all the three investigations are qualitatively similar. In particular, all the three calculations yield considerable radial dependence of rotation rate along the polar axis so that the isorotation contours seem to intersect the rotation axis. If the contours really intersect the axis, then the radial variation of $\Omega(r)$ along the rotation axis would make it impossible to have the Ferraro's isorotation law satisfied in the close neighborhood of the axis. However, there are considerable uncertainties in the estimation of rotation rates near the Sun's center and near the rotation axis (Schou, Christensen-Dalsgaard, & Thomson 1992). Consequently, it is not ruled out that in reality the isorotation contours actually turn near the axis and converge "toward" the center, running *close* to the axis instead of *intersecting* it. This is equivalent to suggesting the presence of toroidal currents near the axis in the central region. On the other hand, the isorotation contours in "CE" are concave outward, suggesting that the large-scale steady field may have a contribution from "external" sources.

Thus, if there is a "steady" and axisymmetric poloidal field which (as required by Cowling's theorem) is current free and isorotating with the contemporary internal rotation of the Sun in "ORC" and "CE," then it must have internal sources (toroidal currents) concentrated near the axis and inside the inner radiative core ("IRC") as well as "external" sources at large distances.

2.2. *The Feasibility of the Existence of Isorotating, Approximately Current-free Poloidal Field with Central as Well as External Sources*

Toroidal currents of very high intensity must have been created in the Sun during its formation by gravitational collapse. These currents must have dissipated during the fully convective (Hayashi) phase of the Sun's subsequent evolution. However, as shown by Spitzer (1956), such a diffusion would also create and maintain electric currents near the axis and the surface. The "steady" parts of the presently surviving internal field and rotation must constitute the slowest decaying solution of the MHD equations with the end configuration of Hayashi phase as the initial configuration. In such a solution the magnetic field must be a solution of the induction equation

(Chandrasekhar 1956a):

$$\frac{\partial T}{\partial t} = \eta \Delta_5 T + \frac{1}{y}\left[\frac{\partial(\Omega, \phi)}{\partial(z, y)} - \frac{\partial(T, y^2 U)}{\partial(z, y)}\right],$$

where $T = y^{-1} B_T$, $y = r \sin \vartheta$, $z = r \cos \vartheta$, η is magnetic diffusivity,

$$\Delta_5 \equiv \left(\frac{\partial^2}{\partial y^2} + \frac{3}{y}\frac{\partial}{\partial y} + \frac{\partial^2}{\partial z^2}\right),$$

and U is a function defining meridional flow field in the same way as ϕ defines the poloidal field.

If in the post-Hayashi evolution the effects of diffusion and that of the "steady" part of the meridional circulation is small compared to that of rotation (see § 4.2), one must have

$$\frac{\partial(\Omega, \phi)}{\partial(z, y)} = 0 .$$

This is in fact Ferraro's law of isorotation in which all points on each field line rotate with the same angular velocity (which may differ from one field line to another). The slow evolution of the "steady" part will be given by the diffusion and meridional circulation. Thus the field lines maintain isorotation with the solar plasma even as the rotation evolves.

It follows from Spitzer's solution again that even in this evolution the currents will be maintained near the axis and near the surface. Whatever currents slowly diffuse into "ORC" and "CE" would decay fast owing to the MHD turbulence (i.e., fluctuations on dynamical time scales) existing there. The currents diffusing into "CE" will also decay by convective turbulence and will be carried away by effects of magnetic buoyancy and coronal expansion.

On these grounds we assume, as a first step, that the large-scale "steady" field is current-free in "ORC" and "CE" and has "central" as well as "external" sources.

There is also the following a posteriori verification for this approximation. The photospheric field structure given by this approximation is almost same as that of a unique superposition of the first two characteristic solutions of the magnetic diffusion equation (Chandrasekhar 1956b), with appropriate parity (see Appendix).

2.3. *The Chosen Form of the Relation between $\phi(r)$ and $\Omega(r)$*

For the relation between $\phi(r)$ and $\Omega(r)$ we assume the linear relation

$$\Omega(r, \vartheta) = \Omega_0 + A\phi(r, \vartheta) , \quad (1)$$

expecting it to serve as the first approximation to any nonlinear relation that might exist.

