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Helioseismic diagnosis of the equation of state

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Abstract. The helioseismic verification of major nonideal effects in the equation of state of solar matter has become well established. The dominant contribution is the Coulomb pressure, conventionally described in the Debye-Hückel approximation. Recently, the increased precision of the helioseismic diagnosis has brought significant observational progress beyond the Debye-Hückel approximation. Obviously, progress in the equation of state serves two purposes. For solar physicists, a better equation of state will lead to reduced uncertainty in solar models. For plasma physicists, it will lead to an astrophysical experiment, in a domain where there is not much laboratory competition.

Key words: helioseismology, equation of state, solar interior

1. Introduction

For equation of state studies we concentrate on the spherically symmetric aspects of solar structure, corresponding to "classical" stellar evolution models. Such models are characterized by a number of simplifying assumptions, as well as by the physical properties of matter in the star, conveniently labelled "micro-physics". The latter include descriptions of the equation of state, the opacity and the nuclear reaction rates; in addition, diffusion, included in several recent calculations, should be considered as part of the micro-physics. The assumptions in the standard calculations, simplifying what might be called the macro-physics, include the neglect of effects of rotation and magnetic fields (implicit in the assumption of spherical symmetry), as well as the assumption that material mixing occurs only in convectively unstable regions, or possibly as a result of diffusion and settling; also, convective energy transport is treated crudely through some form of mixing-length approximation and the contribution to hydrostatic balance from the turbulent motion in the convection zone, usually called turbulent pressure, is ignored.

While we have no direct evidence that the approximations are inadequate in the solar interior, it is obvious that our treatment of layers near the solar surface, and the solar atmosphere, is grossly over-simplified. Even simple estimates indicate that turbulent pressure plays a substantial role in the uppermost parts of the convection zone; hydrodynamical simulations indicate a structure of convection very different from the simple mixing-length description; and magnetic fields dominate at least the upper parts of the atmosphere. As far as the structure of the interior of solar models is concerned, the effects of the resulting errors are largely eliminated through adjustment of the mixing-length parameter, such that the final model has solar radius. However, in analysis of solar oscillation frequencies it must be kept in mind that the near-surface regions are probably inadequately described in solar models.

On the other hand, the bulk of the convection zone, apart from the region near the surface where a substantial superadiabatic gradient is required to drive the convective flux, has a fairly simple structure. Since energy transport is almost entirely through convection, the opacity has no effect; also, the stratification is very nearly adiabatic and is therefore essentially determined by the equation of state. It follows that the structure of the convection zone is characterized by the equation of state, the composition and the (constant) value of the specific entropy. This makes the convection zone particularly well suited for testing the equation of state.

2. Helioseismic probing of the equation of state

Detailed analyses of the observed frequencies must be based on numerically computed frequencies, obtained by solving the equations of stellar oscillations; however, a great deal of insight, as well as quantitative results of considerable precision, have resulted from asymptotic relations for acoustic modes. In its simplest form, the relation for the angular frequency ω can be written as (*e.g.* Gough, 1984)

$$\frac{(n+\alpha)\pi}{\omega} = \int_{r_t}^{R} \left(1 - \frac{L^2}{\omega^2}\frac{c^2}{r^2}\right)^{1/2} \frac{\mathrm{d}r}{c} , \tag{1}$$

where n is the order of the mode and $L = l + 1/2$, l being the degree; c is the adiabatic sound speed, which is a function of the distance r to the center, the lower turning point r_t is located where the the integrand vanishes, and R is the surface radius. Finally, $\alpha(\omega)$ depends on the properties of the region near the solar surface. Equation (1) can be refined, for example by including the effect of the perturbation in the gravitational field and the dependence of the near-surface reflection on degree (*e.g.* Brodsky & Vorontsov, 1991; Gough & Vorontsov, 1995); however, here the present form is sufficient.

