THE ASTROPHYSICAL JOURNAL, 448:437–443, 1995 July 20

"STEADY" AND "FLUCTUATING" PARTS OF THE SUN'S INTERNAL MAGNETIC FIELD: IMPROVED MODEL

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Received 1993 November 16; accepted 1995 January 30

ABSTRACT

Using reasonable assumptions and approximations it is shown that the poloidal component of the "steady" part of the magnetic field in the Sun's radiative core (RC) and convection zone (CE) can be modeled as an analytical solution of the equation for magnetic diffusion in an incompressible medium of constant diffusivity, which is subject to (1) continuity of the normal component across the RC-CE boundary and (2) merging with an asymptotically uniform field of finite strength at large distances, and whose field lines isorotate with the solar plasma. The last requirement enables determination of the values of the parameters in the first two eigenmodes of the diffusion equation.

The resulting model does not have any singularity, separatrix, or closed loop, and yet it yields a much better fit with the helioseismologically determined isorotation contours than the fit given by the earlier model (Gokhale & Hiremath 1993).

The ratio of the range of the travel times of Alfvén waves along the field lines in this model, to their mean value, is comparable to the relative range of the periods of the sunspot cycle. For example, it is 9.5–12.5 yr if $B_0 \sim 0.02$ G.

The model enables us to estimate the "initial" (at zero-age main sequence) relative strengths of the two diffusion eigenmodes as 4:1. The characteristic diffusion timescales of these modes are estimated to be ~ 10.6 and $\sim 2.7 \times 10^9$ yr, respectively.

The model is consistent with (1) nonisorotation in the neighborhood of the RC-CE boundary which may lead to build-up of a strong (e.g., ~ 1 MG) toroidal field on timescales $\sim 10^7$–10^8 yr, and (2) the presence of a torsional MHD perturbation, with the dominant scale of latitudinal variation in CE and the scale of temporal variation, both comparable to the observed scales of the solar magnetic cycle.

Subject headings: diffusion — Sun: magnetic fields — Sun: rotation — waves

1. INTRODUCTION

Theoretically it is possible that the Sun's interior has retained some of its large-scale (global) magnetic field from its protostar phase even after the Hayashi phase (Cowling 1953; Layzer, Rosner, & Doyle 1979; Piddington 1983; Mestel & Weiss 1987; Spruit 1990). This part of the global field (viz., of promordial origin) is expected to vary with time very slowly, i.e., on timescales much longer than the travel times of the slow MHD waves across the Sun.

Recently, we have modeled (Gokhale & Hiremath 1993, hereafter Paper I) this slowly varying ("steady") part of the magnetic field in the Sun's outer radiative core (ORC) and convective envelope (CE), assuming it to be a "current-free" field whose field lines isorotate according to the Sun's internal rotation as determined helioseismologically (Dziembowski, Goode, & Libbrecht 1989). It is true, e.g., for reasons given in Paper I, that the steady field would be current free to a first approximation in the solar interior except near the rotation axis. However, this approximation can be really good only in CE, where buoyancy and convection can lead to fast disposal of the currents but not so in ORC. In fact in the preliminary model (Paper I) we did find it impossible to obtain a good fit for a current-free field to be isorotating with the plasma in the ORC. Moreover, an inward extrapolation of a current-free field model would always lead to multipole type singularities near the center (e.g., dipole and hexapole in the preliminary model), unless the field is trivially uniform everywhere. It is therefore necessary to improve the "preliminary" model by assuming, for the field in ORC and CE, a form of some simple solution of the basic equations. In this paper we present such an "improved" model.

In § 2 we show on the basis of reasonable assumptions that in the first approximation the "steady" part of the poloidal magnetic field in the Sun must be a solution of the equation of magnetic diffusion in a sphere of constant diffusivity and its flux function $\Phi(r, \vartheta)$ must have a functional relation with the angular velocity $\Omega(r, \varphi)$ of rotation of the plasma.

In § 3 we modify Chandrasekhar's solution of magnetic diffusion in a sphere of constant diffusivity (surrounded by a medium of very large diffusivity) to incorporate the boundary condition imposed by the presence of a locally uniform interstellar field. This solution gives us mathematical forms Φ_{RC} and Φ_{CE} for the flux function in RC and CE.

In § 4 we express the forms, Ω^{RC} and Ω^{CE} for $\Omega(r, \vartheta)$ in RC and CE, assuming the relation between Ω and Φ to be linear. The coefficients in this relation and the relative values of the coefficients in Φ_{RC} and Φ_{CE} are then determined from least-squares fits of Ω^{RC} and Ω^{CE} to the values of Ω in RC and CE given by helioseismology.

We find (§ 5) that it is possible to obtain excellent simultaneous least-squares fits for the fields, in both ORC and CE, in their respective assumed forms, satisfying the continuity of the radial component at the common boundary, such that the field lines in each region isorotate with the plasma rotation given by helioseismology. The resulting field geometry is much simpler. Unlike the preliminary model, it is free from any singularity, separatrix, or closed loops.