Here $r = |\boldsymbol{r}|$, ϑ is the colatitude, and Ω_0 and A are constants.

2.4. *The "Data" Used*

Among the aforementioned helioseismological computations of the internal rotation $\Omega(r)$, the one by Christensen-Dalsgaard & Schou yields the smallest uncertainties. We do not find in their paper general expressions which could be used for determining $\Omega(r, \vartheta)$ along directions other than $\vartheta = 0°$, 45° and 90°. Hence, we use the expressions given by Dziembowski et al. to determine the rotation rates at selected sets of points (r, ϑ) and (since the two rotation models are generally similar) take the resulting values of $\Omega(r, \vartheta)$ as if obtained by the method of Christensen-Dalsgaard & Schou, viz., with the correspondingly small uncertainties.

2.5. *The Method of Analysis*

Assuming the current free poloidal field to be symmetric about the axis of rotation, and of odd north-south parity, we write it as

$$\boldsymbol{B} = -\nabla V , \quad (2)$$

where

$$V = V_C(r, \vartheta) + V_E(r, \vartheta) ,$$

wherein

$$V_C(r, \vartheta) = \sum_{l=1} M_l r^{-(l+1)} P_l(\mu) ,$$

$$V_E(r, \vartheta) = -B_0 r P_1(\mu) + \sum_{l=3} M_l r^l P_l(\mu) ,$$

are the magnetic potentials due to the central and the external sources, $\mu = \cos \vartheta$, $P_l(\mu)$ are Legendre polynomials, and the summations are taken only over the odd integral values of l.

Since we expect the *long-lived* external currents to be at very large distances, we expect their field in the local neighborhood of the Sun to be uniform. Hence we keep only the first term in $V_E(r, \vartheta)$ and remove the terms $l \geq 3$. This has been justified a posteriori by the fact that inclusion of terms $l > 1$ or omission of the term $l = 1$ in V_E deteriorates the goodness of the least-squares fit.

Thus we have

$$V(r, \vartheta) = (-B_0 r + M_1 r^{-2}) P_1(\mu) + M_3 r^{-4} P_3(\mu) + M_5 r^{-6} P_5(\mu) + \cdots . \quad (3)$$

This gives the following expression for the magnetic flux function

$$\phi(r, \vartheta) = \pi B_0 R_0^2 [(x^2 + 2\mu_1 x^{-1} + 4\mu_3 x^{-3} + \cdots) \sin^2 \vartheta + (-5\mu_3 x^{-3} + \cdots) \sin^4 \vartheta + \cdots] , \quad (4)$$

where $x = r/R_0$ and $\mu_l = M_l/(B_0 R_0^{l+2})$, $l = 1, 3, 5, \ldots$.

We write equation (1) in terms of the "normalized dimensionless rotation rate":

$$\omega(r_i, \vartheta_i) = [\Omega(r_i, \vartheta_i) - \hat{\Omega}_{\rm obs}]/\sigma_{\Omega \rm obs} = a_0 + a_1 \phi(r_i, \vartheta_i) , \quad (5)$$

where $a_0 = (\Omega_0 - \hat{\Omega}_{\rm obs})/\sigma_{\Omega \rm obs}$ and $a_1 = A/\sigma_{\Omega \rm obs}$, $\hat{\Omega}_{\rm obs}$, and $\sigma_{\Omega \rm obs}$ are the mean and the standard deviation of the whole set of the observationally determined values $\Omega(r_i, \vartheta_i)$ at the chosen set of points (r_i, ϑ_i). Equation (5) can also be written as

$$\Omega = \Omega_0 + \Omega_1 [(x^2 + 2\mu_1 x^{-1} + 4\mu_3 x^{-3} + \cdots) \sin^2 \vartheta + (-5\mu_3 x^{-3} + \cdots) \sin^4 \vartheta + \cdots] , \quad (6)$$

where $\Omega_1 = \pi a_1 B_0 R_0^2 \sigma_{\Omega \rm obs}$.