It is convenient to analyze the frequencies in terms of corrections relative to a reference model. In equation (1) these can be characterized by the change $\delta c(r)$ in the sound speed, evaluated at fixed r, as well as $\delta\alpha(\omega)$ determined by changes near the surface. From equation (1) one finds for the resulting change $\delta\omega$ in the frequency that

$$S_{nl}\frac{\delta\omega_{nl}}{\omega_{nl}} = \mathcal{H}_1\left(\frac{\omega_{nl}}{L}\right) + \mathcal{H}_2(\omega_{nl}) , \tag{2}$$

where

$$S_{nl} = \int_{r_t}^{R} \left(1 - \frac{L^2 c^2}{r^2 \omega_{nl}^2}\right)^{-1/2} \frac{\mathrm{d}r}{c} - \pi \frac{\mathrm{d}\alpha}{\mathrm{d}\omega} \; ; \tag{3}$$

$$\mathcal{H}_1(w) = \int_{r_t}^{R} \left(1 - \frac{c^2}{r^2 w^2}\right)^{-1/2} \frac{\delta c}{c} \frac{\mathrm{d}r}{c} \, , \qquad \mathcal{H}_2(\omega) = \frac{\pi}{\omega} \delta\alpha(\omega) \tag{4}$$

(Christensen-Dalsgaard, Gough & Pérez Hernández, 1988). Thus the frequency change contains a contribution, $\mathcal{H}_1(\omega/L)$, from the sound-speed difference in the bulk of the model and another, $\mathcal{H}_2(\omega)$, from the near-surface changes; because of their different dependence on ω and l these two terms can be separated in an analysis of the data.

The effects of the inadequate treatment of the physics in the model near the surface are contained in the term $\mathcal{H}_2$. The separation in equation (2) in effect removes from the frequency differences the uncertainties introduced by our ignorance about the physics in the superficial layers of the Sun, leaving in $\mathcal{H}_1$ a clean signal which can subsequently be analyzed to determine the correction δc to the sound speed. It might also be noted that under many circumstances the effects of the uncertain region cause a contribution to $\mathcal{H}_2(\omega)$ which varies slowly with ω, allowing further filtering of the signal.

Accurate analysis of the observations requires use of the full, non-asymptotic behaviour of the oscillations. If the Sun is assumed to be in hydrostatic equilibrium, it may be shown that, apart from the superficial region, the frequencies are uniquely determined by, for example, the squared sound speed c^2 and the density ρ. It follows that after linearization the frequency differences between the Sun and the model can be expressed as

$$\frac{\delta\omega_{nl}}{\omega_{nl}} = \int_0^R \left[K_{nl}^{(c,\rho)}(r) \frac{\delta c}{c}(r) + K_{nl}^{(\rho,c)}(r) \frac{\delta \rho}{\rho}(r) \right] dr + Q_{nl}^{-1} \mathcal{G}(\omega_{nl}) \, , \tag{5}$$

Here the last term accounts for the near-surface uncertainty, with E_{nl} being the mode energy, normalized to unit surface displacement; this term is equivalent to the term $\mathcal{H}_2(\omega)$ in the asymptotic relation (2).

While strictly speaking the (thermal) equation of state is merely the relation between pressure p, temperature T and density ρ, in the following we shall use the term equation of state in a slightly broader sense, so that it encompasses as well all thermodynamic quantities. If it is assumed that the equation of state and the abundance Z of heavy elements are known, the pair (c^2, ρ) can be expressed in terms of (u, Y) or (ρ, Y), where Y is the helium abundance and $u = p/\rho$. The transformation uses the fact that $c^2 = \Gamma_1 u$, where $\Gamma_1 = (\partial \ln p / \partial \ln \rho)_{\mathrm{ad}}$ can be determined from p, ρ and the composition, if the equation of state is known. In this way equation (5) is replaced by an analogous equation, expressing $\delta\omega/\omega$ in terms of, for example, $\delta u/u$ and $\delta Y/Y$. Here the kernels multiplying $\delta Y/Y$ are non-zero only in the regions where Γ_1 depends on Y, *i.e.* , essentially only in the ionization zones of hydrogen and helium. This formulation allows determination of δY in these regions; furthermore, it has the advantage that determination of, for example, $\delta u/u$ can be carried out with relatively little interference from the term in δY, unlike the analogous analysis in terms of (c^2, ρ).