The present model yields Alfvén travel times along the different field lines in the range ~9.5–12.5 yr, if $B_0 \sim 0.02$ G (which may be important for oscillatory models of the solar cycle).

The model enables us to determine not only the relative magnitudes of the first two eigenmodes of diffusion in ORC but also their characteristic timescales in terms of the magnetic diffusivity. This information can be used to obtain a crude estimate of the "initial" (ZAMS) relative strengths of the two diffusion eigenmodes.

We also find (§ 5) that the conclusions of the preliminary model about the presence of a nonisorotation near the ORC-CE boundary and of an MHD perturbation (with the scale of latitudinal variation in CE and the scale of temporal variation, both similar to those of the solar cycle) are valid even in the improved model (§ 6).

In § 7 we summarize the important features of the model and discuss their implications.

2. ASSUMPTIONS AND THE RESULTING FORM OF THE BASIC EQUATIONS

We assume that the magnetic fields and the fluid motions are symmetric about the Sun's rotation axis and note that on the relevant timescales the fluid motions can be considered as incompressible. For simplicity the magnetic diffusivity η is assumed to be uniform and constant with a value represented by a suitable average. However, the convection and the magnetic buoyancy operating in CE leads to fast disposal of the electric currents. Hence the effective average value $\eta_{\rm CE}$ of η in the convective envelope must be *several orders of magnitude higher* than the average value $\eta_{\rm RC}$ in the radiative core.

Following Chandrasekhar (1956a), the magnetic field $\boldsymbol{B}$ and the velocity field $\boldsymbol{v}$ can be expressed as

$$\boldsymbol{h} = -\zeta \frac{\partial P}{\partial z} \mathbf{1}_\zeta + (\zeta T)\mathbf{1}_\phi + \zeta^{-1} \frac{\partial(\zeta^2 P)}{\partial \zeta} \mathbf{1}_z \qquad (1)$$

$$\boldsymbol{v} = -\zeta \frac{\partial U}{\partial z} \mathbf{1}_\zeta + (\zeta \Omega)\mathbf{1}_\phi + \zeta^{-1} \frac{\partial(\zeta^2 U)}{\partial \zeta} \mathbf{1}_z \qquad (2)$$

where $\boldsymbol{h} = \boldsymbol{B}/(4\pi\rho)^{1/2}$; ρ is the density; ζ, ϕ, and z are the cylindrical polar coordinates, with their axis along the axis of solar rotation; $\mathbf{1}_\zeta$, $\mathbf{1}_\phi$, and $\mathbf{1}_z$ are the corresponding unit vectors; and T, P, Ω, and U are the scalar functions which are independent of ϕ.

For the reasons given in Paper I, we neglect the meridional motions and write Chandrasekhar's MHD equations in the form

$$\eta \Delta_5 P - \frac{\partial P}{\partial t} = 0 , \qquad (3)$$

$$\eta \Delta_5 T - \frac{\partial T}{\partial t} = -\zeta^{-1}[\Omega, \zeta^2 P] \qquad (4)$$

$$\frac{\partial \Omega}{\partial t} = \zeta^{-3}[\zeta^2 T, \zeta^2 P] \qquad (5)$$

$$[\Delta_5 P, \zeta^2 P] = \zeta \frac{\partial}{\partial z}(T^2 - \Omega^2) \qquad (6)$$

where

$$[f, g] = \frac{\partial f}{\partial z}\frac{\partial g}{\partial \zeta} - \frac{\partial f}{\partial \zeta}\frac{\partial g}{\partial z} , \qquad (7)$$

and

$$\Delta_5 = \frac{\partial^2}{\partial z^2} + \frac{3}{\zeta}\frac{\partial}{\partial \zeta} + \frac{\partial^2}{\partial \zeta^2} . \qquad (8)$$

Next we assume that the "steady" part of the poloidal field is very weak compared to the "steady" part of the rotation and that the strength of the "steady" part of the toroidal field is less than (or at most comparable to) that of the "steady" part of rotation. These assumptions are quite reasonable on physical grounds (e.g., Mestel & Weiss 1987; Spruit 1990) and have the following two consequences.

First, the Lorentz forces in the momentum equations are negligible, and hence the induction equations (3) and (4) are effectively *decoupled* from the momentum equations (5) and (6).

Further, equation (4) itself can be written, to the first approximation, as

$$[\Omega, \zeta^2 P] = 0 \qquad (9)$$

which is same as

$$\Omega = \text{function } (\Phi) \qquad (9)$$

where

$$\Phi(\zeta, z) = \zeta^2 P(\zeta, z) \qquad (10)$$

is the flux function representing the flux of the poloidal field through the circle of axisymmetry passing through (ζ, z).

Physically this means that in the *lowest* order the plasma rotation imposes isorotation on the poloidal field. In the *next* order the small *deviation* from isorotation will give the *time-dependent* part of the toroidal field.