We determine the coefficients Ω_0, Ω_1, μ_1, μ_3, etc., by obtaining weighted least-squares-fits for successive combinations of terms in equation (6) using the following three sets of data points:

TABLE 1

RESULTS OF LEAST-SQUARES FITS OF SUCCESSIVE COMBINATIONS OF TERMS IN EQUATION (6) TO THE THREE DATA SETS

Terms Taken (1)	Ω_0 (nHz) (2)	Ω_1 (nHz) (3)	μ_l (4)	$\Delta\mu$ (5)	$\frac{\mu_3}{\mu_1}$ (6)	χ^2 (7)	Goodness of Fit (8)
Set "CE" ($0.7 \le x \le 1.0$, 99 points)							
P_1, P_3	326	68	$\mu_1 = 0.624$	17%	0.250	17.9	100%[a]
	± 12	± 11	$\mu_3 = 0.156$	30			
P_1, P_3, and P_5	321	57	$\mu_1 = 0.905$	23	0.251	19.0	100[a]
	± 13	± 2	$\mu_3 = 0.227$	26			
			$\mu_5 = 0.001$	...			
Set "ORC + CE" ($0.4 \le x \le 1.0$, 189 points)							
P_1, P_3	332	73	$\mu_1 = 0.490$	31	0.229	156	96
	± 8	± 15	$\mu_3 = 0.112$	22			
P_1, P_3, and P_5	322	64	$\mu_1 = 0.727$	15			
	± 10	± 2	$\mu_3 = 0.187$	16	0.257	143	99
			$\mu_5 = 0.0003$	...			
Set "ORC" ($0.4 \le x \le 0.7$, 72 points)							
P_1, P_3, and P_5	426	− 10	$\mu_1 = -0.349$	...			
	± 150	± 5	$\mu_3 = -0.0129$	...	0.037	57	89
			$\mu_5 = -0.0001$	...			

[a] χ^2: probability $< 10^{-7}$. For each data set, the combinations lower than and higher than those given here yield less satisfactory or unsatisfactory fits. In col. (5), uncertainties exceeding 100% are not given.

"CE": 99 points in the convective envelope ($0.7 \le x \le 1.0$), consisting of 11 equispaced points along each of the directions $\vartheta = 10°, 20°, \ldots 90°$,

"ORC": a set of 72 points in the outer radiative core ($0.4 \le x \le 0.7$) consisting of eight equispaced points along each of the directions just mentioned, and

"ORC + CE": a set of 189 points over the whole range ($0.4 \le x \le 1.0$).

The "weights" have been assigned proportional to the reciprocals of the uncertainties as read off from Christensen-Dalsgaard & Schou (1988).

Values of the coefficients in equation (6) obtained from the least-squares fits for a_0 and a_1 in equation (5), by taking various combinations of terms in equation (3), are given in Table 1, along with the χ^2 probabilities.

2.6. *Estimation of the Goodness of Fit*

For determining the contribution from each point (r_i, ϑ_i) to the value of χ^2 the "observed normalized values" $\omega(i) = \omega(r_i, \vartheta_i)$ must be considered as if obtained from independent experiments. In reality, this condition is not satisfied. However here we use the χ^2 values merely for the purpose of comparing the *relative goodness* of various least-squares fits.

Further, in order that each $\omega(i)$ is a "normal variate" distributed about the corresponding theoretical value $\omega_T(i)$, it is necessary that for each i, $\omega(i)$ and $\omega_T(i)$ are both measured in units of the standard deviation $\sigma_\omega(i)$ of the "distribution" of $\omega(i)$ about $\omega_T(i)$. However, for each i, $|\omega(i) - \omega_T(i)|$ is itself a measure of $\sigma_\omega(i)$. Hence we have

$$\chi^2 = \sum_i |\omega(i) - \omega_T(i)|/\omega_T(i) .$$

We have verified that at least in "CE" the values of the least-squares difference *also* give the same conclusions about the relative goodness of the fits as given by the χ^2 values.