3. Some equation of state issues

From the previous section it is obvious that an *absolutely* accurate equation of state is crucial in the helium-abundance determination. Fortunately, as has been recognized already early by Gough (1984), helioseismology has the potential to probe the equation of state *and* helium abundance at the same time. Several studies have been made to make this point (Gough, 1984; Däppen, 1987; Christensen-Dalsgaard & Däppen, 1992, Kosovichev *et al.*, 1992; Vorontsov *et al.*, 1994; Antia & Basu, 1994; Kosovichev, 1995). These references contain the evidence why the equation of state is a sufficiently sensitive ingredient in an important astrophysical application, and that the present uncertainty in the equation of state is inadmissible.

However, as we discuss in the following, the interest in the equation of state is not merely motivated by astrophysics. Helioseismology has opened a window to study thermodynamic quantities of a Coulomb system. Although the solar plasma is not much non-ideal, the deviations from non-ideality have to be taken seriously precisely because of the narrow observational constraints. Thus, though the solar plasma looks conceptually much simpler than a strongly coupled many-body system, the necessity to compute the deviations from the simple model so accurately will require a serious effort, comparable to the studies of more strongly-coupled plasmas.

3.1 Ideal and nonideal plasmas

The simplest model is a mixture of nuclei and electrons, assumed fully ionized and obeying the classical perfect gas law. However, an *"ideal-gas"* equation of state can be more general. It may include deviations from the perfect gas law, namely ionization, radiation and degenerate of electrons, as long as the underlying microphysics of these additional effects is still ideal, that is, does not contain interactions. The "particles", however, can be be classical or quantum, material or photonic. In such an ideal framework, bound systems (molecules, atoms, ions) are allowed to have internal degrees of freedom (excited states, spin). All such ideal effects can be calculated as exactly as desired.

One measure of non-ideality in plasmas is the so-called coupling parameter Γ (the solar community should note that Γ has nothing to do with the adiabatic gradient Γ_1). In a plasma where particles have average distance $< r >$ from each other, we can define Γ as the ratio of average potential binding energy over mean kinetic energy kT (in the simplest case of hydrogen; generalizations to other elements are straightforward)

$$\Gamma = \frac{\left(\frac{e^2}{<r>}\right)}{kT} \tag{6}$$

Plasmas with $\Gamma >> 1$ are *strongly* coupled, those with $\Gamma << 1$ *weakly* coupled. A famous example of a strongly coupled plasmas is the interior of white dwarfs, where the coupling can become so strong to force crystallization. Weakly coupled are the interiors of stars with masses ranging from the slightly sub-solar ones to the largest.

As one can suspect, Γ is the dimensionless coupling parameter according to which one can classify theories. Weakly-coupled plasmas lend to systematic perturbative ideas (*e.g.* in powers of Γ), strongly coupled plasma need more creative treatments.

Improvements in the equation of state beyond the model of a mixture of ideal gases are difficult. This has both conceptual and technical reasons. As a fundamental conceptual reason we mention the fact that in a plasma environment already the idea of isolated atoms (and compound ions) has to be abandoned. A technical reason is the difficulty encountered when specific non-ideal effect are modelled. The three principal non-ideal effects are related to: (i) the internal partition functions of bound systems, (ii) pressure ionization, and (iii) collective interactions of the charged particles. The internal partition functions contain the difficult problem of excited states, where and how they are to be cut off. They are an important element in determining the ionization balances. Pressure ionization has to be provided by non-ideal interaction terms, because ideal gases would unphysically recombine in the central regions of stars.

