PHYSICAL REVIEW D VOLUME 48, NUMBER 12 15 DECEMBER 1993

Frenet-Serret description of gyroscopic precession

B. R. Iyer*
Raman Research Institute, Bangalore 560080, India

C. V. Vishveshwara†
Indian Institute of Astrophysics, Bangalore 560034, India
(Received 8 June 1993)

The phenomenon of gyroscopic precession is studied within the framework of the Frenet-Serret formalism adapted to quasi-Killing trajectories. Its relation to the congruence vorticity is highlighted with particular reference to the irrotational congruence admitted by the stationary, axisymmetric spacetime. General precession formulas are obtained for circular orbits with arbitrary constant angular speeds. By successive reduction, different types of precessions are derived for the Kerr-Schwarzschild-Minkowski spacetime family. The phenomenon is studied in the case of other interesting spacetimes, such as the de Sitter and Gödel universes as well as the general stationary, cylindrical, vacuum spacetimes.

PACS number(s): 04.20.Me

I. INTRODUCTION

The phenomenon of rotation exhibits interesting and often intriguing physical effects. This is even more so within the framework of the general theory of relativity which leads to novel features. These features, for instance, are built into the structure of spacetime, such as that of a rotating black hole. Dragging of inertial frames is a typical example of rotational effects incorporated into the spacetime structure. Such effects also manifest themselves in the intrinsic aspects of particle motion and related phenomena such as the gyroscope precession. These aspects can be elegantly studied by the invariant geometrical description of particle trajectories that follow the directions of spacetime symmetries, or Killing vector fields, provided of course that the spacetime admits such symmetries. This is accomplished by adopting the Frenet-Serret formalism to characterize the Killing trajectories of a four-dimensional spacetime. Of the three geometric parameters basic to this formalism, the curvature is identified with the particle acceleration, while the two torsions are directly related to the gyroscope precession. Furthermore, the Frenet-Serret tetrad provides a convenient reference frame for the description of all relevant physical phenomena. Therefore, when the formalism is applied to the timelike integral curves of spacetime symmetries, the phenomenon of gyroscope precession can be completely analyzed in a natural and cogent manner.

The Frenet-Serret formalism applied to the Killing trajectories can be extended in a straightforward manner to what we may term as a quasi-Killing congruence. This congruence consists of timelike curves following the direction given by a combination of Killing vectors with nonconstant coefficients. An important example is the irrotational congruence admitted by the Kerr spacetime. With the help of this extended formalism, a broad based framework is provided for the study of gyroscope precession in a variety of circumstances.

The present paper is organized as follows. In Sec. II, we discuss the application of the Frenet-Serret formalism to the quasi-Killing trajectories, precession of gyroscopes transported along them and its relation to the vorticity of the congruence. Section III considers the stationary, axially symmetric spacetimes and concentrates on the globally timelike Killing trajectories followed by stationary observers. Specializing to the Kerr spacetime, the gyroscopic precession with respect to the stationary observer—a direct manifestation of inertial frame dragging—is displayed. By using rotating coordinates, gyroscopic precession along circular orbits with arbitrary constant angular speeds is investigated in Sec. IV. The general formulas derived are first applied to the Kerr spacetime to obtain particle acceleration and gyroscopic precession without approximation. Then by successive specialization, we obtain the Schiff precession, precession in the Schwarzschild spacetime with Fokker–de Sitter precession as a particular example, and Thomas precession in Minkowski spacetime. The irrotational congruence is discussed and the Frenet-Serret parameters are derived for the corresponding trajectories. The general formalism is also applied to de Sitter spacetime. Section V treats in detail the general case of stationary cylindrically symmetric spacetimes where the general quasi-Killing trajectories are helical orbits. Included in this study as special cases are the Gödel universe and the general vacuum metrics as given by Vishveshwara and Winicour. Section VI comprises a summary and concluding remarks.

Starting from Thomas precession, gyroscopic precession has been studied extensively by different approaches both in special and general relativity [1–6]. We have presented here a unified, covariant, geometric treatment of

*Electronic address: bri@rri.ernet.in
†Electronic address: vishu@iiap.ernet.in

this remarkable phenomenon. Furthermore, in this treatment computations can be made in a straightforward and complete manner. We have also highlighted the interrelations among quantities such as vorticity, precession, and Frenet-Serret torsions. In addition to the general formulas, exact expressions have been presented pertaining to special and physically significant spacetimes. It is hoped that the discussions and formalism of this paper offer additional insight into the phenomenon of gyroscopic precession and that the formulas derived can be of use for further elucidation and astrophysical applications.

Our metric signature is $(+,-,-,-)$. Spacetime indices are denoted by latin letters $a, b, \ldots, m, n, \ldots$, and run over 0, 1, 2, 3 while spatial indices are denoted by greek letters $\alpha, \beta, \ldots, \mu, \nu, \ldots$, and run over 1, 2, 3. The corresponding tetrad [triad] indices are indicated by enclosing them in parentheses: $(a)(b)\cdots(m)(n)\cdots[(\alpha)(\beta)\cdots(\mu)(\nu)\cdots]$.

II. THE QUASI-KILLING TRAJECTORIES

A. The Frenet-Serret formalism

In [7, 8] it was shown that the Frenet-Serret formalism has some attractive formal properties in the case of Killing trajectories that find elegant applications in black hole geometries. We show below how these properties obtain in a more general case, which we call the quasi-Killing case. Consider a spacetime with a timelike Killing vector $\boldsymbol{\xi}$ and a set of spacelike Killing vectors $\boldsymbol{\eta}_{(A)}$ $(A = 1, 2, \ldots, m)$. The combination

$$\chi^a \equiv \xi^a + \omega_{(A)}\eta^a_{(A)}, \tag{1}$$

where summation over (A) is implied and

$$\underset{\chi}{\mathcal{L}}\, \omega_{(A)} = 0 \tag{2}$$

is called a quasi-Killing vector. The terminology is justified for our usage since as in the Killing case it follows that, if u^a is the four-velocity associated with χ^a (where it is timelike) obtained by normalizing χ^a,

$$e^a_{(0)} \equiv u^a \equiv e^{\psi}\chi^a, \tag{3}$$

then

$$e^{-2\psi} = \chi_a\chi^a\,, \quad \psi_{,a}\chi^a = 0, \tag{4}$$

and

$$\dot{e}^a_{(0)} \equiv e^a_{(0);b}\, e^b_{(0)} = F^a{}_b\, e^b_{(0)}, \tag{5}$$

where

$$F_{ab} \equiv e^{\psi}\,(\xi_{a;b} + \omega_{(A)}\,\eta_{(A)a;b}). \tag{6}$$

It is easy to show, using the Killing equation and the relation $\xi_{a;b;c} = R_{abcd}\,\xi^d$ for any Killing vector $\boldsymbol{\xi}$, that

$$F_{ab} = -F_{ba}\,, \quad \dot{F}_{ab} = 0. \tag{7}$$

Recall that the Frenet-Serret equations are [7, 8]

$$\begin{aligned} \dot{e}^a_{(0)} &= \kappa\, e^a_{(1)}, \\ \dot{e}^a_{(1)} &= \kappa\, e^a_{(0)} + \tau_1\, e^a_{(2)}, \\ \dot{e}^a_{(2)} &= -\tau_1\, e^a_{(1)} + \tau_2\, e^a_{(3)}, \\ \dot{e}^a_{(3)} &= -\tau_2\, e^a_{(2)}, \end{aligned} \tag{8}$$

where κ is the curvature and τ_1, τ_2 the first and second torsions, respectively. The Frenet-Serret equations (8) together with Eqs. (5)–(7) imply, as in the Killing case, that along trajectories of χ^a the Frenet-Serret invariants κ, τ_1, and τ_2 are constants and the Frenet-Serret basis vectors $e^a_{(i)}$ satisfy a Lorentz-like equation:

$$\dot{\kappa} = \dot{\tau}_1 = \dot{\tau}_2 = 0, \tag{9}$$

$$\dot{e}^a_{(i)} = F^a{}_b\, e^b_{(i)}. \tag{10}$$

Note that $F_{ab} \neq e^{\psi}\,\chi_{a;b}$. As before [7, 8],

$$\kappa^2 = F^2_{ab}\, e^a_{(0)}\, e^b_{(0)}, \tag{11}$$

$$\tau_1^2 = \kappa^2 - \frac{F^4_{ab}\, e^a_{(0)}\, e^b_{(0)}}{\kappa^2}, \tag{12}$$

$$\tau_2^2 = -\frac{(\kappa^2 - \tau_1^2)^2}{\tau_1^2} + \frac{F^6_{ab}\, e^a_{(0)}\, e^b_{(0)}}{\kappa^2\,\tau_1^2}, \tag{13}$$

where

$$(F^n)_{ab} \equiv F_a{}^{a_1}\, F_{a_1}{}^{a_2} \cdots F_{a_{n-1}b}. \tag{14}$$

Moreover

$$\alpha \equiv \tfrac{1}{2}\, F^a{}_b\, F^b{}_a = \kappa^2 - \tau_1^2 - \tau_2^2. \tag{15}$$

Before proceeding further we may mention some examples of quasi-Killing congruences given by (1). In the stationary axisymmetric spacetime $\boldsymbol{\eta}$ can be chosen as the axial Killing vector with ω an arbitrary function of r and θ in adapted coordinates. For instance, ω can be chosen to make the congruence either geodesic or irrotational. Spatially these will represent circular orbits. In cylindrically symmetric spacetimes in addition to the axial Killing vector we can add on the Killing vector generating z translations with coefficients as arbitrary functions of ρ in adapted coordinates. Spatially these will represent helical orbits. In spacetimes admitting other spatial Killing vectors such as de Sitter and Gödel Universes more complicated quasi-Killing congruences can be generated whose spatial projections would not be simple curves such as circles or helices. Along any particular trajectory belonging to a quasi-Killing congruence ω is a constant. With reference to the congruence in which a trajectory is embedded we may call such a curve a quasi-Killing trajectory. Of course, if $\omega_{(A)}$ are constants then $\boldsymbol{\chi}$ defines a Killing trajectory.

