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# SOLAR SUPERGRANULATION AND TWO-DIMENSIONAL HYDRODYNAMIC TURBULENCE

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## Abstract

*In a two-dimensional incompressible fluid, the total energy as well as the total squared vorticity called enstrophy are conserved It is found that the energy spectrum in two-dimensional hydrodynamic turbulence cascades to smaller wavenumbers in the presence of viscous dissipation and therefore the energy is expected to accumulate at the longest wavelengths that the system allows The enstrophy on the other hand cascades to shorter wavelengths and is continuously dissipated This conclusion is reached by finding the inertial range of the turbulent spectrum However, even in the absence of dissipation, one can show that the energy spectrum condensates to largest scales as a consequence of conservation of energy and enstrophy during the cascade through the nonlinear term [(V ∇)V] It is found that if the enstrophy vanishes then the total energy remains constant even in the presence of dissipation Thus the system evolves to a state of minimum enstrophy with constant energy*

*The observed two dimensional nature of the velocity fields in supergranulation permits us to make use of the characteristics of two-dimensional hydrodynamic turbulence Thus it is proposed that supergranulation is produced from granulation by the selective decay process in which the energy tends to accumulate at the largest scales This largest scale is determined from the ratio of energy to enstrophy and presumably determines the scale of the solar supergranulation Inclusion of magnetic field will take us to magneto-hydrodynamic turbulence which also permits the formation of organized structures*

## 1 Introduction

The solar surface shows cellular pattern prominently on two scales the granulation and the supergranulation The granular cells have been interpreted to be the manifestation of convective processes in the hydrogen ionization region and the supergranules have been associated with the helium ionization region of the convection zone, although, there is no direct evidence for the latter association (Simon and Leighton 1964, Simon and Weiss 1968, Howard and Bhatnagar 1969, Howard 1971, Nelson and Musman 1977, 1978, Cloutman 1979a,b and Bray, Loughead and Durrant 1984) In this paper, we have attempted to answer the question Can supergranules be made from granules through the cascading processes in a turbulent medium ? The observed nearly two dimensional velocity field associated with supergranules guides us to investigate the very special properties of the two dimensional hydrodynamic turbulence, which may play an important role in the formation of supergranules

In a two dimensional hydrodynamic turbulence, the energy cascades towards large spatial scales and enstrophy towards small spatial scales where it suffers heavy dissipation It is this property of selective decay that facilitates the formation of large structures, whose dimensions are determined from the ratio of energy and enstrophy The inertial range of turbulent spectrum is derived in section 2 Section 3 deals with the inverse cascade through mode mode interaction The concept of self organization in two dimensional turbulence is discussed in section 4 A model of supergranulation is proposed in section 5

## 2. Inertial Range of the Turbulent Spectrum

The hydrodynamic equations describing the motion of an element in an incompressible fluid are

$$\frac{d\vec{V}}{dt} = \frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \vec{\nabla})\vec{V} = -\vec{\nabla} T + \gamma \nabla^2 \vec{V} \tag{1}$$

$$\vec{\nabla} \cdot \vec{V} = 0 \tag{2}$$

Here V is the velocity, T is the temperature and $\gamma$ is the kinematic viscosity The equation for vorticity vector $\Omega$ can be derived from Equations (1) and (2) as

$$\frac{d\vec{\Omega}}{dt} = \frac{\partial \vec{\Omega}}{\partial t} + (\vec{V} \cdot \vec{\nabla}) \vec{\Omega} \quad \gamma \nabla^2 \vec{\Omega}$$

$$\vec{\Omega} = \vec{\nabla} \times \vec{V} \tag{3}$$

In two dimensions $\vec{V}$ may be expressed by a scalar stream function $\psi$

$$\vec{V} \quad \nabla \times (\psi \hat{Z}) = \quad \nabla \psi \times \hat{Z} \tag{4}$$

$$\Omega = \nabla^2 \psi \hat{Z} \tag{5}$$

Here $\hat{Z}$ is a unit vector Equation (3) can be rewritten in terms of $\psi$ as

$$\frac{\partial}{\partial t} \nabla^2 \psi \quad (\nabla \psi \times \hat{Z}) \nabla (\nabla^2 \psi) \quad \gamma \nabla^2 \psi = 0 \tag{6}$$

where $\quad \nabla \equiv \frac{\partial}{\partial x} \hat{x} + \frac{\partial}{\partial y} \hat{y}$ (7)