3.2 Chemical and physical picture

3.2.1) Chemical Picture

Most realistic equations of state that have appeared in the last 30 years belong to the chemical picture and are based on the free-energy minimization method. This method uses approximate statistical mechanical models (for example the nonrelativistic electron gas, Debye-Hückel theory for ionic species, hard-core atoms to simulate pressure ionization via configurational terms, quantum mechanical models of atoms in perturbed fields, *etc.*). From these models a macroscopic free energy is constructed as a function of temperature T, volume V, and the concentrations $N_1, \ldots, N_m$ of the m components of the plasma. The free energy is minimized subject to the stoichiometric constraint. The solution of this minimum problem then gives both the equilibrium concentrations and, if inserted in the free energy and its derivatives, the equation of state and the thermodynamic quantities.

We note again that this procedure automatically guarantees thermodynamic consistency. As an example, when the Coulomb pressure correction (to the ideal-gas contribution) is taken into account in the free energy (and not merely in the pressure), it affects both the pressure and the equilibrium concentration, *i.e.* , the degrees of ionization. In contrast, the mere inclusion of the pressure correction would be inconsistent with other thermodynamic quantities.

In the chemical picture, perturbed atoms must be introduced on a more-or-less *ad-hoc* basis to avoid the familiar divergence of internal partition functions (see *e.g.* Ebeling, Kraeft & Kremp, 1976). In other words, the approximation of unperturbed atoms precludes the application of standard statistical mechanics, *i.e.* the attribution of a Boltzmann-factor to each atomic state. The conventional remedy of the chemical picture against this is a modification of the atomic states, *e.g.* by cutting off the highly excited states in function of density and temperature of the plasma. Such cutoffs, however, have in general dire consequences due to the discrete nature of the atomic spectrum, *i.e.* jumps in the number of excited states (and thus in the partition functions and in the free energy) despite smoothly varying external parameters (temperature and density). A current formulation in the chemical picture is the formalism based on the Hummer & Mihalas occupation probability formalism (Hummer & Mihalas, 1988; Mihalas, Hummer, & Däppen, 1988; Däppen *et al.*, 1988; Däppen, Andersen & Mihalas, 1987; hereafter 'MHD').

3.2.2) Physical picture

It is clear from the preceding subsection that the advantage of the chemical picture lies in the possibility to model complicated plasmas, and to obtain numerically smooth and consistent thermodynamical quantities. Nevertheless, the heuristic method of the separation of the atomic-physics problem from that of statistical mechanics is not satisfactory, and attempts have been made to avoid the concept of a perturbed atom in a plasma altogether. This has suggested an alternative description, the physical picture. In such an approach one expects that no assumptions about energy-level shifts or the convergence of internal partition functions have to be made. On the contrary, properties of energy levels and the partition functions should come out from the formalism.

There is an impressive body of literature on the physical picture. Important sources of information with many references are the books by Ebeling, Kraeft & Kremp (1976), Kraeft *et al.* (1986), Ebeling *et al.* (1991). However, the majority of work on the physical picture was not dedicated to the problem of obtaining a high precision equation of state for stellar interiors. Such an attempt was made for the first time by a group at Livermore as part of an opacity project (Rogers, 1986; Iglesias & Rogers 1995; Rogers *et al.*1996; and references therein).

To explain the advantages of this approach for partially ionized plasmas, it is instructive to discuss the activity expansion for gaseous hydrogen. The interactions in this case are all short ranged and pressure is determined from a self-consistent solution of the equations (Rogers, 1981)

$$\frac{p}{kT} = z + z^2 b_2 + z^3 b_3 + ... \tag{7}$$

$$\rho = \frac{z}{kT}\left(\frac{\partial p}{\partial z}\right) \tag{8}$$

where $z = \lambda^{-3}\exp(\mu/kT)$ is the activity, $\lambda \equiv h/\sqrt{2\pi m_e kT}$ is the thermal (de Broglie) wavelength of electrons, μ is the chemical potential and T is the temperature. The b_n are cluster coefficients such that b_2 includes all two particle states, b_3 includes all three particle states, *etc.*.