B. Frenet-Serret torsions and gyroscopic precession

The transport law for an observer whose tetrad moves along an arbitrary world line is written as [9]

$$\frac{D}{D\tau}\left(e^a_{(i)}\right) = -\Omega^a{}_b\, e^b_{(i)}, \tag{16}$$

where Ω decomposes into a Fermi-Walker piece and a spatial rotation

$$\Omega^{ab} = \Omega^{ab}_{(\mathrm{FW})} + \Omega^{ab}_{(\mathrm{SR})},$$
$$\Omega^{ab}_{(\mathrm{FW})} \equiv a^a u^b - a^b u^a, \tag{17}$$
$$\Omega^{ab}_{(\mathrm{SR})} \equiv u_c \omega_d \epsilon^{cdab}.$$

In the above, $\boldsymbol{\omega}$ is a vector orthogonal to the four velocity u^a. It is possible to choose the time axis of the tetrad along the four-velocity of the arbitrary world line consistent with the transport law equations (16) and (17) and following [9] we restrict ourselves to such tetrads. If a frame $\mathbf{f}_{(b)}$ is Fermi-Walker transported along the same world line the spatial triad of $\mathbf{e}_{(a)}$ rotates relative to the spatial triad of $\mathbf{f}_{(a)}$ with angular velocity ω, i.e.,

$$\frac{D}{D\tau}\left(\mathbf{e}_{(\mu)} - \mathbf{f}_{(\mu)}\right) = \boldsymbol{\omega} \times \mathbf{e}_{(\mu)}. \tag{18}$$

Comparing the Frenet-Serret equations (8) with the transport equations (16)–(18) it is easy to verify that the Frenet-Serret frame rotates with respect to the Fermi-Walker transported frame by

$$\boldsymbol{\omega}_{(\mathbf{FS})} = \tau_2\, \mathbf{e}_{(1)} + \tau_1\, \mathbf{e}_{(3)}. \tag{19}$$

The Fermi-Walker frame is physically realized by a system of gyroscopes and hence the gyroscopic precession relative to the Frenet-Serret frame, one of the most natural and intrinsic frames associated with an arbitrary curve, is given by $-\boldsymbol{\omega}_{(\mathbf{FS})}$:

$$\boldsymbol{\Omega}_{(\mathbf{g})} = -\boldsymbol{\omega}_{(\mathbf{FS})} = -\left(\tau_2\, \mathbf{e}_{(1)} + \tau_1\, \mathbf{e}_{(3)}\right). \tag{20}$$

Further, using the Frenet-Serret equation (8) one can prove

$$\omega^a_{(\mathbf{FS})} = \tilde{F}^{ab}\, e_{(0)b}, \tag{21}$$

where $\tilde{F}^{ab} \equiv \frac{1}{2\sqrt{-g}}\,\epsilon^{abcd} F_{cd}$ is the dual to F_{cd}. We refer to $\boldsymbol{\omega}_{(\mathbf{FS})}$ as Frenet-Serret rotation. It should be noted that $\boldsymbol{\omega}_{(\mathbf{FS})}$ is defined along one given curve. It is not tied to the existence of a congruence. It gives the rotation of the Frenet-Serret frame relative to the Fermi-Walker transported frame.

We may mention in passing that from Eqs. (8) and (10) we have

$$\kappa e^a_{(1)} = F^a{}_b e^b_{(0)} \tag{22}$$

which indicates that in analogy with electromagnetism $F^a{}_b e^b_{(0)}$ can be interpreted as the gravielectric field as seen by the observer with four-velocity $e^a_{(0)}$. Moreover, the precession equations (18) and (21) exhibit further suggestive resemblance to the electromagnetic "spin precession" equations and indicate that $\tilde{F}^{ab} e_{(0)b}$ is the corresponding gravimagnetic field.

C. Vorticity and gyroscopic precession

Given a trajectory it can be viewed as a member of a suitable chosen congruence of curves. Associated with a congruence of curves is defined the notion of vorticity, which geometrically measures the twisting of the congruence. The gyroscopic precession along a trajectory is related to the vorticity of the congruence. In this section we shall explore this relation in some detail. It was shown in [7] that the Frenet-Serret rotation for a trajectory belonging to the Killing congruence is equal to the vorticity of the congruence. Consequently, the gyroscopic precession for a Killing trajectory is determined by the vorticity of the Killing congruence.

As we shall show below, in this respect, the quasi-Killing case differs from the Killing one. The vorticity of a congruence is defined as

$$\Omega^a \equiv \frac{1}{2\sqrt{-g}}\,\epsilon^{abcd}\, e_{(0)b}\, e_{(0)c;d}$$
$$= \frac{1}{2\sqrt{-g}}\,\epsilon^{abcd}\, e_{(0)b}\left[F_{cd} + e^{\psi}\,\omega_{(A),d}\,\eta_{(A)c}\right] \tag{23}$$
$$= \omega^a_{(\mathbf{FS})} + \tilde{D}^{ab}\, e_{(0)b}, \tag{24}$$

where

$$\tilde{D}^{ab} = \frac{1}{2\sqrt{-g}}\,\epsilon^{abcd}\, D_{cd},$$
$$D_{cd} \equiv e^{\psi}\,\omega_{(A),[d}\,\eta^{(A)}_{c]}, \tag{25}$$

and antisymmetrization is defined as

$$A_{[ab]} \equiv \tfrac{1}{2}\left(A_{ab} - A_{ba}\right).$$

As is well known, physically, vorticity Ω^a represents the angular velocity of the connecting vector with respect to an orthonormal spatial frame Fermi-Walker transported along the congruence [10, 11]. On the other hand, Frenet-Serret rotation $\omega^a_{(\mathbf{FS})}$ represents precession of the intrinsic Frenet-Serret frame with respect to the nonrotating Fermi-Walker frame. In general, for example, in the quasi-Killing case, the two are not the same. Therefore the gyroscopic precession along a quasi-Killing trajectory differs from the rotation of the connecting vector of the corresponding quasi-Killing congruence. However, from Eq. (23) it follows that if $\omega_{(A)}$ are constants, the congruence χ^a becomes Killing, $\Omega^a = \omega^a_{(\mathbf{FS})}$, and the gyroscopic precession is locked on to the rotation of the connecting vector [12].

The above difference between the two cases, namely, Killing and quasi-Killing, may also be understood by examining the Lie derivative of the basis vectors along $\mathbf{e}_{(0)}$ in the two cases. In the Killing case

$$\underset{\mathbf{e}_{(0)}}{\mathcal{L}}\, \mathbf{e}_{(\alpha)} = \kappa\, \mathbf{e}_{(0)}\, \delta^{(1)}_{(\alpha)} \tag{26}$$

so that modulo $\mathbf{e}_{(0)}$ (i.e., if one projects normal to $\mathbf{e}_{(0)}$) the Frenet-Serret frame is Lie dragged along $\mathbf{e}_{(0)}$. In the quasi-Killing case, on the other hand,

$$\underset{\mathbf{e}_{(0)}}{\mathcal{L}}\, \mathbf{e}_{(\alpha)} = \kappa\, \mathbf{e}_{(0)}\, \delta^{(1)}_{(\alpha)} + e^{\psi}\left(\underset{\mathbf{e}_{(\alpha)}}{\mathcal{L}}\, \omega_{(A)}\right)$$
$$\times\left[\left(\boldsymbol{\eta}_{(A)} \cdot \mathbf{e}_{(0)}\right)\mathbf{e}_{(0)} - \boldsymbol{\eta}_{(A)}\right] \tag{27}$$

so that the Frenet-Serret frame is *not* Lie dragged along $\mathbf{e}_{(0)}$. Recall, that by definition the connecting vector is always Lie dragged, i.e.,

$$\underset{\mathbf{e}_{(0)}}{\mathcal{L}}\ \mathbf{c} = 0. \tag{28}$$

In the following sections, we shall discuss particular examples to illustrate the application of the above considerations.

III. STATIONARY AXIALLY SYMMETRIC SPACETIMES

In this section we specialize to spacetimes which are stationary and axially symmetric. Such spacetimes have in addition to the timelike Killing vector $\boldsymbol{\xi}$, a spacelike Killing vector $\boldsymbol{\eta}$ with closed orbits. Assuming further orthogonal transitivity, in coordinates adapted to the Killing vectors $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$, the most general form of the metric may be written as

$$ds^2 = g_{00}dt^2 + 2g_{03}dtd\phi + g_{33}d\phi^2 + g_{11}dr^2 + g_{22}d\theta^2, \tag{29}$$

where g_{ab} are functions of r and θ only.

The contravariant components of the metric may be read off from

$$\left(\frac{\partial}{\partial s}\right)^2 = \frac{g_{33}}{\Delta_3}\left(\frac{\partial}{\partial t}\right)^2 - 2(\frac{g_{03}}{\Delta_3})\frac{\partial}{\partial t}\frac{\partial}{\partial \phi} + \frac{g_{00}}{\Delta_3}\left(\frac{\partial}{\partial \phi}\right)^2 + \frac{1}{g_{11}}\left(\frac{\partial}{\partial r}\right)^2 + \frac{1}{g_{22}}\left(\frac{\partial}{\partial \theta}\right)^2, \tag{30}$$

where

$$\Delta_3 \equiv g_{00}\,g_{33} - g_{03}^2 \tag{31}$$

and

$$\det g_{ab} \equiv g = g_{11}\,g_{22}\Delta_3. \tag{32}$$

After a long but straightforward calculation using Eqs. (11)–(13) and Eqs.(29)–(32) it follows that along trajectories of the timelike Killing vector $\boldsymbol{\xi}$ the Frenet-Serret invariants are given by

$$\kappa^2 = -\tfrac{1}{4}\,g^{ab}(\ln g_{00})_{,a}\,(\ln g_{00})_{,b} = -\frac{1}{4g_{00}^2}\,[g^{11}\,g_{00,1}^2 + g^{22}\,g_{00,2}^2], \tag{33}$$

$$\tau_1^2 = \frac{g_{03}^2}{4\Delta_3}\,\frac{[g^{ab}\,g_{00,a}\,(\ln \frac{g_{03}}{g_{00}})_{,b}]^2}{[g^{ab}\,g_{00,a}\,g_{00,b}]}, \tag{34}$$

$$\tau_2^2 = \frac{1}{4\Delta_3 g_{11}g_{22}}\,\frac{[g_{00,1}\,g_{03,2} - g_{00,2}\,g_{03,1}]^2}{[g^{ab}\,g_{00,a}\,g_{00,b}]}. \tag{35}$$

In this case the Frenet-Serret basis is given by

$$e^a_{(0)} = \frac{1}{\sqrt{g_{00}}}\,(1\,,0\,,0\,,0),$$

$$e^a_{(1)} = -\frac{1}{2\kappa\,g_{00}}\,(0,\,g^{11}\,g_{00,1},\,g^{22}\,g_{00,2},\,0),$$

$$e^a_{(2)} = \frac{1}{\sqrt{g_{00}}\,\sqrt{-\Delta_3}}\,(-g_{03},\,0,\,0,\,g_{00}),$$

$$e^a_{(3)} = \frac{\sqrt{g^{11}\,g^{22}}}{2\kappa\,g_{00}}\,(0,\,-g_{00,2},\,g_{00,1},\,0). \tag{36}$$

Equations (33)–(36) completely describe the world line of a stationary observer and the precession of a gyroscope carried by him. κ and τ_1 are chosen to be positive and τ_2 taken to be the positive square root of the right-hand side of Eq. (35) so that $\mathbf{e}_{(1)}$, $\mathbf{e}_{(2)}$, $\mathbf{e}_{(3)}$, form a right-handed triad. We shall now apply these formulas to the special case of the Kerr spacetime.