3. THE RESULTS

3.1. *Least-Squares Fit for the Convection Zone*

From the values of χ^2 in Table 1 it is clear that in the convective envelope ("CE") the best fit for equation (6) is given by the combination $l = 1$ and 3 (the dipole and the linear hexapole terms) whose strengths in terms of B_0 are given by

$$M_1 = (0.624 \pm 0.16)B_0 R_0^3 \quad \text{and} \quad M_3 = (0.156 \pm 0.046)B_0 R_0^5 . \tag{7}$$

This corresponds to a total "flux" across the solar hemisphere:

$$\phi(R_0, \pi/2) \approx 2.09\pi B_0 R_0^2 ,$$

which gives the following upper limit on B_0:

$$B_0 \le 1 \text{ G} ,$$

since the total observed magnetic flux on the photosphere does not exceed 3×10^{22} Mx (Howard 1974).

3.2. *The Geometrical Structure of the Field*

The field lines of the field given by equations (4) and (5) in the range $0.7 \le r/R_0 \le 1.0$ are plotted in Figure 1. These simulate, quite satisfactorily, the pattern of the isorotation lines given by helioseismology (Libbrecht 1988), except very near the axis, where the uncertainties in the "observed" rotation rates are large.

The field structure also contains a "separatrix" which has two branches (shown by the thick and dashed lines) intersecting at the *neutral* point (r_*, ϑ_*) where

$$r_* = 0.71R_0 \quad \text{and} \quad \vartheta_* = 73° .$$

The revolution of the branch shown by the thick line defines a closed "critical surface," running close to the base of the convection zone in latitudes $\le 45°$.

At S_*, the flux function $\phi(r, \vartheta)$ and the rotation rate $\Omega(r, \vartheta)$ are given by equation (6), along with values of Ω_0, Ω_1 in the best fit in "CE", as

$$\phi_* = \phi(r_*, \vartheta_*) \approx 1.84(\pi B_0 R_0^2) ,$$

and

$$\Omega_* = \Omega(r_*, \vartheta_*) \approx 450 \text{ nHz} .$$

The helioseismological studies cited earlier (Christensen-Dalsgaard & Schou 1988; Dziembowski et al. 1989; Sekii 1989) suggest that the Sun's rotation undergoes a transition from an approximately rigid rotation within "ORC" to a differential rotation within "CE". Within "ORC" the small radial variations in the equatorial plane seem to be time dependent (Goode & Dziembowski 1991), so that the *steady* part of equatorial rotation may be uniform.

4. A SLOW CREATION OF TOROIDAL MAGNETIC FIELD NEAR S_*

4.1. *The Difference between the Values of A_C and A_E*

All the three helioseismological models of rotation referred in § 2.1 suggest that the rotation in the radiative core is more or less rigid. In view of the large uncertainties in the rotation rates within "ORC" near the axis, it is not ruled out that the transition from a "rigid" rotation to the differential rotation occurs at S_* *instead of* occurring at the base of the convection zone. In fact, future improved models of the field and the rotation might reduce the difference between S_* and the base of the convection zone.

Thus we suggest that the isorotation in the steady state takes the form

$$(\Omega - \Omega_*) = A(\phi - \phi_*) , \tag{8}$$

where "A" changes to some *small* value inside S_* from ~ 68 nHz per unit flux outside S_*. This implies a "discontinuity" in $d\Omega/d\phi$ without a discontinuity in Ω.