In contrast to the chemical picture, which is plagued by divergent partition functions, the physical picture has the power to avoid them altogether. An important example of such a fictitious divergence is that associated with the atomic partition function. This divergence is fictitious in the sense that the bound-state part of b_2 is divergent but the scattering state part, which is omitted in the Saha approach, has a compensating divergence. Consequently the total b_2 does not contain a divergence of this type (Ebeling, Kraeft & Kremp, 1976; Rogers, 1977). A major advantage of the physical picture is that it incorporates this compensation at the outset. A further advantage is that no assumptions about energy-level shifts have to be made (see the previous subsection); it follows from the formalism that there are none.

As a result, the Boltzmann sum appearing in the atomic (ionic) free energy is replaced with the so-called Planck-Larkin partition function (PLPF), given by (Ebeling, Kraeft & Kremp, 1976; Kraeft *et al.*, 1986)

$$\mathrm{PLPF} = \sum_{nl}(2l+1)\left[\exp(-\frac{E_{nl}}{kT}) - 1 + \frac{E_{nl}}{kT}\right] \tag{9}$$

The PLPF is convergent without additional cut-off criteria as are required in the chemical picture. We stress, however, that despite its name the PLPF is not a partition function, but merely an auxiliary term in a virial coefficient (see, for example, Däppen *et al.*, 1987).

3.3 The Debye-Hückel approximation

The Debye-Hückel (DH) theory is based on the replacement of the long-distance Coulomb potential by a screened potential, for which an exact expression can be obtained in several ways. The principal dimensionless parameter of the DH approximation is the ratio of the Landau length ($l_D = e^2/kT$) to the Debye radius ($r_D^{-2} = \frac{4\pi e^2}{kT}\sum_\alpha n_\alpha Z_\alpha^2$) ($n_\alpha$ being the particle number density N_α/V of species α). All corrections to the thermodynamic functions of the non-interaction system can be expressed in terms of this value $x = l_L/r_D$ alone (Ebeling, Kraeft & Kremp, 1976). In DH theory they take the form

$$\delta F_{es}/N_c kT = -x/3, \quad \delta p_{es}/n_c kT = -x/6 \, . \tag{10}$$

The subscript "es" denotes that these expressions are estimations; they become exact in the limit of small x. N_c and n_c denote number and number density of charged particles, respectively. Under solar conditions, x culminates at ≈ 0.3 in the outer part of the convection zone and it has another local maximum of ≈ 0.04 in the core. The corresponding negative pressure correction is about 8 %. This is very significant. Another estimate (which is exact for full ionization) can be obtained for Γ_1: $\delta\Gamma_1^{es} = x/36$. Therefore, the Coulomb interaction leads to an increased Γ_1, which, as a result, can exceed the "perfect-gas" value of 5/3 in the solar core.

Originally, the DH formalism included an additional ingredient of a fixed size for positive ions, inside of which the electrostatic potential is assumed to be constant. Such an assumption avoids the formal difficulties in connection with the singularity of the Coulomb potential at short distance, and it leads to a correction of the classical expression of the free energy in the form of a multiplicative factor

$$\tau(y) = 3/y^3(\ln(1+y) - y + y^2/2) \, , \tag{11}$$

where y is ratio of the ion size to r_D. This form of the correction rests on the assumption of a constant ion radius and it is derived from the so-called Debye charging process