Thus the "steady" (*slowly varying*) part of the Sun's internal poloidal field *must satisfy diffusion equation* (3) (with $\eta = \eta_{\rm RC}$ in the radiative core and $\eta = \eta_{\rm CE}$ [$\gg \eta_{\rm RC}$] in the convective envelope), *as well as isorotation*, viz., equation (9).

3. SOLUTION OF EQUATION (3)

3.1. *Chandrasekhar's Solution of the Diffusion Equation*

Chandrasekhar (1956b) has given a solution of the diffusion equation (3), in a sphere of radius R_c, of uniform and constant diffusivity η, which is embedded in a current-free field, in the form

$$P(r, \vartheta, t) = \mathscr{P}(x, \mu) \exp(-t/\tau) , \qquad (11)$$

where $x = r/R_c$, $\mu = \cos \vartheta$, r and ϑ being the radial distance and colatitude of a point, and $\mathscr{P}(x, \mu)$ is a solution of

$$\Delta_5 \mathscr{P} + (\eta\tau)^{-1}\mathscr{P} = 0 \quad \text{in } x \le 1 , \qquad (12a)$$

and

$$\Delta_5 \mathscr{P} = 0 \quad \text{in } x > 1 . \qquad (12b)$$

The solution of equation (12a) which is finite at origin has the form

$$\mathscr{P}_n(x, \mu) = x^{-3/2} J_{n+3/2}(\alpha_n x) C_n^{3/2}(\mu) \quad \text{for } x \le 1 , \qquad (13a)$$

and the solution of equation (12b) which vanishes as $x \to \infty$ has the form

$$\mathscr{P}_n(x, \mu) = x^{-(n+3)} C_n^{3/2}(\mu) \quad \text{for } x > 1 , \qquad (13b)$$

where n is any nonnegative integer, τ has value $\tau_n = R_c^2/\eta\alpha_n^2$, $C_n^{3/2}(\mu)$ is Gegenbaur's polynomial of degree n (Abramowitz & Stegun 1964, p. 794) and $J_{n+3/2}$ represents a Bessel function of order $n + 3/2$.

Hence, the solution of equation (11) is

$$P(r, \vartheta, t) = \sum_{n=0}^{\infty} A_n x^{-3/2} J_{n+3/2}(\alpha_n x) C_n^{3/2}(\mu) \exp(-t/\tau) \quad \text{for } x \leq 1 \quad (14a)$$

and

$$P(r, \vartheta, t) = \sum_{n=0}^{\infty} M_n x^{-(n+3)} C_n^{3/2}(\mu) \exp(-t/\tau) \quad \text{for } x > 1 , \quad (14b)$$

where the values of α_n and the coefficients A_n and M_n are to be determined using the boundary conditions.

3.2. *Modification for Modeling Solar Magnetic Field*

For modeling the "steady" (slowly varying) part of the poloidal field in RC and CE we adopt the above solution with R_c as the RC-CE boundary (i.e., with $R_c = 0.7\ R_\odot$). This is possible because $\eta_{CE} \gg \eta_{RC}$ (see § 2), and equation (12b) is the limiting form of equation (12a) in the limit $\eta \to \infty$.

However, here we introduce the following important modification. In the solution (13b) of equation (12b) the terms with positive powers of x were dropped in order to ensure that the field vanishes as $x \to \infty$. However, the Sun's poloidal field must merge with the interstellar field at large distances which is nonzero and which can be considered as locally uniform. Hence on the right-hand side of equation (13b), we introduce a term B_0 corresponding to a uniform field. Moreover, we assume that the field has *odd* north-south parity (which allows $\mathscr{P}_n$ only with even values of n). Thus, we modify equations (13a) and (13b) as

$$\mathscr{P}_n(x, \mu) = x^{-3/2} \sum_{\substack{n=0 \\ \text{(even)}}}^{\infty} A_n J_{n+3/2}(\alpha_n x) C_n^{3/2}(\mu) \quad \text{for } x \leq 1 \quad (15)$$

and

$$\mathscr{P}_n(x, \mu) = B_0 C_0^{3/2}(\mu) + \sum_{\substack{n=0 \\ \text{(even)}}}^{\infty} M_n x^{-(n+3)} C_n^{3/2}(\mu) \quad \text{for } x > 1 . \quad (16)$$

These solutions give the following forms for the magnetic flux functions in the radiative core and the convective envelope:

$$\Phi_{RC}(x, \vartheta) = 2\pi A_0 R_c^2 x^{1/2} \sin^2 \vartheta \sum_{\substack{n=0 \\ \text{(even)}}}^{\infty} \lambda_n J_{n+3/2}(\alpha_n \alpha) C_n^{3/2}(\mu) , \quad (17)$$

where A_0 is taken as a scale factor for the field, $\lambda_n = A_n/A_0$, and

$$\Phi_{CE}(x, \vartheta) = \pi B_0 R_c^2 \sin^2 \vartheta \left[x^2 C_0^{3/2}(\mu) + \sum_{n \geq 0}^{\text{even}} \hat{\mu}_{n+1} x^{-(n+1)} C_n^{3/2}(\mu) \right], \quad (18)$$

where $\hat{\mu}_{n+1} = M_n/(\pi B_0 R_c^{n+3})$ are strengths of the central multipoles. Equation (18) is equivalent to the form as given in Paper I and is given as follows:

$$\Phi_{CE}(x, \vartheta) = \pi B_0 R_c^2 [(x^2 + 2\mu_1 x^{-1} + 4\mu_3 x^{-3} + \cdots) \sin^2 \vartheta + (-5\mu_3 x^{-3} + \cdots) \sin^4 \vartheta + \cdots] . \quad (19)$$