A. Kerr spacetime

The spacetime describing a rotating black hole is the Kerr solution and its geometry is given by

$$ds^2 = \left(1 - \frac{2Mr}{\Sigma}\right)dt^2 - \frac{\Sigma}{\Delta}dr^2 - \Sigma d\theta^2 + \frac{4Mra\sin^2\theta}{\Sigma}d\phi dt - \left(r^2 + a^2 + \frac{2Mra^2\sin^2\theta}{\Sigma}\right)\sin^2\theta d\phi^2, \tag{37}$$

where

$$\Delta \equiv r^2 + a^2 - 2Mr \;\; ; \;\; \Sigma \equiv r^2 + a^2\cos^2\theta.$$

Substituting the above expressions for g_{ab} in Eqs. (33)–(36) and simplifying we obtain

$$\kappa^2 = \frac{M^2}{\Sigma^5}\,(\Delta\epsilon^2 + 4r^2a^4\cos^2\theta\sin^2\theta)\cdot\frac{1}{(1 - \frac{2Mr}{\Sigma})^2}, \tag{38}$$

$$\tau_1^2 = \frac{M^2a^2\sin^2\theta}{\Sigma}\,\frac{\Delta}{(1 - \frac{2Mr}{\Sigma})^2} \times\frac{1}{(\Delta\epsilon^2 + 4r^2a^4\cos^2\theta\sin^2\theta)}, \tag{39}$$

$$\tau_2^2 = \frac{4M^2a^2r^2\cos^2\theta\,\epsilon^2}{\Sigma^3}\cdot\frac{1}{(\Delta\epsilon^2 + 4r^2a^4\cos^2\theta\sin^2\theta)}, \tag{40}$$

where $\epsilon \equiv r^2 - a^2\cos^2\theta$

$$e^a_{(0)} = \frac{1}{\sqrt{1 - \frac{2Mr}{\Sigma}}}\,(1,\,0,\,0,\,0),$$

$$e^a_{(1)} = \frac{1}{\sqrt{\Sigma(\Delta\epsilon^2 + 4r^2a^4s^2c^2)}}\,(0,\,\Delta\epsilon,\,-2ra^2sc,\,0),$$

$$e^a_{(2)} = \frac{1}{s\sqrt{\Delta(1 - \frac{2Mr}{\Sigma})}} \times\left[-\frac{2Mras^2}{\Sigma},\,0,\,0,\,\left(1 - \frac{2Mr}{\Sigma}\right)\right],$$

$$e^a_{(3)} = \frac{1}{\sqrt{\frac{\Sigma}{\Delta}(\Delta\epsilon^2 + 4r^2a^4s^2c^2)}}\,(0,\,2ra^2sc,\,\epsilon,\,0), \tag{41}$$

where $s \equiv \sin\theta, c \equiv \cos\theta$.

Equations (38)–(41) show that an observer with fixed spatial coordinates, i.e., a world line following t lines, is not only accelerated ($\kappa \neq 0$) but also has an angular velocity relative to the local standards of nonrotation realized by a set of gyroscopes. This is a manifestation of the dragging phenomenon in the Kerr spacetime. For an observer on the equatorial plane $\theta = \pi/2$ it reduces to

$$\tau_1 = \frac{Ma}{r^3}\left(1 - \frac{2M}{r}\right)^{-1}, \tag{42}$$

$$\tau_2 = 0. \tag{43}$$

The bases vectors of the Frenet-Serret frame of the stationary observer ($\boldsymbol{\xi}$ lines) are always pointed to the same fixed stars since they are Lie-dragged along the Killing trajectory. They may be visualized by a set of telescopes locked on to the distant stars. They also form the connecting vectors of the Killing congruence defining stationary observers. Thus the stationary observers will see the gyroscopes precess with respect to the distant stars with an angular velocity per unit proper time given by $-\tau_1$. In Sec. IV A 3 we shall discuss the precession of a gyroscope carried once around a circular orbit as measured by a stationary observer in his rest frame. To measure the precession *relative to a gyroscope* carried by the stationary observer the precession due to dragging mentioned above needs to be taken into account. Of course, for static spacetimes the Frenet-Serret frame of the static observer, locked on to the distant stars, does not precess with respect to the gyroscopes.

It may be worth pointing out that a discussion off the equatorial plane involves no extra work in this formalism. Thus, we give general expressions in all cases when one is off the equatorial plane.

IV. ROTATING COORDINATES AND GYROSCOPIC PRECESSION ALONG CIRCULAR ORBITS WITH CONSTANT ARBITRARY ANGULAR SPEEDS

In Sec. III, we have obtained κ, τ_1, τ_2 for an observer whose world line is along the integral curves of the timelike Killing vector $\boldsymbol{\xi}$ of a stationary spacetime. Such an observer is at a fixed value of r, θ, and ϕ. In this section we show how the use of "rotating" coordinates allows one to adapt the expressions of Sec. III to trajectories belonging to a quasi-Killing congruence that represent observers moving along circular orbits with constant arbitrary angular speeds. This is in the spirit of the method used by Rindler and Perlick [3].

Starting from a stationary axially symmetric metric of the form (29) adapted to the Killing vectors $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$, we note that $\boldsymbol{\xi} + \omega\boldsymbol{\eta}$, where ω is a constant, is also a Killing vector. A coordinate system adapted to $\boldsymbol{\xi}' \equiv \boldsymbol{\xi} + \omega\boldsymbol{\eta}$ is obtained by a coordinate transformation

$$\phi = \phi' + \omega t', \quad t = t' \tag{44}$$

under which the metric becomes

$$ds^2 = g_{0'0'}\, dt'^2 + 2g_{0'3'}\, d\phi'\, dt' + g_{3'3'}\, d\phi'^2 + g_{11}\, dr^2 + g_{22}\, d\theta^2, \tag{45}$$

where

$$g_{0'0'} = g_{00} + 2\omega g_{03} + \omega^2 g_{33} \equiv \mathcal{A}, \tag{46}$$

$$g_{0'3'} = g_{03} + \omega g_{33} \equiv \mathcal{B}, \tag{47}$$

$$g_{3'3'} = g_{33}, \tag{48}$$

$\boldsymbol{\xi}' = (1,0,0,0)$ is a Killing vector of this metric and we can use Eqs. (33)–(35) to obtain κ, τ_1, and τ_2 along this world line. However, $\boldsymbol{\xi}'$ corresponds to $\boldsymbol{\xi} + \omega\boldsymbol{\eta}$ in the unprimed coordinates so that we can compute κ, τ_1, and τ_2 along trajectories $\boldsymbol{\xi} + \omega\boldsymbol{\eta}$ by replacing g_{00}, g_{03}, and g_{33} in Eqs. (38)–(40) by $g_{0'0'}$, $g_{0'3'}$, and $g_{3'3'}$. More importantly the prescription also works in cases where ω is not a constant but only satisfies $\mathcal{L}_{\boldsymbol{\chi}}\, \omega = 0$. This can be seen by noting that the expressions for the Frenet-Serret invariants in the quasi-Killing case do not involve derivatives of ω. One can also check explicitly that the same expressions for κ, τ_1, τ_2 obtains whether one starts from $\boldsymbol{\xi} + \omega\boldsymbol{\eta}$ or if one uses the expressions for $\boldsymbol{\xi}$ and replaces g_{ab} by $g_{a'b'}$ treating ω as a constant. Thus along trajectories of $\boldsymbol{\xi} + \omega\boldsymbol{\eta}$ we obtain

$$\kappa^2 = -\frac{1}{4}\frac{(g^{11}\mathcal{A}_{(1)}^2 + g^{22}\mathcal{A}_{(2)}^2)}{\mathcal{A}^2}, \tag{49}$$

$$\tau_1^2 = \left(\frac{\mathcal{B}^2}{4\Delta_3(g^{11}\mathcal{A}_{(1)}^2 + g^{22}\mathcal{A}_{(2)}^2)}\right) \times \left(\frac{g^{11}\mathcal{A}_{(1)}\mathcal{B}_{(1)} + g^{22}\mathcal{A}_{(2)}\mathcal{B}_{(2)}}{\mathcal{B}} - \frac{g^{11}\mathcal{A}_{(1)}^2 + g^{22}\mathcal{A}_{(2)}^2}{\mathcal{A}}\right)^2, \tag{50}$$

$$\tau_2^2 = \frac{g^{11}\, g^{22}(\mathcal{A}_{(1)}\mathcal{B}_{(2)} - \mathcal{A}_{(2)}\mathcal{B}_{(1)})^2}{4\Delta_3\,(g^{11}\mathcal{A}_{(1)}^2 + g^{22}\mathcal{A}_{(2)}^2)}, \tag{51}$$

where

$$\mathcal{A}_{(a)} \equiv g_{00,a} + 2\omega g_{03,a} + \omega^2 g_{33,a}, \quad a = 1,2, \tag{52}$$

$$\mathcal{B}_{(b)} \equiv g_{03,b} + \omega g_{33,b} \quad b = 1,2. \tag{53}$$

To obtain the Frenet-Serret tetrad associated with $\boldsymbol{\chi}$ we recall that with respect to the primed coordinates $\boldsymbol{\chi}$ is like $\boldsymbol{\xi}$. Thus the Frenet-Serret tetrad in the primed coordinates are obtained by replacing g_{ab} by $g_{a'b'}$ in Eq. (36). The components relative to the original unprimed coordinates are obtained via a vector transformation and finally we find

$$e^a_{(0)} = \frac{1}{\sqrt{\mathcal{A}}}(1,0,0,\omega),$$

$$e^a_{(1)} = -\frac{1}{2\kappa\mathcal{A}}(0,\, g^{11}\mathcal{A}_{(1)},\, g^{22}\mathcal{A}_{(2)},\, 0),$$

$$e^a_{(2)} = \frac{1}{\sqrt{\mathcal{A}}\sqrt{-\Delta_3}}(\mathcal{B},\, 0,\, 0,\, -\mathcal{C}),$$

$$e^a_{(3)} = \frac{\sqrt{g^{11}\, g^{22}}}{2\kappa\mathcal{A}}(0,\, -\mathcal{A}_{(2)},\, \mathcal{A}_{(1)},\, 0), \tag{54}$$

where

$$\mathcal{C} \equiv g_{00} + \omega g_{03}.$$

For later comparison we write down the dual bases below. They are given by

$$\omega^{(0)} = \frac{1}{\sqrt{\mathcal{A}}}[\mathcal{C}\mathbf{dt} - \mathcal{B}\mathbf{d\phi}],$$
$$\omega^{(1)} = \frac{1}{2\kappa\mathcal{A}}[\mathcal{A}_{(1)}\mathbf{dr} + \mathcal{A}_{(2)}\mathbf{d\theta}],$$
$$\omega^{(2)} = \sqrt{\frac{-\Delta_3}{\mathcal{A}}}[\omega\mathbf{dt} - \mathbf{d\phi}],$$
$$\omega^{(3)} = \frac{1}{2\kappa\mathcal{A}}\left[\sqrt{\frac{g_{22}}{g_{11}}}\mathcal{A}_{(2)}\mathbf{dr} - \sqrt{\frac{g_{11}}{g_{22}}}\mathcal{A}_{(1)}\mathbf{d\theta}\right]. \quad (55)$$

We next apply the above formulas to various special cases and retrieve well-known gyroscopic precessions. Many of our formulas are more general in that they are not confined to the equatorial plane but valid off it. The formulas are moreover demonstrated in a unified framework.