From the results of the best fits in "CE" and "ORC + CE", it *appears* that neither Ω_0 nor A changes significantly across S_*. However, from "CE" to "ORC + CE" the estimates of both μ_1 and μ_3 increase significantly by the same ratio (viz., 7/6). Such a change is not physically meaningful. Mathematically, it must be the effect of a change only in the *coefficient* of the terms in μ_1, $\mu_3 \ldots$, i.e., the coefficient of the terms contributed by the "internal" sources. Thus it seems necessary to rewrite equation (6), separating the terms contributed by the "internal" and "external" sources of the field as

$$\Omega(r, \vartheta) = \Omega_0 + A_E \phi_E + A_C \phi_C , \tag{9}$$

where

$$\phi_E = \pi B_0 R_0^2 (x^2 \sin^2 \vartheta) \tag{10}$$

and

$$\phi_C = (\pi B_0 R_0^2)(2\mu_1 x^{-1} \sin^2 \vartheta + \cdots) \tag{11}$$

of $\phi(x, \vartheta)$ correspond to the parts $V_E(r, \vartheta)$ and $V_C(r, \vartheta)$ of the magnetic potential $V(r, \vartheta)$ defined in § 2.5. The apparent increase in μ_1, μ_3 can then be understood as a change in A_C alone, by factor $\sim 7/6$, from "CE" to "ORC + CE." (In fact, it is this jump in A_C which appears in Table 1 as a jump in μ_1, μ_3 owing to the combined effect of the following facts: [1] μ_1, μ_3 were determined only through the products $A_C \mu_1$, $A_C \mu_3$, [2] A_C was not distinguished from A_E; and [3] A_E could change only from 68 to 64 nHz per unit flux from "CE" to "ORC + CE.")

In view of the high relative accuracy of the data in "CE," we assume that the true values of μ_1, μ_3 are as given by the ("best") fit in "CE." Since the values of μ_1, μ_3 must be continuous across S_*, the value of A_C in "ORC + CE" must be $\sim 7/6$ times that given by the corresponding best fit in Table 1. Thus, in "ORC + CE," $A_C \sim 75$ nHz per unit flux giving $(A_C - A_E) \sim 7$ nHz per unit flux.

4.2. *Slow Buildup of Toroidal Field near S_*: Its Form and Time Scale*

Equation (9) can be written as

$$(\Omega - \Omega_*) = A_C(\phi - \phi_*) + (A_E - A_C)(\phi_E - \phi_E^*) .$$

The differences between A_C and A_E in the last term implies a slow winding of the "steady" poloidal field into a toroidal field (B_T) in the neighborhood of S_*. This follows from the law of induction in MHD which takes the following form as a corollary of the form given by Chandrasekhar (1956a) for axisymmetric fields (when the relatively small effects of diffusion and meridional circulation are neglected):

$$\frac{\partial B_T}{\partial t} = \frac{\partial(\Omega, \phi)}{\partial(z, \rho)} = \frac{\partial \Omega}{\partial z}\frac{\partial \phi}{\partial \rho} - \frac{\partial \Omega}{\partial \rho}\frac{\partial \phi}{\partial z} ,$$

where $z = xR_0 \cos \vartheta$ and $\rho = xR_0 \sin \vartheta$.

In the neighborhood of S_* this reduces to:

$$\frac{\partial}{\partial t}(B_T) \simeq (A_C - A_E)\frac{\partial(\phi_C, \phi_E)}{\partial(z, \rho)}$$
$$\simeq -12\pi^2 B_0^2 R_0^2 (A_C - A_E)\mu_1 \sin^3 \vartheta \cos \vartheta$$

Taking $(A_C - A_E) = 7$ nHz per unit flux, we have

$$\frac{\partial}{\partial t}\left(\frac{B_T}{B_0}\right) \simeq (50\pi) \sin^3 \vartheta \cos \vartheta \text{ nHz} . \tag{12}$$

The maximum estimate of this is $\sim 50\pi$ nHz.

Consequently, the time scale of creation of the toroidal field is

$$\tau_{\text{tor}} \simeq (10^9/50\pi)(B_T/B_0) \text{ s} .$$

If $B_T \sim 2 \times 10^6$ G as derived by Goode & Dziembowski (1991), then, since $B_0 < 1$ G,

$$\tau_{\text{tor}} > 3 \times 10^5 \text{ yr} .$$

For the meridional circulation effects to be negligible in the present stage of the Sun's MHD evolution, as assumed by us, τ_{tor} must be $\ll \tau_{\text{mc}}$ (time scale of meridional circulation). If $\tau_{\text{mc}} \sim 10^9$ yr (e.g., Zappala 1972), this requires $B_0 > 3 \times 10^{-4}$ G.

