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Stability of collinear equilibrium points in the generalised photogravitational elliptic restricted three-body problem

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Abstract. We have examined stability of collinear equilibrium points in the generalised photogravitational elliptic restricted three body problem. The problem is generalised in the sense that the smaller primary is considered as an oblate spheroid and the bigger primary is taken as a source of radiation. We have found the characteristic equation of the problem. We conclude that collinear equilibrium points are unstable

Keywords : stability–collinear points–generalised photogravitational–ERTBP

1. Introduction

Radzievskii (1950) investigated the restricted problem of three bodies taking account of the light pressure. Arnold (1961) studied the stability of equilibrium points in the general elliptic case. Broucke (1969) studied the stability of periodic orbits in the elliptic restricted three-body problem.

Bein (1970) gave stationary solutions in simplified resonance case of the restricted three-body problem. Katsiaris (1973) studied the three-dimensional elliptic problem. Hadjidemetriou (1975) investigated the stability of periodic orbits in the three-body problem. Schuerman (1980) studied the restricted three-body problem including radiation pressure. Hadjidemetriou (1988) investigated periodic orbits and stability. Hagel (1989) gave the integration theory for the elliptic restricted three-body problem. Vokrouhlicky, et. al (1993) studied solar radiation pressure perturbations for earth satellites. Khasan (1996) investigated librational solutions to the photogravitational restricted three body problem by considering both primaries as radiating.

Hence, we thought to investigate the stability of collinear equilibrium points in the generalised photogravitational elliptic restricted three-body problem.

The problem is generalised in the sense that the smaller primary is considered as an oblate spheroid. The bigger primary is taken as a source of radiation. We have found the locations of collinear equilibrium points of the problem. We also have calculated the characteristic equation of the problem and conclude that collinear equilibrium points are unstable.

2. Locations of collinear equilibrium points

The equations of motion for the photogravitational elliptic restricted three-body problem in a dimensionless, barycentric, rotating co-ordinate system with the smaller primary as an oblate spheroid and the bigger primary radiating, are as follows

$$\begin{aligned} \xi'' - 2\eta' &= \partial\Omega^*/\partial\xi \\ \eta'' + 2\xi' &= \partial\Omega^*/\partial\eta \\ \zeta'' &= \partial\Omega^*/\partial\zeta \end{aligned} \tag{2.1}$$

where $\Omega^* = [(\xi^2 + \eta^2)/2 + 1/n^2 \{(1 - \mu)q_1/r_1 + \mu/r_2 + \mu A_2/2r^3{}_2\}]/(1 - e^2)^{1/2}$

Also, q_1 is the mass reduction factor constant,

A_2 is the co-efficient of oblateness and n is the mean motion given by the relation

$$n^2 = \frac{(1+3A_2/2)\,(1+e^2)^{1/2}}{a(1 - e^2)}$$

By an analysis similar to Mc Cuskey (1963) for the existence and positions of equilibrium points we have.

$$\frac{\partial\Omega^*}{\partial\xi} = 0 \qquad \frac{\partial\Omega^*}{\partial\eta} = 0$$

which gives,

$$\xi - \frac{1}{n^2}\left[\frac{(1 - \mu)q_1(\xi + \mu)}{r^3{}_1} + \frac{\mu(\xi + \mu - 1)}{r^3{}_2} + \frac{3\mu A_2(\xi + \mu - 1)}{2r^5{}_2}\right] = 0 \tag{2.2}$$

and $$1 - \frac{1}{n^2}\left[\frac{(1 - \mu)q_1}{r^3{}_1} + \frac{\mu}{r^3{}_2} + \frac{3\mu A_2}{2r^5{}_2}\right] = 0 \tag{2.2}$$

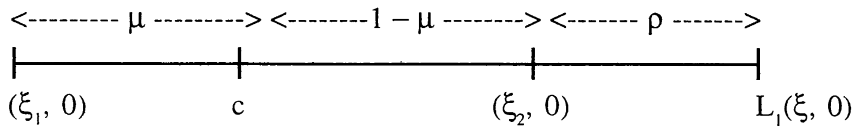
where $r_1{}^2 = (\xi + \mu)^2 + \eta^2 + \zeta^2$