It should be recalled that in the above ω can be any arbitrary function of r and θ. This allows one to discuss precession along a geodesic (where ω is not a constant) by chosing ω such that $\kappa = 0$. Although the Frenet-Serret equations are reduced to only one, τ_1 and τ_2 are defined through ω, $\mathbf{e}_{(2)}$, and $\mathbf{e}_{(3)}$ by considering the geodesic as a particular limit of the congruence obtained by keeping ω constant corresponding to the geodetic value. The Schwarzschild metric corresponds to $a = 0$ while flat spacetime corresponds to $M = a = 0$. In addition if trajectories are confined to the equatorial plane then $\theta = \pi/2$.

A. Kerr black hole

1. The general case

The general procedure outlined above may be applied to obtain the acceleration and gyroscopic precession in the case of an observer following a quasi-Killing trajectory in the Kerr spacetime. A straightforward computation yields

$$\kappa^2 = \frac{\mathcal{K}_1}{\Sigma\mathcal{K}_2}, \quad (56)$$
$$\tau_1^2 = \frac{\Delta}{\Sigma}\frac{\mathcal{K}_3}{\mathcal{K}_1\mathcal{K}_2}s^2, \quad (57)$$
$$\tau_2^2 = \frac{M^2\mathcal{K}_4}{\Sigma^5\mathcal{K}_1}c^2, \quad (58)$$

where

$$\mathcal{K}_1 = \Delta\left[\frac{M\epsilon}{\Sigma^2}(1 - a\omega s^2)^2 - rs^2\omega^2\right]^2 + c^2s^2\left[\frac{2Mr}{\Sigma^2}\{(r^2 + a^2)\omega - a\}^2 + \Delta\omega^2\right]^2, \quad (59)$$

$$\mathcal{K}_2 = \left[1 - (r^2 + a^2)s^2\omega^2 - \frac{2Mr}{\Sigma}(1 - a\omega s^2)^2\right]^2, \quad (60)$$

$$\mathcal{K}_3 = \left[\left\{\frac{M\epsilon}{\Sigma^2}(1 - a\omega s^2) - rs^2\omega^2\right\}\left[r\omega - \frac{2Mr^2}{\Sigma}(1 - a\omega s^2)\omega - \frac{M\epsilon}{\Sigma^2}(1 - a\omega s^2)\{(r^2 + a^2)\omega - a\}\right]\right.$$
$$\left. + c^2\left\{\frac{2Mra}{\Sigma^2}(1 - a\omega s^2)^2 - \omega\right\}\left\{\frac{2Mr}{\Sigma^2}[(r^2 + a^2)\omega - a]^2 + \Delta\omega^2\right\}\right], \quad (61)$$

$$\mathcal{K}_4 = \left[\frac{2Mr\epsilon a}{\Sigma}(1 - a\omega s^2)^2 - \epsilon\omega(r^2 + a^2)(1 - a\omega s^2) + 2as^2\omega r^2\{(r^2 + a^2)\omega - a\}\right]^2. \quad (62)$$

The bases are given by Eq. (54) with $\mathcal{A}$, $\mathcal{A}_{(1)}$, $\mathcal{A}_{(2)}$, $\mathcal{B}$, and $\mathcal{C}$ given by

$$\mathcal{A} = 1 - \omega^2s^2(r^2 + a^2) - \frac{2Mr}{\Sigma}(1 - a\omega s^2)^2,$$
$$\mathcal{A}_{(1)} = \frac{2M\epsilon}{\Sigma^2}(1 - a\omega s^2)^2 - 2r\omega^2s^2,$$
$$\mathcal{A}_{(2)} = -2cs\left[\Delta\omega^2 + \frac{2Mr}{\Sigma^2}\{(r^2 + a^2)\omega - a\}^2\right],$$
$$\mathcal{B} = \frac{2Mras^2}{\Sigma}(1 - a\omega s^2) - (r^2 + a^2)\omega s^2,$$
$$\mathcal{C} = 1 - \frac{2Mr}{\Sigma}(1 - a\omega s^2). \quad (63)$$

As explained earlier for $\omega = 0$, Eqs. (56)–(62) reduce to Eqs. (38)–(41) for motion along $\boldsymbol{\xi}$, the global timelike Killing vector defining a stationary observer.

2. *The equatorial plane $\theta = \pi/2$*

On the equatorial plane the above expressions reduce to

$$\kappa^2 = \frac{\Delta M^2}{r^6} \frac{\left[(a\omega - 1)^2 - \frac{r^3\omega^2}{M}\right]^2}{\left[1 - (r^2 + a^2)\omega^2 - \frac{2M(a\omega-1)^2}{r}\right]^2}, \tag{64}$$

$$\tau_1^2 = \frac{1}{r^2} \frac{\left\{\frac{Ma}{r^2} - \left[\frac{(r^2+2a^2)M}{r^2} - r(1 - \frac{2M}{r})\right]\omega + \frac{Ma(3r^2+a^2)\omega^2}{r^2}\right\}^2}{\left[1 - (r^2 + a^2)\omega^2 - \frac{2M(a\omega-1)^2}{r}\right]^2}, \tag{65}$$

$$\tau_2^2 = 0. \tag{66}$$

The bases are given by Eqs. (54)–(63) with $s = 1$, $c = 0$.

We note that the gyroscopic precession is about $\mathbf{e}_{(3)}$ which is normal to the orbital plane and the precession frequency is given by τ_1 as above.

3. *Geodesic motion and Schiff precession*

Along a geodesic, $\kappa = 0$, whence

$$\omega^{-1} = a \pm \sqrt{\frac{r^3}{M}}. \tag{67}$$

This yields the Keplerian frequency in the Kerr case and the Frenet-Serret invariant for motion along this geodesic is

$$\tau_1^2 = \frac{M}{r^3}. \tag{68}$$

In Eq. (67) the + (−) signs correspond to corotating (counterrotating) orbits. The range of values of r for which these orbits are timelike have been discussed in Sec. 2.2 of [3]. Their analysis shows that the range of r for which counterrotating orbits are timelike requires that the absolute value of a be less than the modulus of $\sqrt{\frac{r^3}{M}}$. It should be noted that as ω approaches the Keplerian value the combination $\mathcal{A}_{(1)}/\kappa\mathcal{A}$ is still well defined leading to $\mathbf{e}_{(1)}$ and $\mathbf{e}_{(3)}$ independent of ω. This allows us to extract the geodetic case as a special instance of our more general motion.

The gyroscopic precession frequency, $\mp|\tau_1|$, is thus $\mp\sqrt{\frac{M}{r^3}}$. This precession is about $\mathbf{e}_{(3)}$ which coincides with the z direction. The orbiting (corotating) observer measures precession relative to $\mathbf{e}_{(1)}$ which coincides with her radius vector which rotates with angular velocity ω given by Eq. (67). The precession angle per unit proper time as computed in the rotating coordinates is therefore

$$\Delta\phi' = \mp\frac{M}{r^3}\sqrt{g_{0'0'}}\,\frac{2\pi}{|\omega|}, \tag{69}$$

where ω is the angular frequency of rotation per unit coordinate time. This agrees with the results of Rindler and Perlick [3]. The base line with respect to which the precession is calculated in the rotating coordinates by them coincides with the Frenet-Serret vector $\mathbf{e}_{(1)}$ on the equatorial plane. Consequently, it leads to the same precession angle $\Delta\phi'$. In order to compute the precession relative to a stationary geometry ("frame of fixed stars") we need to subtract from the precession at the end of one revolution the amount through which $\mathbf{e}_{(1)}$ has rotated with respect to the stationary observer, namely, 2π radians. Following this procedure we arrive at the gyroscopic precession in the Kerr spacetime:

$$\Delta\phi = \mp 2\pi\left[\left(1 - \frac{3M}{r} \pm 2a\sqrt{\frac{M}{r^3}}\right)^{\frac{1}{2}} - 1\right]. \tag{70}$$

In the linear approximation this reduces to the Schiff precession. This agrees with the standard results quoted in literature including Ref. [3].

As discussed earlier in Sec. III A one may want to compute the precession of the orbiting gyroscope with respect to the fiducial gyroscope of the stationary observer. In one revolution of the orbiting gyroscope the latter precesses due to dragging by an amount

$$\Delta\phi_{(\mathbf{drag})} = (-\tau_1)\sqrt{g_{00}}\,\frac{2\pi}{|\omega|}, \tag{71}$$

where τ_1 is given by Eq. (42). This leads to [13]

$$\Delta\phi_{(\mathbf{drag})} = -\frac{2\pi Ma}{r^3}\left(\sqrt{\frac{r^3}{M}} \pm a\right)\left(1 - \frac{2M}{r}\right)^{-\frac{1}{2}}. \tag{72}$$

B. Schwarzschild black hole

The Schwarzschild metric may be obtained from the Kerr metric by setting $a = 0$. Correspondingly the most general case of gyroscopic precession follows from the Kerr expression for $a = 0$.

1. *General Schwarzschild case*

The $a = 0$ limit of Eqs. (56)–(63) yields

$$\kappa^2 = r^2 \frac{[(1-\frac{2M}{r})(\frac{M}{r^3} - \omega^2 s^2)^2 + \omega^4 s^2 c^2]}{(1-\frac{2M}{r} - r^2\omega^2 s^2)^2}, \tag{73}$$

$$\tau_1^2 = \omega^2 s^2 \frac{(1-\frac{2M}{r})[(\frac{M}{r^3} - \omega^2 s^2)(1-\frac{3M}{r}) - \omega^2 c^2]^2}{(1-\frac{2M}{r} - r^2\omega^2 s^2)^2[(1-\frac{2M}{r})(\frac{M}{r^3} - \omega^2 s^2)^2 + \omega^4 s^2 c^2]}, \tag{74}$$

$$\tau_2^2 = \frac{\omega^2 M^2 c^2}{r^6[(1-\frac{2M}{r})(\frac{M}{r^3} - \omega^2 s^2)^2 + \omega^4 s^2 c^2]}. \tag{75}$$

The Frenet-Serret frame is given by

$$e^a_{(0)} = \frac{1}{\sqrt{1-\frac{2M}{r} - r^2\omega^2 s^2}}\,(1,0,0,\omega),$$

$$e^a_{(1)} = \frac{1}{[(1-\frac{2M}{r})(\frac{M}{r^3} - \omega^2 s^2)^2 + \omega^4 s^2 c^2]^{1/2}}\left(0, \left(1-\frac{2M}{r}\right)\left(\frac{M}{r^3} - \omega^2 s^2\right), \frac{\omega^2 cs}{r}, 0\right),$$

$$e^a_{(2)} = \frac{1}{rs\sqrt{(1-\frac{2M}{r})(1-\frac{2M}{r} - \omega^2 r^2 s^2)}}\left(\omega r^2 s^2, 0, 0, -\left(1-\frac{2M}{r}\right)\right),$$

$$e^a_{(3)} = \frac{\sqrt{1-\frac{2M}{r}}}{r[(1-\frac{2m}{r})(\frac{M}{r^3} - \omega^2 s^2)^2 + \omega^4 c^2 s^2]^{1/2}}\left(0,\ csr\omega^2,\ \frac{M}{r^3} - \omega^2 s^2,\ 0\right). \tag{76}$$

2. *The equatorial plane*

Most commonly, the precession is computed for orbits in the equatorial plane for which $\theta = \pi/2$. Equations (73)–(75) reduce when $\theta = \frac{\pi}{2}$ to

$$\kappa^2 = \frac{r^2(1-\frac{2M}{r})(\frac{M}{r^3} - \omega^2)^2}{(1-\frac{2M}{r} - r^2\omega^2)^2}, \tag{77}$$

$$\tau_1^2 = \omega^2 \frac{(1-\frac{3M}{r})^2}{(1-\frac{2M}{r} - r^2\omega^2)^2}, \tag{78}$$

$$\tau_2^2 = 0. \tag{79}$$

The bases vectors of the Frenet-Serret frame obtain by inserting $s = 1, c = 0$ in Eq. (76). The ω independence of $\mathbf{e}_{(1)}$ and $\mathbf{e}_{(3)}$ mentioned earlier may be noted more transparently in this instance.