3.3. *The Boundary Conditions*

At $x = 1$, i.e., $r = R_c$, the flux function $\Phi(x, \vartheta)$ and the radial component B_r of the magnetic field must be continuous. Following Chandrasekhar's (1956b) method, with the modification necessitated by the presence of the uniform field term in Φ_{CE}, the boundary conditions can be reduced to the following pairs of equations for each value of n.

For $n = 0$:

$$A_0 J_{3/2}(\alpha_0) = (1 + \hat{\mu}_1) B_0$$

and

$$A_0[\tfrac{1}{2} J_{3/2}(\alpha_0) + \alpha_0 J'_{3/2}(\alpha_0)] = (2 - \hat{\mu}_1) B_0 ,$$

where $J'_n(x) = (d/dx) J_n(x)$. These two equations give

$$3J_{3/2}(\alpha_0)/\alpha_0 J_{1/2}(\alpha_0) = (1 + \hat{\mu}_1) \quad (20)$$

and

$$A_0 = 3B_0/\alpha_0 J_{1/2}(\alpha_0) . \quad (21)$$

For $n = 2$:

$$A_2 J_{7/2}(\alpha_2) = \hat{\mu}_3 B_0 ,$$

$$A_2[\tfrac{1}{2} J_{7/2}(\alpha_2) + \alpha_2 J'_{7/2}(\alpha_2)] = -3\hat{\mu}_3 B_0 .$$

These give

$$J_{5/2}(\alpha_2) = 0 \quad (22)$$

and

$$A_2 = \hat{\mu}_3 B_0 / J_{7/2}(\alpha_2) . \quad (23)$$

4. EVALUATION OF COEFFICIENTS USING EQUATION (9)

4.1. *The Data and the Method*

The "data" used here are the same as those used in Paper I, viz., the values of the angular velocity of the Sun's internal rotation as determined by Dziembowski et al. (1989), with uncertainties as quoted by Christensen-Dalsgaard & Schou (1988). The method of analysis is also somewhat similar. As before, the law of isorotation (eq. [9]) is assumed to be linear and written in the form

$$\Omega_{mod}(x, \vartheta) = \Omega_0 + \Omega_1 \Phi(x, \vartheta) , \quad (24)$$

and the parameter Ω_0 and the products of the coefficient "Ω_1" with the coefficients in $\Phi(x, \vartheta)$ are determined by least-squares fitting $\Omega_{mod}(x, \vartheta)$ to the helioseismologically determined plasma rotation $\Omega_{obs}(x, \vartheta)$. The difference is in the assumptions about the fields in ORC and in CE which yield different forms for $\Phi(x, \vartheta)$ and in the order in which the various parameters are determined.

Since the forms of $\Phi(x, \vartheta)$ as well as the levels of uncertainties in Ω_{obs} are different in CE and in ORC, we write equation (24) separately in CE and ORC as

$$\Omega_{mod}^{CE}(x, \vartheta) = \Omega_0^{CE} + \Omega_1^{CE} \Phi_{CE}(x, \vartheta) , \quad (25)$$

and

$$\Omega_{mod}^{ORC}(x, \vartheta) = \Omega_0^{ORC} + \Omega_1^{ORC} \Phi_{ORC}(x, \vartheta) ,$$

where Φ_{CE} and Φ_{ORC} represent flux functions in CE and ORC given in equations (17) and (18), respectively.

As in Paper I, we define a dimensionless rotation $\omega_{obs}(x, \vartheta)$ as

$$\omega_{obs}(x, \vartheta) = [\Omega_{obs}(x, \vartheta) - \hat{\Omega}]/\sigma_{\Omega_{obs}} ,$$

where $\hat{\Omega}$ is the mean and $\sigma_{\Omega_{obs}}$ is the standard deviation in the set of the values of $\Omega_{obs}(x, \vartheta)$ used. However, instead of determining $\hat{\Omega}$ and $\sigma_{\Omega_{obs}}$ separately in ORC and CE, we have determined these for the combined data from ORC and CE.