$r^2{}_2 = (\xi + \mu - 1)^2 + \eta^2 + \zeta^2$

The Lagrangian points of the x-axis are given by setting $\eta = 0$, $\zeta = 0$ in the equation (2.2). We have

$$\begin{aligned} &2n^2\xi(\xi + \mu)^2 (\xi + \mu - 1)^4 - 2(1 - \mu)q_1 (\xi + \mu - 1)^4 \\ &\quad - 2\mu(\xi + \mu)^2 (\xi + \mu - 1)^2 - 3\mu A_2(\xi + \mu)^2 = 0 \end{aligned} \tag{2.4}$$

For three libration points on the x-axis we have one root of ξ be greater than ξ_2, another root lies between ξ_1 and ξ_2 and other root is less than ξ_1



Case 1:

Let $\xi > \xi_2$ and consider $\xi - \xi_2 = \rho$; so $\xi - \xi_1 = 1 + \rho$; also $\xi = 1 + \rho - \mu$; Substituting these values in the equation (2.4) we have

$$2n^2\rho^7 + 2n^2(3 - \mu)\rho^6 + 2n^2(3 - 2\mu)\rho^5 + \{2n^2(1 - \mu) - 2(1 - \mu)\, q_1 - 2\mu\}\rho^4 - 4\mu\rho^3 - (2\mu + 3\mu A_2)\rho^2 - 6\mu A_2\rho - 3\mu A_2 = 0 \qquad (2.5)$$

Let γ_1 be the value of ρ in the classical case. i.e. when $e = 0$, $A_2 = 0$ and $q_1 = 1$

For the presence of these terms let the value of ρ be slightly changed and let the new value of ρ be defined by

$\rho = \gamma_1 + \delta_1$, and $\delta_1 << 1$;

Next let $q_1 = 1 - \beta_1$, $\beta_1 << 1$

Substituting the value of ρ in the equation (2.5) we have

$$\delta_1(P_1 + Q_1\beta_1 + R_1A_2) = L_1 + M_1\beta_1 + N_1A_2 \qquad (2.6)$$

where $P_1 = 14n^2\gamma_1^6 + 12n^2(3 - \mu)\gamma_1^5 + 10n^2(3 - 2\mu)\gamma_1^4 + 4\{2n^2(1 - \mu) - 2\}\gamma_1^3 - 12\mu\gamma_1^2 - 4\mu\gamma_1$;

$Q_1 = 8\,(1 - \mu)\gamma_1^3$;

$R_1 = -\,6\mu(1 + \gamma_1)$;

$L_1 = -2n^2\gamma_1^7 - 2n^2(3 - \mu)\gamma_1^6 - 2n^2(3 - 2\mu)\gamma_1^5 - \{2n^2(1 - \mu) - 2\}\gamma_1^4 + 4\mu\gamma_1^3 + 2\mu\gamma_1^2$;

$M_1 = -2\,(1 - \mu)\gamma_1^4$;

and $N_1 = 3\mu(1 + 2\gamma_1 + \gamma_1^2)$,

Now, from the equation (2.6) we have

$$\delta_1 = \frac{L_1 + M_1\beta_1 + N_1A_2}{P_1 + Q_1\beta_1 + R_1A_2}$$

$$= [\,(L_1 + M_1\beta_1 + N_1A_2)\{1 + (Q_1/P_1)\beta_1 + (R_1/P_1)A_2\}^{-1}]\,/P_1$$

$$= [(L_1 + M_1\beta_1 + N_1A_2)\{1 - (Q_1/P_1)\beta_1 - (R_1/P_1)A_2\}/P_1$$

[neglecting the higher order terms]

$$= [L_1 - (Q_1L_1/P_1)\beta_1 - (R_1L_1/P_1)A_2 + M_1\beta_1 + N_1A_2] /P_1$$

$$= [L_1 + (M_1 - Q_1L_1/P_1)\beta_1 + (N_1 - R_1L_1/P_1)A_2]/P_1 \quad (2.7)$$

since $n^2 = (1 + 3A_2/2)(1 + e^2/2)(1 + e^2)/a$

$= (1 + 3e^2/2 + 3A_2/2)/a$ [Neglecting the higher order terms, since A_2 and e^2 are small]

we have $P_1 = 14n^2\gamma_1^6 + 12n^2(3 - \mu)\gamma_1^5 + 10n^2(3 - 2\mu)\gamma_1^4 + 4\{2n^2(1 - \mu) - 2\}\gamma_1^3 - 12\mu\gamma_1^2 - 4\mu\gamma_1$;