The gyroscopic precession in this case is

$$\tau_1 = \omega\left(1-\frac{3M}{r}\right)\left(1-\frac{2M}{r} - r^2\omega^2\right)^{-1} \tag{80}$$

and

$$\Delta\phi = -2\pi\left[\left(1-\frac{3M}{r}\right)\left(1-\frac{2M}{r} - r^2\omega^2\right)^{-1/2} - 1\right]. \tag{81}$$

3. *Fokker-de Sitter precession*

Along a geodesic $\kappa = 0$ so that we recover the Keplerian frequency

$$\omega^2 = \frac{M}{r^3}.$$

In this case,

$$\tau_1^2 = \omega^2$$

so that the orbital gyroscopic precession frequency is ω, the same as the angular speed ω. In one orbital revolution, the gyroscope rotates by

$$\Delta\phi = -2\pi\left[\left(1-\frac{3M}{r}\right)^{1/2} - 1\right]. \tag{82}$$

C. Minkowski spacetime

1. *The general case*

This corresponds to $M = 0$ in Eqs. (73)–(75) whence

$$\kappa^2 = \frac{r^2\omega^4 s^2}{(1-r^2\omega^2 s^2)^2}, \tag{83}$$

$$\tau_1^2 = \frac{\omega^2}{(1-r^2\omega^2 s^2)^2}, \tag{84}$$

$$\tau_2^2 = 0, \tag{85}$$

while Eqs. (76) reduce to

$$e^a_{(0)} = \frac{1}{\sqrt{1-r^2 s^2\omega^2}}(1,0,0,\omega),$$

$$e^a_{(1)} = \left(0, -s, \frac{c}{r}, 0\right),$$

$$e^a_{(2)} = \frac{1}{rs\sqrt{1-\omega^2 r^2 s^2}}(\omega r^2 s^2, 0, 0, -1),$$

$$e^a_{(3)} = \left(0, c, -\frac{s}{r}, 0\right). \tag{86}$$

Note that τ_2 vanishes identically. Therefore, the precession is about the normal to the orbital plane as should be expected from the symmetry of the situation.

2. Thomas precession

The above expressions reduce on the $\theta = \pi/2$ plane to

$$\kappa^2 = \frac{r^2\omega^4}{(1-r^2\omega^2)^2}, \tag{87}$$

$$\tau_1^2 = \frac{\omega^2}{(1-r^2\omega^2)^2}, \tag{88}$$

$$\tau_2^2 = 0, \tag{89}$$

leading to the familiar expression for Thomas precession:

$$\Delta\phi = -2\pi\,[(1-r^2\omega^2)^{-1/2} - 1]. \tag{90}$$

As expected the "Keplerian" analogue is $\omega = 0$ in which case there is no precession at all.

In the above sections we have shown how the general Frenet-Serret formalism can be adapted to retrieve the results discussed in, e.g., Ref. [3]. Motions more general than geodetic or confined to the equatorial plane are easy to include and formulas corresponding to these cases have also been exhibited.

D. Globally hypersurface orthogonal stationary trajectories (GHOST's)

The Kerr spacetime admits an important congruence which conforms to our definition of quasi-Killing vector fields. Observers adapted to this congruence have been called locally nonrotating observers (LNRO's) or zero angular momentum observers (ZAMO's). Considerable insight into the physical significance of phenomena occurring in the Kerr spacetime is gained by studying them with reference to the above observers [1]. In the broader context of orthogonal transitivity, it was shown [14] that this congruence consists of what we may term as globally hypersurface orthogonal stationary trajectories or GHOST's, with t=const being the hypersurfaces to which they are orthogonal. Therefore, the vorticity of the congruence identically vanishes, so that the connecting vector between two adjacent trajectories does not precess with respect to the Fermi-Walker transported gyroscopes.

The quasi-Killing vector corresponding to the LNRO, ZAMO, or GHOST is defined by

$$\boldsymbol{\chi} = \boldsymbol{\xi} + \omega\boldsymbol{\eta}, \tag{91}$$

$$\omega = -\frac{\xi\cdot\eta}{\eta\cdot\eta} = -\frac{g_{03}}{g_{33}}. \tag{92}$$

We note that $\boldsymbol{\chi}$ is timelike down to the event horizon on which it becomes null.

As mentioned earlier, the vorticity of this congruence is zero, so that

$$\Omega^a_{\mathtt{GHOST}} = 0 = \frac{1}{2\sqrt{-g}}\,\epsilon^{abcd}\,u_b u_{c;d} = \frac{1}{2\sqrt{-g}}\,\epsilon^{abcd}\,u_b F_{cd} + \frac{e^{2\psi}}{2\sqrt{-g}}\,\epsilon^{abcd}\,\xi_b\eta_c\omega_{,d} \tag{93}$$

$$= \omega^a_{(\mathbf{FS})} + \Omega^a_{(\mathbf{prec})}\,. \tag{94}$$

Consequently, the connecting vector between two neighboring trajectories belonging to this congruence does not precess relative to the Fermi-Walker transported gyroscopes. Further, $\boldsymbol{\omega}_{(\mathbf{FS})}$ is the precession of the Frenet-Serret frame with respect to the gyroscopes. Therefore, precession of the gyroscopes relative to the Frenet-Serret frame is given by $-\boldsymbol{\omega}_{(\mathbf{FS})}$ which is equal to $\boldsymbol{\Omega}_{(\mathbf{prec})}$ *only for the irrotational congruence.* The expression for $\boldsymbol{\omega}_{(\mathbf{FS})}$ is the same as one would obtain if the particular trajectory is treated as a member of the *Killing congruence* obtained by taking ω of the trajectory as a constant for the entire congruence. This also means that the Frenet-Serret frame is rigidly attached to the connecting vector associated with this *Killing congruence.* The expression $\Omega_{(\mathbf{prec})}$ is exactly the same as given in [1] which is the precession of the gyroscope with respect to the locally orthogonal triad of [1] adapted to the irrotational congruence. To sum up, because of the vanishing vorticity, the connecting vector between adjacent observers following the irrotational congruence is locked on to the inertial system of gyroscopes and does not precess with respect to the latter. However, gyroscopes do precess relative to the Frenet-Serret frame and the latter are not inertial. This precession frequency in this case is given by two equivalent expressions one of them involving derivatives of ω. As we shall show later, the Frenet-Serret triad coincides with the triad defined in [1] on the equatorial plane but differs from it by a constant spatial rotation for $\theta \neq \frac{\pi}{2}$.

We may note in passing that in the case of a LNRO Thorne and MacDonald [15] write down the Fermi-Walker time derivative of any vector orthogonal to u^a as

$$D_\tau \mathbf{M} = \alpha^{-1}[\underset{\mathbf{t}}{\mathcal{L}}\,\mathbf{M} + \omega\underset{\mathbf{m}}{\mathcal{L}}\,\mathbf{M} + \tfrac{1}{2}(\mathbf{m}\times\nabla\omega)\times\mathbf{M}], \tag{95}$$

where

$$\mathbf{m} \equiv \boldsymbol{\eta} \text{ and } \alpha = \sqrt{\boldsymbol{\chi}\cdot\boldsymbol{\chi}}. \tag{96}$$

If $\mathbf{M}$ is taken as the Frenet-Serret spatial triad then $\underset{\mathbf{t}}{\mathcal{L}}\,\mathbf{M} = \underset{\mathbf{m}}{\mathcal{L}}\,\mathbf{M} = 0$ and the formula reduces to the precession of the triad relative to the gyroscopes with frequency $\frac{1}{2}(\mathbf{m}\times\nabla\omega)$ which is an equivalent form of our expression for the precession frequency.

We now consider the congruence within the framework of our formalism. It should be clear from our discussion of the quasi-Killing congruence and the section on the use of rotating coordinates that all our formulas are applicable when referring to *a particular fixed curve* of any congruence—in particular the GHOST. Consequently, to calculate the precession of gyroscopes relative to a GHOST we proceed exactly as in the earlier cases and use for ω the expression appropriate to a GHOST, i.e., $\omega = -g_{03}/g_{33}$. Since none of our formulas involve differentiation of ω the same expressions Eqs. (49)–(54) give the formula for precession of a gyroscope relative to the Frenet-Serret frame of the GHOST. Thus we obtain

$$\kappa^2 = -\frac{1}{4}\frac{g^{ab}(\frac{\Delta_3}{g_{33}})_{,a}(\frac{\Delta_3}{g_{33}})_{,b}}{(\frac{\Delta_3}{g_{33}})^2}, \tag{97}$$

$$\tau_1^2 = \frac{g_{33}^2}{4\Delta_3}\frac{[g^{ab}(\frac{\Delta_3}{g_{33}})_{,a}(\frac{g_{03}}{g_{33}})_{,b}]^2}{g^{ab}(\frac{\Delta_3}{g_{33}})_{,a}(\frac{\Delta_3}{g_{33}})_{,b}}, \tag{98}$$

$$\tau_2^2 = \frac{g_{33}^2}{4\Delta_3 g_{11}g_{22}}\frac{[(\frac{\Delta_3}{g_{33}})_{,1}(\frac{g_{03}}{g_{33}})_{,2} - (\frac{\Delta_3}{g_{33}})_{,2}(\frac{g_{03}}{g_{33}})_{,1}]^2}{g^{ab}(\frac{\Delta_3}{g_{33}})_{,a}(\frac{\Delta_3}{g_{33}})_{,b}}, \tag{99}$$

$$\begin{aligned} e^a_{(0)} &= \left(\frac{g_{33}}{\Delta_3}\right)^{1/2}\left(1,0,0,-\frac{g_{03}}{g_{33}}\right),\\ e^a_{(1)} &= -\frac{1}{\sqrt{g^{11}(\frac{\Delta_3}{g_{33}})^2_{,1} + g^{22}(\frac{\Delta_3}{g_{33}})^2_{,2}}}\left(0, g^{11}\left(\frac{\Delta_3}{g_{33}}\right)_{,1}, g^{22}\left(\frac{\Delta_3}{g_{33}}\right)_{,2}, 0\right),\\ e^a_{(2)} &= -\left(0,0,0,\frac{1}{\sqrt{-g_{33}}}\right),\\ e^a_{(3)} &= \frac{\sqrt{g^{11}g^{22}}}{\sqrt{g^{11}(\frac{\Delta_3}{g_{33}})^2_{,1} + g^{22}(\frac{\Delta_3}{g_{33}})^2_{,2}}}\left(0, -\left(\frac{\Delta_3}{g_{33}}\right)_{,2}, \left(\frac{\Delta_3}{g_{33}}\right)_{,1}, 0\right). \end{aligned} \tag{100}$$

These observers accelerate ($\kappa \neq 0$) and their Frenet-Serret frames precess with respect to the gyroscopes ($\tau_1, \tau_2 \neq 0$).