If the viscosity is small i e the Reynolds number is large, the time evolution of the velocity field is determined by the second term in Equation (6) For large Reynolds number the various spatial Fourier components interact strongly and a turbulent state develops The equation for mode coupling is obtained from equation (6) by expressing $\psi$ as

$$\psi = \frac{1}{2} \left[ \sum_k \psi_k(t)\, e^{i \vec{k} \cdot \vec{x}} + c\, c \right] \tag{8}$$

where k is a two dimensional wave vector and $\psi_k$ is the Fourier amplitude Equation (6) can be rewritten as

$$\frac{d\psi_k}{dt} + K^2 \gamma \psi_k - \frac{1}{2} \sum_{K = K' + K''} \Lambda^{K}_{K\,K'} \psi_{K'} \psi_{K''} \tag{9}$$

where

$$\Lambda^{K}_{K\,K'} = \frac{1}{K^2} (\vec{K}' \times \vec{K}'') \hat{Z} (K'^2 \quad K''^2) \tag{10}$$

The conservation of total energy and enstrophy can be easily proved from equations (1) (2) and (3) Taking the scalar product of $\vec{V}$ with equation (1) and using vector algebra, one gets

$$\frac{\partial}{\partial t}\left(\frac{V^2}{2}\right) + \nabla \left[\vec{V} \frac{V^2}{2} + \vec{V}\, T \right] = \gamma \nabla (\vec{V} \times \vec{\Omega}) \quad \gamma \Omega^2 \tag{11}$$

If the fluid is urrounded by either a periodic boundary or a rigid boundary so that the normal component of the velocity $V_n$ vani hes on the boundary equation (11) gives

$$\frac{\partial W}{\partial t} = \frac{\partial}{\partial t} \int \frac{V^2}{2} d^3 r = \oint \gamma (\vec{V} \times \vec{\Omega}) \, d\vec{s} \quad \int \gamma \, \Omega^2 \, d^3 r \tag{12}$$

Taking the scalar product of equation (3) with $\Omega$ one finds

$$\frac{\partial}{\partial t}\left(\frac{\Omega^2}{2}\right) + \nabla\left(\frac{\Omega^2}{2}\vec{V}\right) = \gamma \, \nabla [\Omega \times (\nabla \times \Omega)] \quad \gamma(\nabla \times \Omega)^2 \tag{13}$$

The conservation of en trophy is obtained as

$$\frac{\partial U}{\partial t} \equiv \frac{\partial}{\partial t} \int \frac{\Omega^2}{2} d^3 r \quad \oint \gamma \vec{\Omega} \times (\vec{V} \times \vec{\Omega}) d\vec{s} \quad \int \gamma \, (\nabla \times \Omega)^2 \, d^3 r \tag{14}$$

Now, the inertial range of the turbulent spectrum can be determined by using Kolmogorov law If $V_k$ i the Fourier amplitude of the velocity field, the rate at which the pectrum cascades is given by $KV_k$ (the second term in equation (11)) The Omnidirectional energy spectrum W(K) is defined uch that $\int W(K)\,dk$ give the total energy Therefore W(K)K has the dimensions of $V_K^2$ Kolmogorov argued that in a qua i steady tate, there should be a stationary flow of energy in k space from the source to the ink, i e the energy density flow $(\rho V_k{}^2)(KV_k)$ hould be constant and is equal to the dissipation rate $\epsilon$ of the energy density at the sink

$$\rho V_k^3 K \quad \epsilon \tag{15}$$

and $\quad KW(K) \quad V_k^2$

gives $\quad W(K) - C\left(\frac{\epsilon}{\rho}\right)^{\frac{2}{3}} K^{\frac{5}{3}}$ (16)

where C is a univer al dimensionless constant In three dimensional turbulence only the energy is conserved in the inertial range and the energy spectrum cascades towards large wavenumbers where it suffers viscous dissipation