We then fit $\omega_{obs}(x, \vartheta)$ in CE and ORC to the forms

$$\omega^{CE}_{mod}(x, \vartheta) = \omega^{CE}_0 + a^{CE}\Phi_{CE}(x, \vartheta) \tag{26a}$$

and

$$\omega^{ORC}_{mod}(x, \vartheta) = \omega^{ORC}_0 + a^{ORC}\Phi_{ORC}(x, \vartheta) . \tag{26b}$$

4.2. *Procedure for Determining the "Steady" Parts of the Rotation and the Poloidal Field*

As before, the uniform field at the large distances, B_0, is essentially a scaling factor for the Sun's field. Hence we choose

$$B_0 = 1 \text{ "unit."}$$

First the parameters Ω^{CE}_0 $(= \hat{\Omega} + \omega^{CE}_0 \sigma_{\Omega_{obs}})$, Ω^{CE}_1 $(= \pi B_0 R^2_c a^{CE} \sigma_{\Omega_{obs}})$, $\hat{\mu}_1$, and $\hat{\mu}_3$ for "steady" parts of rotation and field in CE are computed by *least-squares fitting* of $\omega^{CE}_{mod}(x, \mu)$ to $\omega^{CE}_{obs}(x, \mu)$. As shown in Paper I, such a two-term fit is the best fit in CE (it is not enough to take only the first term in Φ_{CE}, and there is no gain in including the third term). The goodness of the fit and the uncertainties in the parameters are also computed in the same way as in Paper I.

Next, equations (20)–(23) are used to determine the values of the parameters α_0, A_0, $\alpha_{2,i}$, and $A_{2,i}$, $i = 1, 2, 3, \ldots$ etc.

Last, the parameters Ω^{ORC}_0 $(= \hat{\Omega} + \omega^{ORC}_0 \sigma_{\Omega_{obs}})$ and Ω^{ORC}_1 $(= \pi B_0 R^2_c a^{ORC} \sigma_{\Omega_{obs}})$ are determined by fitting $\omega^{ORC}_{mod}(x, \vartheta)$ to $\omega^{ORC}_{obs}(x, \vartheta)$.

5. THE MODEL FOR THE "STEADY" PARTS OF ROTATION AND POLOIDAL MAGNETIC FIELD

5.1. *The "Steady" Parts of Rotation and Poloidal Field in CE*

The above-mentioned computations yield the following results for the "steady" part of the *field* in CE:

$$\hat{\mu}_1 = 3.638 \pm 0.212 , \quad \hat{\mu}_3 = 0.621 \pm 0.063 . \tag{27}$$

Here we note that this value of $\hat{\mu}_1$ is exactly equal to $\hat{\mu}'_1$ and $\hat{\mu}_3$ is almost exactly equal to $\hat{\mu}'_3$, where $\hat{\mu}'_1$, $\hat{\mu}'_3$ are the values obtained from values of μ_1 and μ_3 determined in Paper I (by converting from formula [19] to formula [18]). This confirms once again that the field whose field lines isorotate with the plasma in CE is current free up to the accuracy of Ω^{CE}_{obs}.

For the "steady" part of the *rotation* in CE, we obtain

$$\Omega^{CE}_0 = 325.9 \pm 4.1 \text{ nHz} , \quad \Omega^{CE}_1 = 33.2 \pm 1.9 \text{ nHz per unit flux} . \tag{28}$$

The corresponding χ^2 is $= 42.95$, and the *confidence for goodness of fit* is $= 99.9943\%$.

5.2. *The "Steady" Parts of Rotation and Poloidal Field in ORC*

With the value of $\hat{\mu}_1$ given in equation (27), equations (20) and (21) yield unique values $\alpha_0 = 2.904$ and $A_0 = 9.374$. On the contrary, equation (22) has many roots, and the corresponding values of A_2 are given by equation (23). Different pairs of values of α_2 and A_2, along with the unique values of α_0 and A_0, yield the results as given in Table 1.

According to Table 1, the different values of α_2 yield very low values of χ^2 (all with probabilities $<10^{-7}$), the smallest being given by the smallest positive root $\alpha_{2,1}$ of α_2. The differences between the values of χ^2 are so small that it is difficult to identify the most appropriate value. However, this is mainly because of the large uncertainties in Ω_{obs} near the rotation axis, where the polynomials $C^{3/2}_n(\mu)$ have the largest amplitudes. Hence the difficulty can be avoided by rewriting Φ_{ORC} and Φ_{CE} in terms of $\sin^2 \vartheta$ and $\sin^4 \vartheta$ (since $\sin \vartheta$ has large values near the equator where helioseismic data are more reliable). When this is done and the condition of continuity of magnetic flux is rechecked, we find that the continuity condition is satisfied by only the *smallest* value of α_2, viz., 5.763.

Hence the corresponding parameters for the "steady" part of the field and the *rotation* are

$$\alpha_0 = 2.904 , \quad A_0 = 9.374 , \quad \alpha_2 = 5.763 , \quad A_2 = 1.955 , \tag{29}$$

and

$$\Omega^{ORC}_0 = 450.2 \pm 5.0 \text{ nHz} , \quad \Omega^{ORC}_1 = -3.9 \pm 0.6 \text{ nHz per unit flux} . \tag{30}$$

5.3. *The Geometrical Structure of the Field and the Currents*

The field lines of the field given by equations (17) and (18) in the region $0.0 \le r/R_\odot \le 1.0$ are plotted in Figure 1. In CE $(0.7 \le r/R_\odot \le 1)$ the field structure is similar to that in Paper I. However in RC it is entirely different from that derived in Paper I. It is interesting to note that in RC the field is much simpler: it has *no closed loops* and *no singularity* (see eq. [17]).