The above expression can be calculated explicitly for the Kerr solution. This gives

$$\kappa^2 = \frac{M^2[\mathcal{L}_1^2 + \Delta\mathcal{L}_2^2 s^2c^2]}{\Sigma^3\Delta\mathcal{L}_3^2}, \tag{101}$$

$$\tau_1^2 = \frac{M^2a^2s^2[\Sigma\mathcal{L}_1\mathcal{L}_4 + 2ra^2\mathcal{L}_2\Delta s^2c^2]^2}{\Sigma^5\mathcal{L}_3^2[\mathcal{L}_1^2 + \mathcal{L}_2^2\Delta s^2c^2]}, \tag{102}$$

$$\tau_2^2 = \frac{4M^2r^2a^6s^4c^2\Delta[\mathcal{L}_1 + (r^2+a^2)\mathcal{L}_4]^2}{\Sigma^5\mathcal{L}_3^2[\mathcal{L}_1^2 + \mathcal{L}_2^2\Delta s^2c^2]}, \tag{103}$$

$$\begin{aligned} e^a_{(0)} &= \sqrt{\frac{\mathcal{L}_3}{\Delta}}\left(1,0,0,\frac{2Mra}{\Sigma\mathcal{L}_3}\right),\\ e^a_{(1)} &= \frac{M}{\kappa\Sigma^2\mathcal{L}_3}(0,\mathcal{L}_1,\mathcal{L}_2sc,0),\\ e^a_{(2)} &= -(0,0,0,\sqrt{\mathcal{L}_3}s),\\ e^a_{(3)} &= \frac{M}{\kappa\Sigma^2\mathcal{L}_3}\left(0,-\sqrt{\Delta}\mathcal{L}_2sc,\frac{\mathcal{L}_1}{\sqrt{\Delta}},0\right), \end{aligned} \tag{104}$$

where

$$\mathcal{L}_1 = r^4 - a^4 + \frac{2a^2s^2r^2\Delta}{\Sigma}, \tag{105}$$

$$\mathcal{L}_2 = \frac{2ra^2(r^2+a^2)}{\Sigma}, \tag{106}$$

$$\mathcal{L}_3 = r^2 + a^2 + \frac{2Mra^2s^2}{\Sigma}, \tag{107}$$

$$\mathcal{L}_4 = 2r^2 + (r^2+a^2)\frac{\epsilon}{\Sigma}. \tag{108}$$

Specializing Eqs. (55) to the GHOST it is easy to see after a little computation that the Frenet-Serret frame coincides with the LNRO frame in [1] if $\theta = \pi/2$. The Frenet-Serret frame is in general oriented so that $\mathbf{e}_{(1)}$ is along the direction of the acceleration which is not along the r direction, if the orbit is not confined to the equatorial plane.

E. de Sitter universe

We next apply the formulas to the case of the de Sitter universe whose metric we take in the form

$$ds^2 = \left(1-\frac{r^2}{\alpha^2}\right)dt^2 - \left(1-\frac{r^2}{\alpha^2}\right)^{-1}dr^2 - r^2d\theta^2 - r^2\sin^2\theta\, d\phi^2. \tag{109}$$

Along trajectories of $\boldsymbol{\xi} + \omega\boldsymbol{\eta}$ in this case we obtain

$$\kappa^2 = r^2\frac{[(1-\frac{r^2}{\alpha^2})(\frac{1}{\alpha^2}+\omega^2s^2)^2 + \omega^4s^2c^2]}{(1-\frac{r^2}{\alpha^2} - r^2\omega^2s^2)^2}, \tag{110}$$

$$\tau_1^2 = \frac{\omega^2s^2(1-\frac{r^2}{\alpha^2})(\frac{1}{\alpha^2}+\omega^2)^2}{(1-\frac{r^2}{\alpha^2} - r^2\omega^2s^2)^2[(1-\frac{r^2}{\alpha^2})(\frac{1}{\alpha^2}+\omega^2s^2)^2 + \omega^4s^2c^2]}, \tag{111}$$

$$\tau_2^2 = \frac{\omega^2c^2}{\alpha^4[(1-\frac{r^2}{\alpha^2})(\frac{1}{\alpha^2}+\omega^2s^2)^2 + \omega^4s^2c^2]}, \tag{112}$$

$$e^a_{(0)} = \frac{1}{\sqrt{\mathcal{S}_1}}(1,\ 0,\ 0,\ \omega),$$
$$e^a_{(1)} = \frac{1}{\sqrt{\mathcal{S}_2}}\left(0,\ \left(1-\frac{r^2}{\alpha^2}\right)\left(\frac{1}{\alpha^2}+\omega^2 s^2\right),\ \frac{-\omega^2 sc}{r},\ 0\right),$$
$$e^a_{(2)} = \frac{1}{\sqrt{\mathcal{S}_1}}\left(\frac{\omega r s}{\sqrt{1-\frac{r^2}{\alpha^2}}},\ 0,\ 0,\ -\frac{\sqrt{1-\frac{r^2}{\alpha^2}}}{rs}\right),$$
$$e^a_{(3)} = \frac{\sqrt{1-\frac{r^2}{\alpha^2}}}{\sqrt{\mathcal{S}_2}}\left(0,\ \omega^2 sc,\ \frac{1}{r}(\frac{1}{\alpha^2}+\omega^2 s^2),\ 0\right), \qquad (113)$$

where

$$\mathcal{S}_1 = \sqrt{1-\frac{r^2}{\alpha^2}-\omega^2 r^2 s^2}, \qquad (114)$$

$$\mathcal{S}_2 = \left(1-\frac{r^2}{\alpha^2}\right)\left(\frac{1}{\alpha^2}+\omega^2 s^2\right)^2 + \omega^4 s^2 c^2. \qquad (115)$$

It is easy to see that there is no analogue of the Keplerian orbits. This is related to the fact that the "potential g_{00}" is proportional to a positive power of r rather than a negative power as in the Schwarzschild case. An arbitrary ω leads to precession analogous to Thomas precession in flat space but more complicated due to the curvature of the spatial sections.

V. STATIONARY CYLINDRICALLY SYMMETRIC SPACETIMES

In this section we extend the treatment of the previous section to spacetimes which in addition to the Killing vector $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ have yet another Killing vector $\boldsymbol{\mu}$ representing translation invariance in the z direction. A well-known example is the Gödel solution as well as metrics representing solutions with cylindrical symmetry. As discussed in the beginning all our earlier results obtain in this instance and hence we write down without proof the main expressions.

We start with the standard form of the line element in this case as given by

$$ds^2 = g_{00}dt^2 + 2g_{03}\,dt\,d\phi + g_{33}\,d\phi^2 + 2g_{02}\,dt\,dz + g_{22}dz^2 + g_{11}d\rho^2, \qquad (116)$$

where g_{ab} are functions of ρ only, since we are in coordinates adapted to the Killing vectors $\boldsymbol{\xi}$, $\boldsymbol{\eta}$, and $\boldsymbol{\mu}$. In this case we have

$$g \equiv \det(g_{ab}) = g_{11}\Delta_{23}, \qquad (117)$$

where

$$\Delta_{23} \equiv g_{00}g_{33}g_{22} - g_{22}g_{03}^2 - g_{33}g_{02}^2. \qquad (118)$$

Further,

$$g^{ab} = \frac{1}{\Delta_{23}}\begin{pmatrix} g_{22}\,g_{33} & -g_{03}\,g_{22} & -g_{02}\,g_{33} & 0 \\ -g_{03}\,g_{22} & \Delta_2 & g_{02}g_{03} & 0 \\ -g_{02}\,g_{33} & g_{02}g_{03} & \Delta_3 & 0 \\ 0 & 0 & 0 & \Delta_{23}/g_{11} \end{pmatrix}, \qquad (119)$$

$$\Delta_3 \equiv g_{00}\,g_{33} - g_{03}^2\ , \quad \Delta_2 \equiv g_{00}\,g_{22} - g_{02}^2. \qquad (120)$$

Proceeding as before we first compute κ, τ_1, and τ_2 for an observer whose world line is $\boldsymbol{\xi}$. We obtain

$$\kappa^2 = -\frac{g^{11}}{4}\frac{g_{00,1}^2}{g_{00}^2}, \qquad (121)$$

$$\tau_1^2 = \frac{-g^{11}}{4\Delta_{23}g_{00}^2}[\,g_{00}(g_{02}g_{03,1}-g_{03}g_{02,1})^2 - g_{22}(g_{03}g_{00,1}-g_{00}g_{03,1})^2 - g_{33}(g_{02}g_{00,1}-g_{00}g_{02,1})^2], \qquad (122)$$

$$\tau_2^2 = 0. \qquad (123)$$

For completeness we also compute the Frenet-Serret bases for these metrics. It is given by

$$e^a_{(0)} = \frac{1}{\sqrt{g_{00}}}\,(1,\ 0,\ 0,\ 0\,),$$
$$e^a_{(1)} = (\,0,\ \sqrt{-g^{11}},\ 0,\ 0\,),$$
$$e^a_{(2)} = -\frac{\sqrt{-g^{11}}}{2\sqrt{g_{00}}\Delta_{23}\tau_1}\,(a_2,\ 0,\ b_2,\ c_2),$$
$$e^a_{(3)} = \frac{\sqrt{-g^{11}}}{2g_{00}\sqrt{\Delta_{23}}\tau_1}\,(a_3,\ 0,\ b_3,\ c_3), \qquad (124)$$

where

$$a_2 = \frac{1}{g_{00}}[\,g_{22}g_{03}(g_{03}g_{00,1}-g_{00}g_{03,1}) + g_{33}g_{02}(g_{02}g_{00,1}-g_{00}g_{02,1})], \qquad (125)$$

$$b_2 = g_{33}(g_{00}g_{02,1}-g_{02}g_{00,1}) + g_{03}(g_{02}g_{03,1}-g_{03}g_{02,1}), \qquad (126)$$

$$c_2 = g_{22}(g_{00}g_{03,1}-g_{03}g_{00,1}) + g_{02}(g_{03}g_{02,1}-g_{02}g_{03,1}), \qquad (127)$$

$$a_3 = g_{03}g_{02,1} - g_{02}g_{03,1}, \qquad (128)$$
$$b_3 = g_{00}g_{03,1} - g_{03}g_{00,1}, \qquad (129)$$
$$c_3 = g_{02}g_{00,1} - g_{00}g_{02,1}. \qquad (130)$$