In two dimensional turbulence there is an additional invariant, the en trophy Hence two types of inertial ranges are expected, one for energy and the other for enstrophy The enstrophy density is given by $K^2V_k^2$, the inertial range for enstrophy requires that

$$(\rho \, K^2 V_k^2)(KV_k) = \epsilon' = \text{constant} \tag{17}$$

The energy spectrum in this range is given by

$$W(K) = C'\left(\frac{\epsilon'}{\rho}\right)^{\frac{2}{3}} K^{3} \tag{18}$$

Kraichnan (1967) showed that if $W(K) \sim K^{3}$, there is no energy cascade and if $W(K) \sim K^{\frac{5}{3}}$, there is no enstrophy cascade Hence a source at $K = K_s$ will set up two inertial ranges $K > K_s$ and $K < K_s$ Since enstrophy because of its stronger K dependence $(K^2 V_k^2)$, is dissipated at large K at a rate faster than the energy, the $K > K_s$ region would be the inertial range for enstrophy, which implies that $K < K_s$ region would be the inertial range for energy Thus the energy spectrum has two parts :

$$W(K) \sim K^{3}, \quad K > K_s \tag{19}$$

$$\sim K^{\frac{5}{3}}, \quad K < K_s \tag{20}$$

Kraichnan argues that since there is no energy cascade for $K > K_s$, the energy should cascade towards the smaller wavenumbers for $K < K_s$ An inverse cascade is expected The enstrophy cascades toward the large wavenumber regime is $K > K_s$

Now, in the small wave number regime, there is no energy dissipation and an inertial range for energy may not be established if $K < K$

### 3 Inverse Cascade through Mode Mode Coupling

Let there be a source at $K = K_s$ with energy $W$ Through mode mode coupling this would decay to two modes with wavenumbers $\vec{K}_1$ and $\vec{K}_2$ Since energy and enstrophy are conserved one can calculate the energies $W_1$ and $W_2$ of the modes $\vec{K}_1$ and $\vec{K}_2$ as

$$W_s = W_1 + W_2 \tag{21}$$

$$K_s^2 W_s \quad K_1^2 W_1 + K_2^2 W_2 \tag{22}$$

From these

$$W_1 = \frac{K_2^2 \quad K_s^2}{K_2^2 \quad K_1^2} W_s \tag{23}$$

$$W_2 \quad \frac{K_s^2 \quad K_1^2}{K_2^2 \quad K_1^2} W_s \tag{24}$$

For $W_1$ $W_2 > 0$, we see that

$$K_2^2 > K_s^2 > K_1^2$$

and for $K_2^2 > K_1^2$ (25)

Thus $K_s$ decays to two modes with wavenumber $K_1 < K_s$ and to another mode with $K_2 > K_s$ For maximum decay rate, Hasegawa and Kodama (1978) find that $p = [K_1^2/K_s^2] - (\sqrt{2} \quad 1)$,

$$K_2^2 - K_s^2 + K_1^2 \ , W_1 = p W_s \text{ and } W_2 = (1 \quad p)W_s$$

In the next step of the cascade, the mode $K_1^2$ decays to mode at $pK_1^2 - p^2K_s^2$ and $(1+p)K_1^2$ $p(1+p)K_s^2$ The mode at $K_2^2$ decays to $pK_2^2 = p(1+p)K_s^2$ and $(1+p)K_2^2 = (1+p)^2K^2$ The corresponding energy partitions are $p^2W_s$, $2p(1 \ p)W_s$ and $(1 \ p)^2W_s$ for wavenumber at $p^2K_s^2$, $p(1+p)K_s^2$ and $(1+p)K_s^2$ respectively Continuing to the $n$th step the energy distribution is given by a binomial distribution for a parameter $(r/n)$ such that

$$W\left(K^2 = p^{n \ r}(1+p)^r K_s^2\right) = \binom{n}{r} p^{n \ r}(1 \ p)^r W_s \tag{26}$$