Finally for τ_{tor} to be less than the age of the Sun (4.5×10^9 yr), one must have

$$B_0 > 10^{-4} \text{ G} .$$

This gives the *lower* limit on the order of magnitude of B_0.

It is interesting to note that in the low latitudes (which are significant for observations), the leading term in the ϑ-dependence of the toroidal field created near S_* is

"sin ϑ cos ϑ," i.e., same as that deduced by Dziembowski & Goode (1991).

5. TIME-DEPENDENT TORSIONAL MHD PERTURBATIONS

5.1. *Likely Presence of Perturbations and its Interpretations*

We see from Table 1 that in the radiative core "ORC," the inclusion of the term $l = 5$ is necessary for obtaining even a satisfactory fit. Since this fit involves large uncertainties, we study the results for the set "ORC + CE." For this set, the inclusion of the term $l = 5$ improves the goodness of fit from 96% to 99%, by lowering the χ^2. Since $\Omega(r_i, \vartheta_i)$ cannot be considered as independent random variables, we are unable to estimate the significance of this lowering of χ^2. However, we believe that the best fit in "ORC + CE" will require terms $l > 5$, since inclusion of the term $l = 5$ *substantially reduces* the percentage of uncertainties in μ_1 and μ_3.

One may wonder how the helioseismologically determined $\Omega(r, \vartheta)$, which is truncated at $\sin^4 \vartheta$, can show improvement in the goodness of fit with inclusion of the term $l = 5$ which is equivalent to extending the fitting formula up to $\sin^6 \vartheta$. However, the inclusion of the term $l = 5$ also implies inclusion of terms in x^{-5} in the coefficients of $\sin^2 \vartheta$ and $\sin^4 \vartheta$. This explains how the fit improves and indicates that at the time of the helioseismological observations there must be (over and above the rotation field represented by the terms $l = 1$ and 3) a "residual rotation" $\delta\Omega(x, \vartheta)$ varying as x^{-5}. The substantial reduction in the uncertainties in μ_1, μ_3 indicates that the "residual rotation" is not an artifact of noise or errors.

If the isorotating magnetic field did not contain a residual term $\delta \boldsymbol{B}(x, \vartheta)$, then $\delta\Omega$ would be time dependent. Even in case the $\delta \boldsymbol{B}$ required for the steady state exists in "ORC," the inclusion of the corresponding term in $\phi(x, \vartheta)$ deteriorates the fit in "CE" (cf. Table 1), implying nonisorotation in "CE." Thus, either the residual rotation in "ORC," or the field structure in "CE," or both would be nonsteady. Thus, a *time-dependent* torsional MHD perturbation seems to exist in the region "ORC + CE." A very rough estimate of the amplitude of the "$l = 5$" component in the $\delta\Omega$ (based on the highly uncertain value of μ_5 in Table 1) is $\sim \Omega_1 \mu_5/\mu_1$ i.e., ~ 0.07 nHz. (The variation of ~ 30 nHz in the equatorial rotation rate at $x = 0.4$ derived observationally by Goode & Dziembowski [1991] may be the rms fluctuation caused by superposition of $\sim 10^5$ such harmonic terms in MHD perturbations. The presence of perturbations must be contributing substantially to the uncertainties in the determination of μ_1, μ_3, etc.)

5.2. *Time Scales of Variations of MHD Modes in the Perturbations*

The "frequencies" (ν) of the various MHD modes in these perturbations will be given (along with the other MHD equations) by the induction equation: $\nu \sim |\text{curl}\ (\boldsymbol{V}_0 \times \delta \boldsymbol{B}_P)| / |\delta \boldsymbol{B}_T|$, where $\boldsymbol{V}_0$ is the "zero-order" flow defined by the approximately "steady" part of the rotation. The order of magnitude of the operator curl $(\boldsymbol{V}_0 \times \ldots)$ has already been estimated in equation (12) to be ~ 150 nHz. This can give the periods of the modes in the range ~ 2–20 yr, if $\delta B_T/\delta B_P$ is in the range ~ 10–100. (Similar orders of magnitude for the periods of MHD perturbations have been suggested by Layzer, Rosner, & Doyle [1979] using a different approach.)