Here the toroidal currents in RC will be much less concentrated near the axis than the previous model in which the diffusion was neglected.

TABLE 1

RESULTS OF THE LEAST-SQUARES FITS FOR $\alpha_0 = 2.904$, $A_0 = 9.374$ WITH THE FIRST FEW VALUES OF α_2 AND THE CORRESPONDING VALUES OF A_2

Values of α_2	A_2	Ω_0 (nHz)	$\Delta\Omega_0$ (nHz)	Ω_1 (nHz per unit flux)	$\frac{\Delta\Omega_1}{\Omega_1}$	χ^2
5.763......	1.955	450	5.0	−3.89	0.17	11.58
9.095......	−2.389	439	3.0	−0.31	0.16	13.51
12.32......	2.754	437	2.4	0.11	0.18	13.69
15.51......	−3.079	434	1.8	0.98	0.16	13.93

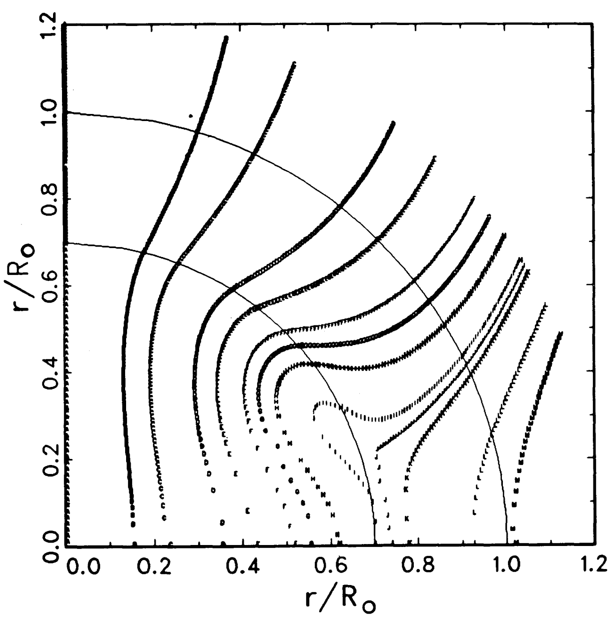

FIG. 1.—Structure of the "steady" part of the magnetic field as given by flux functions (17) and (18) with values of parameters given in eqs. (27) and (29). The field lines correspond to flux values A: 0.0, B: 0.51, C: 1.02, D: 2.04, E: 2.55, F: 3.06, G: 3.27, H: 3.47, I: 3.71, J: 3.76, K: 3.82, L: 4.08, and M: 4.29, in units of $\pi B_0 R_c^2$.

5.4. *Admissibility of Global Modes of Slow MHD Oscillations*

In the context of models of solar cycle based on global MHD (magnetohydrodynamic) oscillations, it is of interest to calculate the Alfvén wave travel times along the field lines in the present and the previous models of the "steady" part of the poloidal field. Using the density values from the standard solar model (Bahcall's model [1989, p. 90] in ORC and Spruit's model [1977, p. 26] in CE), we have determined the time taken by Alfvén waves to travel along various field lines (from one "end" on the solar surface to the other) in the previous model (Paper I) and the present model. In units of $R_c/[B_0/(4\pi\rho_0)^{1/2}]$, where ρ_0 is the central density, the travel times vary from $\sim 0.97 \times 10^{-6}$ to $\sim 7.75 \times 10^{-6}$ for the field lines in the previous model. In the present model it varies from $\sim 2.42 \times 10^{-6}$ for the field line very near the axis to $\sim 1.84 \times 10^{-6}$ for the outermost field line that just touches the RC-CE boundary. The former gives a maximum-to-minimum ratio of ~ 8, making it impossible for that model to have global modes of slow MHD oscillations. On the other hand, the present model gives for Alfvén travel times a range of only $\sim 20\%$ about the mean value. This makes it possible, in the present model, for the field structure across the core to have global slow MHD oscillations with a relative bandwidth in frequency comparable to the observed relative bandwidth of the solar cycle frequency (Gokhale et al. 1992). (For the mean travel time to be ~ 11 yr, i.e., the period to be ~ 22 yr, the value of B_0 should be ~ 0.02 G.)

5.5. *Estimation of Diffusion Timescales and the "Initial" Amplitudes of Diffusion Eigenmodes*

From the values of α_0 and α_2 obtained in § 5.2, we estimate the diffusion timescales for the two terms in equation (17) as

$$\tau_0 = R_c^2/(\eta_{\rm RC}\,\alpha_0^2) \approx 10.6 \times 10^9 \text{ yr}$$

$$\tau_2 = R_c^2/(\eta_{\rm RC}\,\alpha_2^2) \approx 2.7 \times 10^9 \text{ yr}$$

taking $\eta_{\rm RC} = 34.6 \text{ cm}^2 \text{ s}^{-1}$.