Equation (26) gives the energy spectrum which result from a series of cascades at a fixed ratio of $(K_1^2/K_s^2) = p$ at each step, where $K_1^2 + K^2 = K_2^2$ It can be easily shown that the energy spectrum condensates at $K \to 0$ as $n \to \infty$ Peak of a binomial distribution occurs at $(r/n) \to (1 \ p)$ as $n \to \infty$

Therefore the $K^2 = K_p^2$ as $n \to \infty$ is found from equation (26) as

$$K_p^2 \quad \lim_{n \to \infty} p^{n \ r}(1+p)^r K_s^2$$

$$= \lim_{n \to \infty} \left[p^{(1 \quad r/n)}(1+p)^{r/n}\right]^n K_s^2 \tag{27}$$

Letting $(r/n) \to (1 \ p)$ as $n \to \infty$

$$K_p^2 = \lim_{n \to \infty} [p^p (1 + p)^{1-p}]^n K_s^2$$

$$\to 0$$

Since $p^p (1 + p)^{1-p} < 1$ for $0 < p < 1$

Thus the peak of the energy distribution moves to $K \to 0$ as $n \to \infty$. Hence an inverse cascade and condensation of the spectrum at $K \to 0$ is expected from this model. Inverse cascade obtained this way is a consequence of conservation of energy and enstrophy.

## 4 Self Organisation in Two Dimensional Turbulence

Kraichnan's hypotheses of inverse cascade and inertial range spectra (Kraichnan 1967) have been tested by solving equation (9) numerically (Batchelor 1969, Lilly 1969, Fornberg 1977), as shown in figure (1). The smooth structure of the stream function shown in figure (2a) is a consequence of the inverse cascade of the energy to longer wavelengths, while the chaotic state of the vorticity shown in figure (2b) is a result of the enstophy cascading to smaller wavelengths. The creation of large scale structure in the stream function in two dimensional fluids has also been observed in laboratory experiments. The condensation of energy at the longest wavelengths permitted either due to the finite size of the container or due to the periodic boundary condition has been reproduced in computer simulations (Hossain et al 1983).

From equation (12), if the enstrophy $\int \Omega^2 \, d^3r$ vanishes during normal cascade, $(\partial W/\partial t)$ approaches zero even in the presence of viscosity. This together with the experimental evidence for the inverse cascade indicates that the system will evolve to a state of minimum enstrophy with constant energy. Such a dissipation process is called selective dissipation (Kraichnan and Montgomery 1980). Thus the large scale structure appears as a result of minimization of enstrophy with the constraint of constant energy. This is expressed as

$$\partial U - \lambda \delta W = 0 \tag{28}$$

or $$\delta \int (\nabla \times V)^2 \, d^3r - \lambda \delta \int V^2 \, d^3r = 0 \tag{29}$$

Integrating by parts one gets

$$\int \delta \vec{V} \, [(\vec{\nabla} \times \vec{\nabla} \times \vec{V}) - \lambda \vec{V}] \, d^3r + \oint (\delta \vec{V} \times \vec{\Omega}) \cdot d\vec{s} = 0 \tag{30}$$

For periodic boundary conditions or for a viscous boundary such that $\Omega = 0$ at the boundary, equation (30) gives