Thus, it is not ruled out that the MHD modes in the perturbations detected in § 5.1 have time scales comparable to those of solar activity, including the "period" of the solar cycle.

5.3. *A Note on the Latitudinal Structure of the Perturbations*

The uncertainties $\Delta\mu_1$, $\Delta\mu_3$ in μ_1, μ_3 in the "steady" part of the field may be expected to be comparable to the amplitudes δb_1, δb_3 of the terms $l = 1$ and $l = 3$ in the "fluctuating" part δB of the real field. If δb_1 and/or δb_3 were $\geq \delta b_5$ (the amplitude of the term $l = 5$), then inclusion of the term "$l = 5$" in the fit could not have reduced $\Delta\mu_1$, $\Delta\mu_3$ in "ORC + CE" as seen in Table 1. This suggests that the term $l = 5$ may be dominant in the time-dependent part of the field. It is significant in this context that the terms $l = 5$ and 7 are already known to be dominant in the "22 yr cycle" of the photospheric magnetic field (Stenflo 1988) and of the internal field inferred from the sunspot data (Gokhale et al. 1992).

6. CONCLUSIONS AND DISCUSSION

We have shown that a large-scale magnetic field which can remain in steady state with the contemporary internal rotation of the Sun can be expressed in terms of a potential given by the terms $l = 1$ and $l = 3$ in equation (3).

Here the linear relation in equation (1) serves as a universal first approximation to all nonlinear relations. Our attempts (not described here) to replace equation (1) by an exponential relation between Ω and ϕ have shown that such a relation cannot give any good fit. The goodness of fit for the linear relation obtained here shows that nonlinear terms are not needed at least with the existing uncertainties in the determination of Ω.

The best estimates of the strengths of the terms $l = 1$ and $l = 3$ obtained from "CE," where the uncertainties in Ω are smallest, are given in equation (7).

The field structure shows (see Fig. 1) *distinctly different patterns* in the *high-* ($\geq 30°$) and *low-* ($< 30°$) latitude zones, which can provide a natural framework for the remarkably different behaviors of the surface fields observed in these zones.

The structure also shows a clear separation of a part of the flux of the "steady" field "trapped" near the Sun's center and the other part of the flux which seems to be in the process of diffusing out of the Sun, across the convection zone. One may wonder if this separation also defines "in principle" the separation of the zero-order flux connected to "interstellar" field and that constituting the "interplanetary" part of the Sun's large-scale field.

Our analysis also indicates possible existence of a time-dependent torsional MHD perturbation and the possibility that the time scales of the modes in this perturbation may turn out to be comparable to those in the solar magnetic cycle. It is interesting that the most likely dominant term in this perturbation is $l = 5$, which is also the dominant term in the solar magnetic cycle.

The "zero-order" ("steady") field contributed by the internal sources falls off with negative powers of the radial distance, whereas the first order perturbations, if originating as MHD waves in the central regions, will *grow* as they approach the surface. Additional amplification of the perturbed field may be provided by processes in the convection zone. Moreover, nonaxisymmetry of the perturbations, formation of flux tubes and eruption of toroidal flux by magnetic buoyancy will cause considerable noise at the surface. Consequently, it will be very difficult to detect the "steady" field at and above the surface.

It is therefore difficult also to determine the value of B_0. One may detect the steady field in long-term space observations at high heliographic latitudes. It might also be detected and

determined as "d.c." components in the spherical harmonic Fourier analysis of a long enough time series of the magnetograms, very much longer than 22 yr. A preliminary indication of such a field is already seen in an analysis of magnetogram data by J. O. Stenflo (1992, private communication).