It should be noted that since $\tau_2 = 0$, $\mathbf{e}_{(3)}$ cannot be obtained by the usual Frenet-Serret process of differentiation but in this case has been obtained just by orthonormality with the $\mathbf{e}_{(i)}$ $(i = 0, 1, 2)$. Adapting the procedure of Sec. II to the quasi-Killing congruence

$$\zeta^a \equiv \xi^a + \omega\eta^a + v\mu^a \equiv e^{-\psi} e_0^a, \tag{131}$$

where

$$\underset{\zeta}{\mathcal{L}}\,\omega = \underset{\zeta}{\mathcal{L}}\,v = 0 \tag{132}$$

we obtain

$$\dot{e}^a_{(m)} = F^a{}_b\, e^b_{(m)}, \tag{133}$$

where

$$F_{ab} \equiv e^{\psi}(\xi_{a;b} + \omega\eta_{a;b} + v\mu_{a;b}), \tag{134}$$

and

$$\Omega^a = \omega^a + \tilde{H}^{ab} e_{(0)b}, \tag{135}$$

where

$$\omega^a = \tilde{F}^{ab} e_{(0)b},$$

and

$$H_{dc} \equiv e^{\psi}\,[\omega_{[,d}\eta_{c]} + v_{[,d}\mu_{c]}]. \tag{136}$$

The general quasi-Killing trajectories along $\boldsymbol{\zeta}$ represent helical orbits. Nevertheless, the computation of κ, τ_1, τ_2 for ζ^a involves a similar trick as before. Under the coordinate transformation

$$t = t', \tag{137}$$

$$\phi = \phi' + \omega t', \tag{138}$$

$$z = z' + vt', \tag{139}$$

the metric transforms to

$$g_{0'0'} = g_{00} + 2\omega g_{03} + \omega^2 g_{33} + 2v g_{02} + v^2 g_{22} \equiv \mathcal{D}, \tag{140}$$

$$g_{0'3'} = g_{03} + \omega g_{33} \equiv \mathcal{B} \tag{141}$$

$$g_{0'2'} = g_{02} + v g_{22} \equiv \mathcal{E}, \tag{142}$$

$$g_{3'3'} = g_{33},\; g_{2'2'} = g_{22}, \tag{143}$$

where $g_{a'b'}$ are independent of t', ϕ', and z'. The Killing vector $\boldsymbol{\zeta} = (1,0,0,0)$ corresponds in the old coordinates to $\boldsymbol{\xi} + \omega\boldsymbol{\eta} + v\boldsymbol{\mu}$ and consequently we can use Eqs. (121)–(123) to evaluate κ, τ_1, τ_2 by using $g_{a'b'}$ instead of g_{ab} in the equations. This gives

$$\kappa^2 = -\frac{g^{11}\,\mathcal{D}^2_{(1)}}{4\,\mathcal{D}^2}, \tag{144}$$

$$\tau_1^2 = \frac{-g^{11}}{4\Delta_{23}\mathcal{D}^2}\Big[\mathcal{D}(\mathcal{E}\mathcal{B}_{(1)} - \mathcal{B}\mathcal{E}_{(1)})^2 - g_{22}(\mathcal{B}\mathcal{D}_{(1)} - \mathcal{D}\mathcal{B}_{(1)})^2 - g_{33}(\mathcal{E}\mathcal{D}_{(1)} - \mathcal{D}\mathcal{E}_{(1)})^2\Big], \tag{145}$$

$$\tau_2^2 = 0, \tag{146}$$

where

$$\mathcal{D}_{(1)} \equiv g_{00,1} + 2\omega g_{03,1} + \omega^2 g_{33,1} + 2v g_{02,1} + v^2 g_{22,1}, \tag{147}$$

$$\mathcal{E}_{(1)} \equiv g_{02,1} + v g_{22,1}, \tag{148}$$

$$e^a_{(0)} = \frac{1}{\sqrt{\mathcal{D}}}(1,\,0,\,v,\,\omega),$$
$$e^a_{(1)} = (0,\,\sqrt{-g^{11}},\,0,\,0),$$
$$e^a_{(2)} = -\frac{\sqrt{-g^{11}}}{2\sqrt{\mathcal{D}}\Delta_{23}\tau_1}(a_2',\,0,\,b_2' + va_2',\,c_2' + \omega a_2'),$$
$$e^a_{(3)} = \frac{\sqrt{-g^{11}}}{2\mathcal{D}\sqrt{\Delta_{23}}\tau_1}(a_3',\,0,\,b_3' + va_3',\,c_3' + \omega a_3'), \tag{149}$$

where a_i' and b_i' $(i = 2,3)$ refer to a_i and b_i with g_{ab} replaced by the corresponding $g_{a'b'}$.

The geodesics are determined by $\kappa = 0$, i.e., $\mathcal{D}_{(1)} = 0$ and in this case τ_1^2 simplifies to

$$\tau_1^2 = \frac{g_{11}}{4\Delta_{23}\mathcal{D}}[\,(g_{02}^2 - g_{22}\mathcal{A})\mathcal{B}^2_{(1)} + (g_{03}^2 - g_{33}\mathcal{F})\mathcal{E}^2_{(1)} - 2\mathcal{E}\mathcal{B}\,\mathcal{B}_{(1)}\mathcal{E}_{(1)}], \tag{150}$$

where

$$\mathcal{F} \equiv g_{00} + 2v g_{02} + v^2 g_{22}. \tag{151}$$

A further simplification obtains if the spacetimes under consideration satisfy

$$g_{02,\rho} = g_{22,\rho} = 0. \tag{152}$$

In this case the Keplerian orbits for $\boldsymbol{\zeta}$ are determined by $\mathcal{A}_{(1)} = 0$ as for the $\boldsymbol{\chi}$ congruence and τ_1^2 reduces to

$$\tau_1^2 = -\frac{g^{11}\mathcal{B}^2_{(1)}(g_{02}^2 - g_{22}\mathcal{A})}{4\Delta_{23}\mathcal{D}}. \tag{153}$$

The further restriction $g_{02} = 0$ finally leads to a form useful for the Gödel case:

$$\tau_1^2 = \frac{g^{11}\mathcal{B}^2_{(1)}\mathcal{A}}{4\Delta_3(\mathcal{A} + v^2 g_{22})}. \tag{154}$$

The general expressions above may be simplified in three particular cases: (i) $g_{02} = 0$, (ii) $g_{02,\rho} = g_{22,\rho} = 0$, (iii) $g_{02} = 0$, $g_{22,\rho} = 0$. The Gödel universe belongs to category (iii) while the cylindrical vacuum metrics of [16] belong to class (i).

We next apply these general considerations to the Gödel case and finally, to the general cylindrically symmetric vacuum metrics.

A. Gödel universe

The Gödel universe is described by the line element

$$ds^2 = 4R^2\,[\,dt^2 + 2\sqrt{2}\,S^2 d\phi dt - (S^2 - S^4)\,d\phi^2 - dr^2 - dz^2], \tag{155}$$

where $S \equiv \sinh r, C \equiv \cosh r$.

1. General spiraling ζ trajectories

Adapting the formulas of the previous section for observers moving along $\boldsymbol{\zeta}$ we have

$$\kappa^2 = \frac{\omega^2 S^2 C^2\left[2\sqrt{2} - \omega(1-2S^2)\right]^2}{4R^2\mathcal{G}_1^2}, \tag{156}$$

$$\tau_1^2 = \frac{1}{4R^2\mathcal{G}_1^2}\{\mathcal{G}_2^2 + v^2(\mathcal{G}_3^2 + \mathcal{G}_4^2\mathcal{G}_1)\}, \tag{157}$$

$$\tau_2^2 = 0, \tag{158}$$

where

$$\mathcal{G}_1 \equiv 1 + 2\sqrt{2}\omega S^2 - \omega^2 S^2(1-S^2) - v^2, \tag{159}$$

$$\mathcal{G}_2 \equiv \omega(1-2S^2) - \sqrt{2}(1+\omega^2 S^4) + v^2\{\sqrt{2} - \omega(1-2S^2)\}, \tag{160}$$

$$\mathcal{G}_3 \equiv \omega S\sqrt{1-S^2}\{2\sqrt{2} - \omega(1-2S^2)\}, \tag{161}$$

$$\mathcal{G}_4 \equiv \sqrt{2} - \omega(1-2S^2). \tag{162}$$

2. Geodesics

Along a geodesic $\kappa = 0$ yielding for the "Keplerian" frequency

$$\omega = \frac{2\sqrt{2}}{1-2S^2}, \tag{163}$$

the same as for the $\boldsymbol{\xi}$ orbits [3]. However, the gyroscopic precession frequency contains the signature of the z motion and is given by

$$\tau_1^2 = \frac{1-4S^2C^2}{2R^2[1-4S^2C^2 - v^2(1-2S^2)^2]}. \tag{164}$$

3. The χ motion

The $v = 0$ limit of the Eqs. (156)–(162) leads us to the motion along $\boldsymbol{\chi}$ lines and we obtain

$$\kappa^2 = \frac{\omega^2 S^2 C^2}{4R^2}\left[\frac{2\sqrt{2} - \omega(1-2S^2)}{1+2\sqrt{2}\omega S^2 - \omega^2 S^2(1-S^2)}\right]^2, \tag{165}$$

$$\tau_1^2 = \frac{(\sqrt{2} - \omega(1-2S^2) + \sqrt{2}\,\omega^2 S^4)^2}{4R^2[1+2\sqrt{2}\omega S^2 - \omega^2 S^2(1-S^2)]^2}. \tag{166}$$

The precession frequency for motion along circular geodesics takes the simple form

$$\tau_1^2 = \frac{1}{2R^2} \tag{167}$$

yielding for the precession

$$\Delta\phi = -\pi[(1-\sinh^2 2r)^{1/2} - 2] \tag{168}$$

in agreement with earlier results [3].

Finally in the case of stationary observers ($\boldsymbol{\xi}$ lines) κ=0, i.e., the t lines are geodesics. In this case we obtain

$$\tau_1^2 = \frac{1}{2R^2}. \tag{169}$$

Thus the Frenet-Serret frame of the stationary observers precesses relative to the gyroscopes and reveals the rotation intrinsic to the Gödel universe. Following the procedure outlined in Sec. IV A 3 the precession due to dragging is

$$\Delta\phi_{(\mathbf{drag})} = \pi(1 - 2\sinh^2 r) \tag{170}$$

in agreement with [3].