Substituting the value of n^2 we have -

$$P_1 = 14\gamma_1^6/a + 12(3 - \mu)\gamma_1^5/a + 10(3 - 2\mu)\gamma_1^4/a + 8\{(1 - \mu)/a - 1\}\gamma_1^3 - 12\mu\gamma_1^2 - 4\mu\gamma_1 + e^2\{21\gamma_1^6/a + 18(3 - \mu)\gamma_1^5/a + 15(3 - 2\mu)\gamma_1^4/a + 12(1 - \mu)\gamma_1^3/a\} + A_2\{21\gamma_1^6/a + 18(3 - \mu)\gamma_1^5/a + 15(3 - 2\mu)\gamma_1^4/a + 12(1 - \mu)\gamma_1^3/a\} ;$$

$$\therefore P_1^{-1} = X_1 + Y_1e^2 + Y_1A_2$$

Where $X_1 = [14\gamma_1^6/a + 12(3 - \mu)\gamma_1^5/a + 10(3 - 2\mu)\gamma_1^4/a + 8\{(1 - \mu)/a - 1\}\gamma_1^3 - 12\mu\gamma_1^2 - 4\mu\gamma_1]^{-1}$

$$Y_1 = -\frac{\{21\gamma_1^6/a + 18(3 - \mu)\gamma_1^5/a + 15(3 - 2\mu)\gamma_1^4/a + 12(1 - \mu)\gamma_1^3/a\}}{[14\gamma_1^6/a + 12(3 - \mu)\gamma_1^5/a + 10(3 - 2\mu)\gamma_1^4/a + 8\{(1 - \mu)/a - 1\}\gamma_1^3 - 12\mu\gamma_1^2 - 4\mu\gamma_1]}$$

similarly $L_1 = - 2n^2\gamma_1^7 - 2n^2(3 - \mu)\gamma_1^6 - 2n^2(3 - 2\mu)\gamma_1^5 - \{2n^2(1 - \mu) - 2\}\gamma_1^4 + 4\mu\gamma_1^3 + 2\mu\gamma_1^2$;

$= U_1 + V_1e^2 + V_1A_2$ [Substituting the value of n^2].

where $U_1 = -2\{\gamma_1^7/a + (3 - \mu)\gamma_1^6/a + (3 - 2\mu)\gamma_1^5/a + (1 - \mu)\gamma_1^4/a - \gamma_1^4 - 2\mu\gamma_1^3 - \mu\gamma_1^2\}$

$V_1 = -3\{\gamma_1^7 + (3 - \mu)\gamma_1^6 + (3 - 2\mu)\gamma_1^5 + (1 - \mu)\gamma_1^4\}/a$

Hence substituting the values of L_1, M_1, N_1, P_1, Q_1 and R_1 in the equation (2.7) we have

$$\delta_1 = U_1X_1 + (U_1Y_1 + V_1X_1)e^2 + 2X_1(\mu - 1)\gamma_1^3(\gamma_1 + 4U_1X_1)\beta_1 + \{U_1Y_1 + V_1X_1 + 3X_1\mu(1 + \gamma_1)(1 + \gamma_1 + 2U_1X_1)\}A_2$$

we have $\rho = \gamma_1 + \delta_1$

$$= \gamma_1 + U_1X_1 + (U_1Y_1 + V_1X_1)e^2 + 2X_1(\mu - 1)\gamma_1^3(\gamma_1 + 4U_1X_1)\beta_1 + \{U_1Y_1 + V_1X_1 + 3X_1\mu(1 + \gamma_1)(1 + \gamma_1 + 2U_1X_1)\}A_2$$

where γ_1 is the value of ρ in the classical case.

$$U_1 = -2\{\gamma_1^7/a + (3 - \mu)\gamma_1^6/a + (3 - 2\mu)\gamma_1^5/a + (1 - \mu)\gamma_1^4/a - \gamma_1^4 - 2\mu\gamma_1^3 - \mu\gamma_1^2\}$$

$$V_1 = -3\{\gamma_1^7 + (3 - \mu)\gamma_1^6 + (3 - 2\mu)\gamma_1^5 + (1 - \mu)\gamma_1^4\}/a$$

$$X_1 = [14\gamma_1^6/a + 12(3 - \mu)\gamma_1^5/a + 10(3 - 2\mu)\gamma_1^4/a + 8\{(1 - \mu)/a - 1\}\gamma_1^3 - 12\mu\gamma_1^2 - 4\mu\gamma_1]^{-1}$$

$$Y_1 = \frac{\{21\gamma_1^6/a + 18(3 - \mu)\gamma_1^5/a + 15(3 - 2\mu)\gamma_1^4/a + 12(1 - \mu)\gamma_1^3/a\}}{[14\gamma_1^6/a + 12(3 - \mu)\gamma_1^5/a + 10(3 - 2\mu)\gamma_1^4/a + 8\{(1 - \mu)/a - 1\}\gamma_1^3 - 12\mu\gamma_1^2 - 4\mu\gamma_1]}$$