This enables us to estimate the ratio of the "initial" (ZAMS) relative amplitudes of these two terms A_2^* and A_0^* as

$$\frac{A_2^*}{A_0^*} = \frac{A_2}{A_0} \exp\left[\frac{\eta_{\rm RC}(\alpha_2^2 - \alpha_0^2)t}{R_c^2}\right] \approx 0.276$$

where "t" is the present epoch.

5.6. *The Slow Field Winding at the ORC-CE Boundary*

Equations (25) yield different values for the angular velocity Ω on the two sides of the ORC-CE boundary, suggesting a steep radial gradient across the boundary. The values of the apparent "jump,"

$$[\Omega] = [\Omega_{\rm mod}^{\rm ORC}(1, \vartheta) - \Omega_{\rm mod}^{\rm CE}(1, \vartheta)] ,$$

as given by the present model, vary from ~ 125 nHz at $\vartheta = 10°$ to ~ 50 nHz at $\vartheta = 90°$.

How much of the "jump" $[\Omega]$ is time dependent (varying on short timescales) and how much is "steady" (varying on long timescales) is not known at present. If a substantial part of this is sufficiently long lived, then the timescale τ_ϕ of field winding will be given by

$$\tau_\phi = |B_\phi|/(|B_p|\,[\Omega]) ,$$

where $|B_\phi|$ is the strength of the long-lived part of the toroidal field and $|B_p|$ is strength of the poloidal field. The values of $|B_p|$ at $r = R_c$ (i.e., at the ORC-CE boundary) range from $\sim 36B_0$ at $\vartheta = 10°$ to $\sim 6B_0$ at 90° (equator).

If $B_\phi \sim 10^6$ G (e.g., Dziembowski & Goode 1991) and $B_0 \sim 10^{-2}$ G as suggested in § 5.4, the timescales of winding range from $\sim 10^7$ yr in the polar regions to $\sim 10^8$ yr in the neighborhood of the equator.

To the order of the present approximation in which the dominant term, viz. $[\Omega, \zeta^2 P]$ in equation (4), is equated to zero, equations (3) and (4) will both have the same form. Hence the function $T(r, \vartheta, t)$ describing the slowly varying toroidal field will also have the form similar to that of P in equation (11), e.g.,

$$T(r, \vartheta, t) = \mathscr{T}(x, \mu) \exp(-t/\tau)$$

with $\mathscr{T}$ given by an equation similar to equation (13a), *except* in the spherical shell near $r = R_c$ across which the "jump" $[\Omega]$ occurs. Within this shell $\mathscr{T}$ will be given by the equation

$$\Delta_5 \mathscr{T} + (\eta\tau)^{-1}\mathscr{T} = -\zeta^{-1}[\Omega, \zeta^2 \mathscr{P}_n]$$

subject to the boundary conditions at the boundaries of the shell. However, details of such a solution remain to be worked out.

6. RESIDUAL ROTATION: EXISTENCE OF TIME-DEPENDENT PERTURBATIONS

Having determined the "steady" parts of the rotation and the magnetic field by the above procedure, we attempt to determine the forms for the nonisorotating (i.e., time-dependent) parts of the field and rotation by least-squares fitting the *residual rotation rate*, viz.,

$$\delta\Omega(x, \vartheta) = \Omega_{\rm obs}(x, \vartheta) - \Omega_{\rm mod}(x, \vartheta) , \quad (31)$$

to the *next* term in equations (26a) and (26b)—i.e., to the term in $\hat{\mu}_5$ in equation (18)—in CE and to the term λ_4 in equation (17) in ORC. The goodness of the fits and the uncertainties in the estimates of the parameters are also computed as before.

6.1. *Time-dependent Parts in CE*

Such an attempt gives

$$\hat{\mu}_5 = 0.0125 \pm 0.0335 ,$$

and *a very large value of* χ^2 showing that the residual rotation in CE cannot be used to determine accurately the parameter $\hat{\mu}_5$. This implies that either the residual rotation $\delta\Omega$ is not of the form $x^{-5}C_4^{3/2}(\cos \vartheta)$, or it contains observational errors which are larger than the real values of the residual rotation, or both.

For further insight, we add a "third" term of this form, with the above value of $\hat{\mu}_5$, to the first two terms already determined in § 5.1 and recalculate the χ^2 for the fit between $\Omega_{mod}(x, \vartheta)$ and $\Omega_{obs}(x, \vartheta)$. The new value of χ^2 is somewhat larger (43.97) than its two-term value of 42.95. Thus, even if a *part* of the residual rotation constitutes a *true* rotational perturbation, it may *not* be of the form $x^{-5}C_4^{3/2}(\cos \vartheta)$.