4. GHOST

We conclude by a consideration of precession along GHOST's. The angular velocity of these observers correspond to

$$\omega = -\frac{g_{03}}{g_{33}} = \frac{\sqrt{2}}{1-S^2} \tag{171}$$

leading to

$$\kappa^2 = \frac{S^2}{R^2C^2(1-S^2)^2} \tag{172}$$

and

$$\tau_1^2 = \frac{S^4}{2R^2(1-S^2)^2}. \tag{173}$$

B. Stationary cylindrically symmetric vacuum spacetimes

The stationary cylindrically symmetric vacuum spacetimes have been given in an elegant compact form by Vishveshwara and Winicour [16] as

$$ds^2 = e^{2\varphi}(d\tau^2 + d\sigma^2) + \lambda_{00}dt^2 + 2\lambda_{03}dtd\phi + \lambda_{33}d\phi^2, \tag{174}$$

where

$$\lambda_\alpha = A_\alpha\tau^{1+b} + B_\alpha\tau^{1-b} \quad \alpha = 00,\ 03,\ 33, \tag{175}$$

$$e^{2\varphi} = c\tau^{b^2-1}, \quad \tau = \sqrt{2}\rho, \quad \sigma = \sqrt{2}z. \tag{176}$$

The coefficients A_α and β_α satisfy the algebraic relations

$$A_{00}A_{33} - A_{03}^2 = B_{00}B_{33} - B_{03}^2 = 0, \tag{177}$$

$$A_{00}B_{33} + A_{33}B_{00} - 2A_{03}B_{03} = -\tfrac{1}{2}. \tag{178}$$

The mass per unit length m and angular momentum per unit length j are given by

$$m = \tfrac{1}{4} + \tfrac{1}{2}b(A_{33}B_{00} - A_{00}B_{33}), \tag{179}$$

$$j = \tfrac{1}{2}b(A_{03}B_{33} - A_{33}B_{03}). \tag{180}$$

1. The ζ trajectories

For completeness we write down the Frenet-Serret invariants for the line element listed above. They turn out to be

$$\kappa^2 = -\frac{1}{4c\tau^{b^2+1}}\left[\frac{(1+b)\tau^b A_{0'0'} + (1-b)\tau^{-b}B_{0'0'} + (b^2-1)v^2 c\tau^{b^2-2}}{A_{0'0'}\tau^b + B_{0'0'}\tau^{-b} + v^2 c\tau^{b^2-2}}\right]^2, \tag{181}$$

$$\tau_1^2 = \frac{1}{\tau^3\mathcal{W}^2}\left[\frac{-2\mathcal{W}_1^2}{c\tau^{b^2-2}} + v^2(\mathcal{W}\mathcal{W}_3^2 - \mathcal{W}_2^2)\right], \tag{182}$$

where

$$A_{0'0'} \equiv A_{00} + 2\omega A_{03} + \omega^2 A_{33}, \tag{183}$$

$$B_{0'0'} \equiv B_{00} + 2\omega B_{03} + \omega^2 B_{33}, \tag{184}$$

$$A_{0'3'} \equiv A_{03} + \omega A_{03}\ ;\ B_{0'3'} \equiv B_{03} + \omega B_{03}, \tag{185}$$

$$\mathcal{W} \equiv A_{0'0'}\tau^b + B_{0'0'}\tau^{-b} + 2v^2 c\tau^{b^2-2}, \tag{186}$$

$$\mathcal{W}_1 \equiv b(A_{0'0'}B_{0'3'} - A_{0'3'}B_{0'0'}) - v^2 c\tau^{b^2-2}\{(1+b)(2-b)A_{0'3'}\tau^b + (1-b)(2+b)B_{0'3'}\tau^{-b}\}, \tag{187}$$

$$\mathcal{W}_2 \equiv \sqrt{A_{33}\tau^b + B_{33}\tau_{-b}}[(1+b)(2-b)A_{0'0'}\tau^b + (1-b)(2+b)B_{0'0'}\tau^{-b}], \tag{188}$$

$$\mathcal{W}_3 \equiv (1+b)(2-b)A_{0'3'}\tau^b + (1-b)(2+b)B_{0'3'}\tau^{-b}. \tag{189}$$

2. Observers with arbitrary constant angular velocity along χ

In this case we obtain

$$\kappa^2 = -\frac{1}{4c\tau^{b^2+1}}\left[\frac{(1+b)A_{0'0'}\tau^b + (1-b)B_{0'0'}\tau^{-b}}{A_{0'0'}\tau^b + B_{0'0'}\tau^{-b}}\right]^2, \tag{190}$$

$$\tau_1^2 = -\frac{2b^2}{c\tau^{b^2+1}}\left[\frac{(A_{0'0'}B_{0'3'} - A_{0'3'}B_{0'0'})}{A_{0'0'}\tau^b + B_{0'0'}\tau^{-b}}\right]^2. \tag{191}$$

Note

$$\begin{aligned} b(A_{0'0'}B_{0'3'} - A_{0'3'}B_{0'0'}) &= b[(A_{00}B_{03} - A_{03}B_{00} + \omega(A_{00}B_{33} - A_{33}B_{00}) + \omega^2(A_{03}B_{33} - A_{33}B_{03})] \\ &= \tfrac{1}{2}[2b(A_{00}B_{03} - A_{03}B_{00}) - \omega(4m-1) + 4j\omega^2]. \end{aligned} \tag{192}$$

3. Keplerian geodesics

These are determined by $\kappa = 0$ yielding

$$\omega = \frac{[-(1+b)A_{03}\tau^b + (1-b)B_{03}\tau^{-b}] \pm (\frac{1-b^2}{2})^{\frac{1}{2}}}{(1+b)A_{33}\tau^b + (1-b)B_{33}\tau^{-b}}. \tag{193}$$

Note that real roots are possible only for $b^2 < 1$ which is consistent with the fact that the potential g_{00} is a function of negative powers of τ only for these values. The precession is obtained to be

$$\tau_1^2 = -\frac{(1-b^2)}{4c\,\tau^{1+b^2}}. \tag{194}$$

4. GHOST

Finally we look at these special trajectories for the cylindrically symmetric vacuum metrics. In this instance

$$\omega = -\frac{g_{03}}{g_{33}} = -\frac{A_{03}\,\tau^b + B_{03}\,\tau^{-b}}{A_{33}\,\tau^b + B_{33}\,\tau^{-b}} \tag{195}$$

and the acceleration becomes

$$\kappa^2 = -\frac{1}{4c\tau^{b^2+1}}\left[\frac{(1-b)A_{33}\tau^b + (1+b)B_{33}\tau^{-b}}{A_{33}\tau^b + B_{33}\tau^{-b}}\right]^2. \tag{196}$$

The precession takes the form

$$\tau_1^2 = -\frac{8j^2}{c\tau^{b^2+1}(A_{33}\tau^b + B_{33}\tau^{-b})^2}. \tag{197}$$

The gyroscopic precession given by τ_1 is proportional in this case to the specific angular momentum j. From our previous discussion we know that the connecting vector between two adjacent trajectories of the "irrotational" congruence does not precess with respect to the Fermi-Walker transported gyroscope. In general, the Frenet-Serret triad does precess with respect to the gyroscope or equivalently in this case with respect to the connecting vector. However, if $\tau_1 = 0$ (τ_2 is identically zero) then the Frenet-Serret triad is also nonprecessing with respect to the gyroscope or the connecting vector. This happens when the angular momentum of the source $j = 0$. Thus, the observer can decide if the source is rotating or not by checking whether or not his Frenet-Serret triad, which is Lie transported, precesses with respect to the gyroscope or the connecting vector. Gyroscopic precession along trajectories of the irrotational congruence reveal directly the rotation of the central source.

5. The stationary observers

For completeness we write down the parameters when $\omega = 0$, i.e., the $\boldsymbol{\xi}$ lines. We have

$$\kappa^2 = -\frac{1}{4c\tau^{b^2+1}}\left[\frac{(1+b)A_{00}\tau^b + (1-b)B_{00}\tau^{-b}}{A_{00}\tau^b + B_{00}\tau^{-b}}\right]^2, \quad (198)$$

$$\tau_1^2 = -\frac{2b^2}{c\tau^{b^2+1}}\left[\frac{A_{00}B_{03} - A_{03}B_{00}}{A_{00}\tau^b + B_{00}\tau^{-b}}\right]^2. \quad (199)$$

Equation (199) describes the rotation of the stationary gyroscope due to dragging. Once again this illustrates the very general effect of spacetime rotation on local experiments.

VI. CONCLUSION

As has been mentioned earlier, gyroscopic precession is a phenomenon that has been extensively studied both in flat and curved spacetimes by different methods. The orbits of the gyroscopes in these instances are given by combinations of Killing directions admitted by the spacetimes under consideration. It is found that in these circumstances, the invariant geometrical description of the Frenet-Serret formalism provides a covariant and elegant framework for the study of precession. By extending the formalism applied to Killing trajectories to quasi-Killing trajectories, a large number of cases can be studied in a unified manner. Furthermore, this treatment makes it possible to relate the precession of a gyroscope to the vorticity of a congruence when the gyroscope is transported along a given member of that congruence. An important example is the irrotational congruence admitted by stationary, axisymmetric spacetimes like the Kerr. It is worth pointing out, however, that gyroscopic precession is directly determined by the Frenet-Serret rotation in general. Another aspect of our treatment is the unified description of precession applicable to a whole family of spacetimes. Specifically, precession for orbits with arbitrary constant angular speed has been worked out for the Kerr metric. Starting from this, particular examples have been worked out for the entire Kerr-Schwarzschild-Minkowski spacetimes. Expressions presented are general, exact, and not confined to the equatorial plane. In deriving these results rotating coordinate systems have been used to generate circular orbits from static trajectories. Also, additional interesting cases such as Gödel, de Sitter universe, and general vacuum cylindrical spacetimes have been investigated. It would be interesting to explore in detail the implications of the general results obtained here and their possible astrophysical applications.

[1] C.W. Misner, K.S. Thorne, and J.A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).

[2] K.S. Thorne, R.M. Price, and D.A. MacDonald, *Black Holes - The Membrane Paradigm* (Yale University Press, New Haven, Connecticut, 1986).

[3] W. Rindler and V. Perlick, Gen. Relativ. Gravit. **22**, 1067 (1990).

[4] D. Tsoubelis, A. Economou, and E. Stoghianidis, Phys. Rev. D **36**, 1045 (1987).

[5] N. Ashby and B. Shahid-Saless, Phys. Rev. D **42**, 1118 (1990).

[6] D. Wilkins and M.W. Jacobs, Phys. Rev. D **46**, 3395 (1992).

[7] E. Honig, E.L. Schücking, and C.V. Vishveshwara, J. Math. Phys. **15**, 774 (1974).

[8] B.R. Iyer and C.V. Vishveshwara, Class Quantum. Grav. **5**, 961 (1988).

[9] See *Gravitation* [1], Ex. 6.8, p. 174.

[10] R.T. Jantzen, P. Carini, and D. Bini, Ann. Phys. (N.Y.) **215**, 1 (1992).

[11] S.W. Hawking and G.F.R. Ellis, *The Large Scale Structure of Spacetime* (Cambridge University Press, Cambridge, England, 1973).

[12] Along a general congruence the evolution of the connecting vector is governed by the quantities vorticity, shear and expansion. In the case of a Killing congruence only the vorticity is nonzero. None of the three quantities is zero in the case of an arbitrary quasi-Killing congruence. However, axisymmetric stationary spacetimes, such as the Kerr spacetime, admit an irrotational quasi-Killing congruence for which the vorticity is zero. Expansion would be zero if the coefficient ω_A is Lie derived by the Killing vector η_A associated with it.

[13] Equation (72) corrects a misprint in the formula for precession due to drag on p. 1079 of Ref. [3].

[14] R.D. Greene, E.L. Schücking, and C.V. Vishveshwara, J. Math. Phys. **16**, 153 (1975).

[15] K.S. Thorne and D.A. MacDonald, Mon. Not. R. Astron. Soc. **198**, 339 (1982).

[16] C.V. Vishveshwara and J. Winicour, J. Math. Phys. **18**, 1280 (1977).