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{V}) - \lambda \vec{V} = 0 \tag{31}$$

which can be solved by using the stream function which is determined by

$$\nabla^2 \psi + \lambda \psi = 0 \tag{32}$$

Since $\lambda$ gives the ratio of enstrophy to energy, equation (32) should be solved for a minimum eigenvalue $\lambda$. If the fluid has a periodic boundary condition with the periods a and b in the x and y directions, then

$$\psi = \psi_0 \cos \frac{2\pi x}{a} \cos \frac{2\pi y}{b} \tag{33}$$

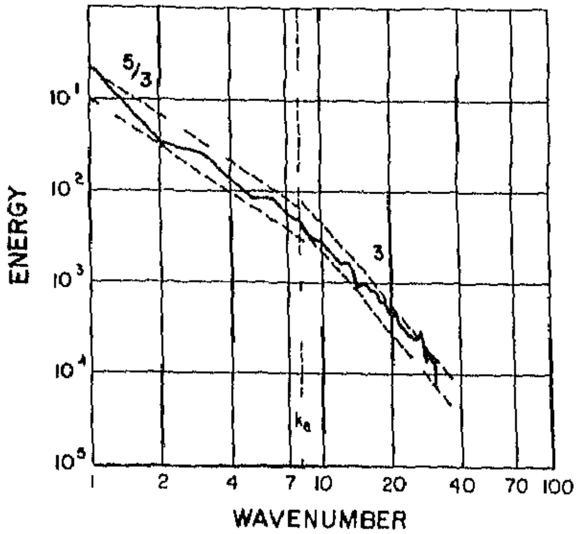


**Fig.1** An omnidirectional energy of two dimensional Navier Stokes turbulence obtained numerically (Lilly 1969) The initial spectrum which is dominated by the source spectrum at the source wave $k_e$ is shown to relax to the inertial range spectra for enstrophy at $k > k_e$ and energy at $k < k_e$

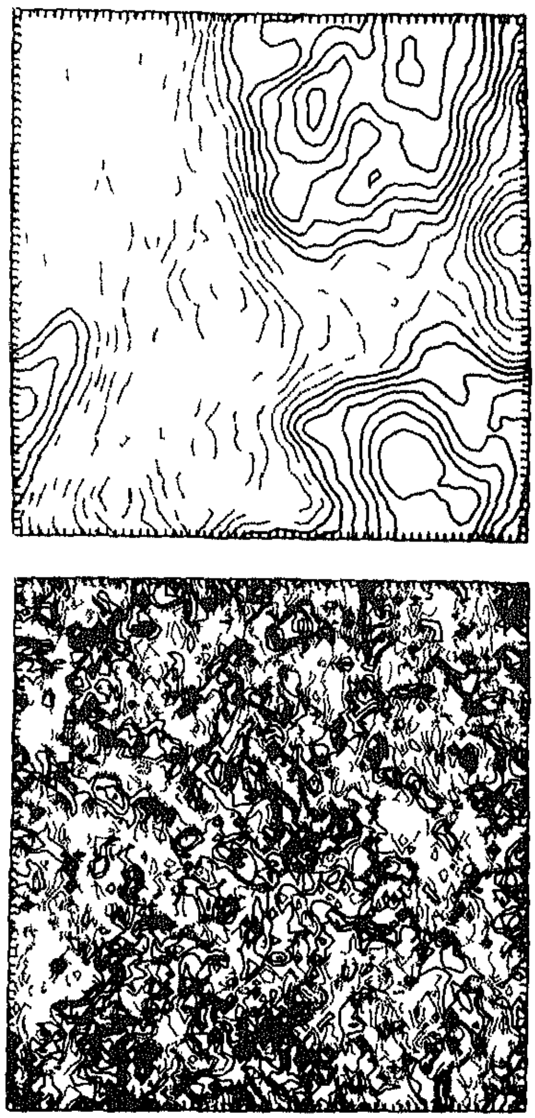

**Fig.2** (a) The stream function and (b) the vorticity at the 2360th time step of simulated Navier Stokes turbulence (Lilly 1969)