μ = mass parameter,

$\beta_1 = 1 - q_1$, where q_1 is radiation parameter,

A_2 = oblateness parameter,

e = eccentricity,

a = semi major axis

3. Stability of collinear equilibrium points

The motion of an infinitesimal particle will be stable near the Collinear equilibrium points when given a very small displacement and small velocity, the particle oscillates for a considerable time around the said points.

Let α, β denote small displacements of the infinitesimal particle from the Collinear equilibrium point L_1 then,

$$\xi = \xi_0 + \alpha$$

$$\eta = \eta_0 + \beta$$

Now $\Omega^*_\xi = \Omega^*_\xi(\xi, \eta) = \Omega^*_\xi(\xi_0 + \alpha, \eta_0 + \beta)$

Expanding by Taylor's expansion and considering only first order terms, we have

$$\Omega^*_\xi = \Omega^{*0}_\xi + \alpha\Omega^{*0}_{\xi\xi} + \beta\Omega^{*0}_{\xi\eta}$$

$$\Omega^*_\eta = \Omega^{*0}_\eta + \alpha\Omega^{*0}_{\eta\xi} + \beta\Omega^{*0}_{\eta\eta}$$

Where Ω^{*0}_ξ is the value of Ω^*_ξ at the point (ξ_0, η_0) and similarly the other values $\Omega^{*0}_{\xi\xi}$, $\Omega^{*0}_{\xi\eta}$, Ω^{*0}_η, $\Omega^{*0}_{\eta\xi}$ and $\Omega^{*0}_{\eta\eta}$ are the respective values at the point (ξ_0, η_0).

At the equilibrium point (ξ_0, η_0), we have

$$\Omega^{*0}_{\xi} = \Omega^{*0}_{\eta} = 0$$

Hence from (3.1) we have

$$\Omega^{*}_{\xi} = \alpha\Omega^{*0}_{\xi\xi} + \beta\Omega^{*0}_{\xi\eta}$$

$$\Omega^{*}_{\eta} = \alpha\Omega^{*0}_{\eta\xi} + \beta\Omega^{*0}_{\eta\eta} \tag{3.2}$$

The equation of motion then takes the form as

$$\ddot{\alpha} - 2\dot{\beta} = \alpha\Omega^{*0}_{\xi\xi} + \beta\Omega^{*0}_{\xi\eta}$$

$$\ddot{\beta} - 2\dot{\alpha} = \alpha\Omega^{*0}_{\eta\xi} + \beta\Omega^{*0}_{\eta\eta} \tag{3.3}$$

To solve the equation (3.3) let $\alpha = Ae^{\lambda t}$ and $\beta = Be^{\lambda t}$, where A, B and λ are parameters to be found. Substituting the values of α, β, $\dot{\alpha}$, $\dot{\beta}$, $\ddot{\alpha}$ and $\ddot{\beta}$ in the equation (3.3) we have

$$A(\lambda^2 - \Omega^{*0}_{\xi\xi})e^{\lambda t} + B(-2\lambda - \Omega^{*0}_{\xi\eta})e^{\lambda t} = 0$$

$$A(2\lambda - \Omega^{*0}_{\xi\eta})e^{\lambda t} + B(\lambda^2 - \Omega^{*0}_{\eta\eta})e^{\lambda t} = 0 \tag{3.4}$$

These will have a nontrivial solution if

$$\begin{vmatrix} \lambda^2 - \Omega^{*0}_{\xi\xi} & -2\lambda - \Omega^{*0}_{\xi\eta} \\ 2\lambda - \Omega^{*0}_{\xi\eta} & \lambda^2 - \Omega^{*0}_{\eta\eta} \end{vmatrix}$$

$$\text{or } \lambda^4 - (\Omega^{*0}_{\eta\eta} + \lambda^{*0}_{\xi\xi} - 4)\lambda^2 + \Omega^{*0}_{\eta\eta}\Omega^{*0}_{\xi\xi} - (\Omega^{*0}_{\xi\eta})^2 = 0 \tag{3.5}$$