The MHD perturbations convert rotational energy into magnetic energy and, together with the other "activity"-generating processes such as flux tube formations, magnetic buoyancy, etc., they can provide a nonradiative mechanism of energy transport away from the central regions on MHD time scales. The superposition of these perturbations can also provide diffusion of magnetic flux and angular momentum away from the axis, on diffusion time scales. On the same time scales, the "discontinuity" in the isorotation law at S_* provides a mechanism for a slow conversion of the "steady" poloidal flux into toroidal flux and also a mechanism of converting rotational energy into magnetic energy.

According to the Rayleigh criterion (Chandrasekhar 1961), the first two terms on the right-hand side of equation (9), and the dipole term in the field of internal sources, represent stable rotation. The next ($l = 3$) term indicates the presence of instability. The time scale and the length scale of this instability will have to be determined by detailed calculations including effects of gravity, etc. It will be interesting to see if this instability provides the basic excitation of the above mentioned MHD perturbations.

Helioseismological determination of rotation, in some physically significant form, such as equation (6), will be useful for studying the interaction of rotation and magnetic field inside the Sun. Such determination in all phases of the solar magnetic cycle will throw more light on the properties of the steady as well as the varying components of the Sun's internal field.

Clearly, for a given rotation field, the structure of the isorotating magnetic field is not unique, but will depend on the assumptions underlying the chosen mathematical form of the magnetic flux function. In the present model the uniqueness is obtained by assumptions given in § 2. One may replace the present assumptions by others that one may believe to be more realistic. However, the real test of any model will depend upon how well it agrees with properties of the real steady field when it will be possible to determine the latter observationally. Till then, for reasons given in § 2, and on the basis of the interesting properties discussed above, we believe that our present model can serve as a fairly good first approximation to the real steady field of the Sun.

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APPENDIX

AGREEMENT BETWEEN THE PHOTOSPHERIC FIELD DISTRIBUTION IN THE PRESENT MODEL AND THAT IN A SOLUTION OF THE DIFFUSION EQUATION FOR MAGNETIC FIELDS

Here we show that the values of μ_1 and μ_3 obtained by us in § 3 are such that $\phi(1, \vartheta)$ can be expressed as

$$\phi(1, \vartheta) \equiv \psi_0(1, \vartheta) + \Lambda\psi_2(1, \vartheta) \tag{A1}$$

where ψ_0 and ψ_2 are the first two even-parity characteristic solutions of the diffusion equation given by Chandrasekhar (1956b), and Λ is a constant.

Our model (eqs. [4] and [7]) gives

$$\phi(1, \vartheta) = \alpha_0 + \alpha_2(\cos^2 \vartheta) + \alpha_4(\cos^4 \vartheta) , \tag{A2}$$

where $\alpha_0 = (1 + 2\mu_1 - \mu_3) = 2.092$, $\alpha_2 = -(1 + 2\mu_1 - 6\mu_3) = -1.312$, and $\alpha_4 = -5\mu_3 = -0.78$.

Using the definition of Gegenbauer polynomials, which occur in Chandrasekhar's solution, equation (A1) can be written as

$$\phi(1, \vartheta) \equiv (1 - 3\Lambda/2) + (-1 + 9\Lambda) \cos^2 \vartheta + (-15\Lambda/2) \cos^4 \vartheta . \tag{A3}$$

Equations (A2) and (A3) together require that

$$\frac{-15\Lambda/2}{9\Lambda - 1} = \frac{\alpha_4}{\alpha_2} = 0.5945 \tag{A4}$$

and

$$\frac{9\Lambda - 1}{1 - 3\Lambda/2} = \frac{\alpha_2}{\alpha_1} = -0.6272 . \tag{A5}$$

It is important to note that equations (A4) and (A5) yield almost identical values of Λ, viz., $\Lambda = 0.0462628$ and 0.0462571, respectively.

Thus for the values of μ_1 and μ_3 obtained using assumptions in § 2.2, a unique value of Λ exists such that on the photosphere the ϑ dependence of the flux function $\phi(1, \vartheta)$ can be written in the form (A1).

This shows that near the photosphere the steady field obtained by assuming a current-free form is almost same as that obtainable by assuming it to be in the form of a solution of the diffusion equation with appropriate parity.

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