On the other hand, as shown in Paper I, inclusion of a term $P_5(\cos \vartheta)$ in the magnetic potential, i.e., a term $x^{-5} \sin^6 \vartheta$ in the form of equation (19) not only improves the fit but also reduces the uncertainties in μ_1 and μ_3 (and thereby in $\hat{\mu}_1$, $\hat{\mu}_5$). Thus if $\delta\Omega$ in CE contains a real rotational perturbation, it is closer in form to the term $x^{-5} \sin^6 \vartheta$ in equation (19) than to the term $x^{-5}C_4^{3/2}(\cos \vartheta)$ in equation (18).

The overall conclusion is that the residual rotation in CE is time dependent and may be of the form $x^{-5} \sin^6 \vartheta$.

6.2. *Time-dependent Parts in ORC*

A similar attempt to fit the residual rotation in ORC yields

$$\alpha_4 = 8.182$$

and

$$\lambda_4 = 0.0027 \pm 0.0018 .$$

The corresponding χ^2 is also very large. Thus the residual perturbation is not of the form of the higher order term in the "steady" part of the rotation. However, the smallness of the uncertainty in λ_4 suggests that the residual rotation may contain a small part having the form of

$$x^{1/2}J_{11/2}(\alpha_4 x)C_4^{3/2}(\cos \vartheta) ,$$

though mixed with substantial observational errors.

Here again, for further checking we add a "third" term of this form, with these values of α_4 and λ_4, to the two terms already determined in § 5.2 and recalculate the χ^2 for the fit of $\Omega_{mod}(x, \vartheta)$ with $\Omega_{obs}(x, \vartheta)$. The new value of χ^2 (11.56) is in fact slightly smaller than that (11.58) given by the two-term fit. At present we do not know a reliable method to determine if this lowering of χ^2 is significant or not. However, if it is, it would imply presence of a small part of the above form in the rotation in ORC.

The overall conclusion is that a small part of the "residual" rotation in ORC may be "steady," but most of the residual rotation in ORC is time dependent.

6.3. *The Torsional Oscillations*

From §§ 6.1 and 6.2, we see that in any case the "residual" rotation in ORC and CE cannot fit those terms which extend the common solution of equations (3) and (9) beyond the fourth power in sin ϑ. These "residual" rotation fields will therefore contain time-dependent forms. Since the displacements due to these are without compressions (by virtue of their axisymmetry) and are perpendicular to the steady component of the poloidal magnetic field, they must constitute torsional MHD oscillations. From § 5.4 it is clear that their periods will be in the range ~ 1 to ~ 100 yr if B_0 lies within the limits set in Paper I and ~ 22 yr if $B_0 \sim 0.02$ G.

7. CONCLUSIONS AND DISCUSSION

The present model incorporates important improvements over the preliminary model of the steady part of the poloidal magnetic field in the Sun's interior in the following respects: (1) the central singularity and the complexity due to the "separatrix" are eliminated and (2) the field lines in this model have a much better fit with the helioseismologically inferred isorotation contours in the outer radiative core.

The Alfvén wave travel times along different field lines in the present model have less spread than those in the preliminary model and would therefore provide a better basis for developing a model of the solar cycle in terms of global MHD oscillations.

The removal of the central singularity enables us to draw inferences about the rotation near the center. The present model suggests that the rotation near the center is uniform with angular velocity $\Omega_0 = 450$ nHz.

The opposite signs of Ω_1^{ORC} and Ω_1^{CE} represent evolutions of the fields in ORC and CE in separate frames of rotation, rotating with angular velocities Ω_0^{ORC} and Ω_0^{CE}, respectively.

As in the preliminary model, here also the observed rotation enables us to determine the leading two terms in the magnetic flux function. This is essentially because the latitudinal dependence of rotation is known with enough accuracy only up to $\sin^4 \vartheta$.

Regarding the slow winding of the poloidal field into a strong toroidal field and presence of MHD perturbations with timescales comparable to those of the solar cycle, the conclusions of the present model are qualitatively similar to those of the preliminary model.

The assumption of field diffusion provides a starting point for studying the complexity of the "initial" (ZAMS) configuration of the magnetic field (though the diffusion rate itself will be modified by the possible presence of MHD oscillations throughout the evolution of the field and rotation).

The strength of the field in the model is normalized to B_0, whose upper and lower limits have been estimated in Paper I as $\sim 10^{-4}$ G and 1 G, respectively. Recent analysis by Stenflo (1993) of the magnetogram data during 1960–1993 indicates an upper limit of $\sim 6 \times 10^{21}$ Mx on the flux of "steady" part of the poloidal field. This corresponds to a limit of ~ 0.1 G on B_0.

It must be emphasized that at any time the "rms" field near the surface will be much stronger due to possible simultaneous presence of many global MHD oscillations.

More data especially on rotation in high latitudes and near the axis with higher resolution and accuracy will be needed for further refinements of the model.

Direct fitting of the acoustic frequency splittings to a rotation law of the form given in equation (25) will be more convenient for studying the interaction of rotation and magnetic field inside the Sun.

We are thankful to an anonymous referee for helpful comments and suggestions.

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