To find the stability of collinear equilibrium points, let $\Omega^{*0}_{\xi\eta}$, $\Omega^{*0}_{\xi\xi}$, $\Omega^{*0}_{\eta\eta}$ are of the following forms.

$$\Omega^{*0}_{\xi\eta} = h_1 e^2 + h_2 A_2 + h_3 \beta_1$$

$$\Omega^{*0}_{\xi\xi} = S_1{}^2 + S_2 e^2 + S_3 A_2 + S_4 \beta_1$$

and

$$\Omega^{*0}_{\eta\eta} = - T_1{}^2 + T_2 e^2 + T_3 A_2 + T_4 \beta_1$$

In classical case, $0 < \mu < 1/2$ and $\Omega^{*0}_{\xi\eta} = 0$, $\Omega^{*0}_{\xi\xi} > 0$, $\Omega^{*0}_{\eta\eta} < 0$; But in our case three possibilities may arise.

Case 1:

$$\Omega^{*0}_{\xi\eta} = 0,\ \Omega^{*0}_{\xi\xi} > 0,\ \Omega^{*0}_{\eta\eta} < 0;$$

which is same as the classical case and is unstable according to Szebehely (1967).

Case 2 :

$$\Omega^{*0}_{\xi\eta} > 0,\ \Omega^{*0}_{\xi\xi} > 0,\ \Omega^{*0}_{\eta\eta} < 0;$$

From the equation (3.5) we have the characteristic equation which is of the following from

$$\lambda^4 - (\Omega^{*0}_{\eta\eta} + \Omega^{*0}_{\xi\xi} - 4)\lambda^2 + \Omega^{*0}_{\eta\eta}\Omega^{*0}_{\xi\xi} - (\Omega^{*0}_{\xi\eta})^2 = 0$$

In this case $\Omega^{*0}_{\eta\eta}\Omega^{*0}_{\xi\xi} - (\Omega^{*0}_{\xi\eta})^2 < 0$

Also, the characteristic equation can be written in the following from

$$\Lambda^2 + 2\beta_2\Lambda - \beta_3^{\,2} = 0 \qquad (3.6)$$

where $\beta_2 = 2 - (\Omega^{*0}_{\xi\xi} + \Omega^{*0}_{\eta\eta})/2$

$\beta^2_3 = (\Omega^{*0}_{\xi\eta})^2 - \Omega^{*0}_{\xi\xi}\Omega^{*0}_{\eta\eta}$

and $\lambda^2 = \Lambda$

$$\therefore\ \lambda = \pm (\Lambda)^{1/2} \qquad (3.7)$$

Now from the equation (3.6) we have

$$\Lambda = -\beta_2 \pm (\beta_2^{\,2} + \beta_3^{\,2})^{1/2}$$

Let $\Lambda_1 = -\beta_2 + (\beta_2^{\,2} + \beta_3^{\,2})^{1/2}$

$\Lambda_2 = -\beta_2 - (\beta_2^{\,2} + \beta_3^{\,2})^{1/2}$

For positive or negative value of β_2, Λ_1 is always positive and Λ_2 is always negative, i.e. they are of opposite sign.

Again from the equation (3.7) we have

$$\lambda_{1,2} = \pm(\Lambda_1)^{1/2} = \pm \text{ real (since } \Lambda_1 \text{ is positive)}$$

and

$$\lambda_{3,4} = \pm(\Lambda_2)^{1/2} = \pm \text{ imaginary (since } \Lambda_2 \text{ is negative)}$$

Hence, for only one real positive value of $\lambda = \lambda_1$ (say) the solution $\xi = Ae^{\lambda t}$ and $\eta = Be^{\lambda t}$ will be unbounded.

Therefore the equilibrium point is unstable.

Case 3 :

$\Omega^{*0}_{\xi\eta} < 0,\ \Omega^{*0}_{\xi\xi} > 0,\ \Omega^{*0}_{\eta\eta} < 0;$

In this case also $\Omega^{*0}_{\xi\xi}\Omega^{*0}_{\eta\eta} - (\Omega^{*0}_{\xi\eta})^2 < 0$;

By the same process of case 2, the equilibrium point is also unstable.

Similarly, we can show that L_2 and L_3 are also unstable.

Conclusion

We conclude that collinear equilibrium points are unstable which give the same result of the classical case.

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