The self organized state obtained here is also a stationary solution of the dynamical equation (1) Substituting equation (31) into (1) and setting $[(\partial\vec{V}/\partial t) = 0]$ and $\gamma = 0$ one gets

$$\nabla \left[ \frac{V^2}{2} + T + \frac{\Omega^2}{2\lambda} \right] = 0 \tag{34}$$

This gives the temperature profile $T(x\ y)$

## Application to Solar Supergranulation

The observed two dimensional nature of the velocity field in the supergranules permits us to use the result of the sections (2), (3) and (4) Based on this, we like to propose and test the following model for formation of supergranular cell on the solar surface

(1) The supergranulation is produced as a result of redistribution of energy associated with ganulation

(2) The redistribution of energy takes place in a region with predominantly horizontal velocity fields i e between middle chromosphere and photosphere below which the velocity field becomes three dimensional and isotropic

(3) The redistribution of energy responsible for supergranulation is through the inverse cascade of energy towards larger scales, a consequence of the mode mode inter action in a two dimensional system with two invariants, the energy and the en strophy

(4) The largest spatial scale is determined from the ratio of energy and enstrophy From equations (32) and (33), we find

$$\lambda = \left(\frac{2\pi}{a}\right)^2 + \left(\frac{2\pi}{b}\right)^2$$

where a and b are the dimensions of the organized structures, here the supergranular cell $\lambda$ is the ratio of enstrophy to energy Therefore for $a \sim b \sim l$, the size of the cell, one finds

$$L = \left(\frac{8\pi^2}{\lambda}\right)^{\frac{1}{2}} = \left(\frac{8\pi^2 W}{U}\right)^{\frac{1}{2}}$$

For horizontal velocities $V \sim 0\ 5$ km/sec, the energy per unit density, per unit volume i $\sim \frac{1}{2}\ (0\ 5 \times 10^5)^2\ (\text{cm sec}^{-1})^2$ Therefore to get $L \sim 30{,}000$ Km, the value of enstrophy per unit density per unit volume is required to be $10^{-8}\ (\text{sec}^{-2})$ It is instructive to compare this number with the square of the average velocity gradient in the supergranular cell i e with

$$\left(\frac{V}{l}\right)^2 \sim \left[\frac{0\ 5 \times 10^5}{3 \times 10^9}\right]^2 \sim 028 \times 10^{-8}\ (\text{sec}^{-2})$$

Thus the required value of the enstrophy corresponds to a stronger velocity circulation

(5) The spatial variation of temperature within the supergranular cell is given by equations (34) and (33)

(6) The distributions in energy and enstrophy would give a range of spatial scales, the largest of which may correspond to the giant cells

## Tests for the validity of the model

1. If the energy input for supergranulation is at the granular scale ($K_s$) then the energy spectrum should show a break at $K_s$, the spectrum should go as $K^{3}$ for $K > K_s$ and as $K$ for $K < K_s$ Duvall (1987) has proposed two experiments to check the spectral behaviour (i) Doppler shift measurements which have the advantage of providing a high precision map of motions over the surface The disadvantage is that one gets only one component of the horizontal motion as the Doppler effect gives only the line of sight velocity (ii) the tracer measurement in which small magnetic elements can be followed and both horizontal components can be measured The disadvantage is that one does not obtain a very dense grid of tracers and this would yield a noisy measurement Under the assumption that the two components of the horizontal motion are approximately equal, the Doppler method looks quite promising

2. The observed spatial variation of temperature when compared with the prediction of equation (34) will provide another test of our proposal

## Conclusion

The inverse cascade of energy in two dimensional hydrodynamic turbulence favour the formation of large organized structures Application of this idea to the production of supergranulation seems to account for the observed spatial scale of the cellular motion Other tests of the validity of this model remain to be investigated

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