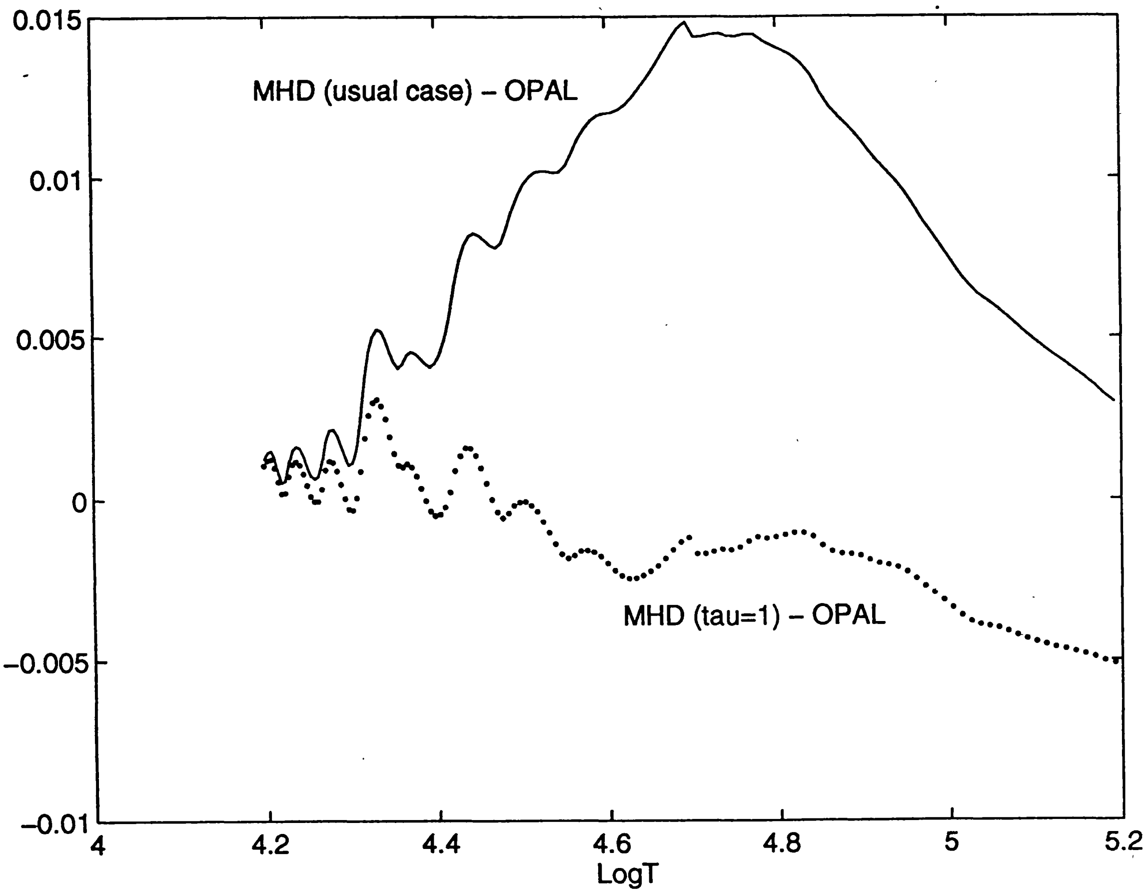

Figure 1. Relative pressure differences (in per cent) between the usual (solid line) MHD and OPAL, and between the ($\tau = 1$)-MHD and OPAL. See text for further details.

(see for example, Eyring *et al.* , 1964). In astrophysical applications, the τ-correction has been adopted in the form given by Harris *et al.* (1960), which used a particular approximation for the ion size, namely the Landau length, that is, the minimal distance of approach of charged particles. The argument for the correction function τ became therefore $x = l_L/r_D$ (hereafter we neglect electron degeneracy and restrict ourselves to plasmas with singly charged particles).

It is clear, that such an estimation of the ion size (more exactly, the switch-off parameter of the classical DH free energy) is rather crude. However, Harris's (1960) definition became rather popular. For instance, it was advocated by Graboske *et al.* (1969), and it was later incorporated in the MHD equation of state. However, it is easy to realize that the Landau length overestimates the ion size (ions have higher binding energies than kT, after all), and therefore the influence of this specific τ-correction (which reduced the classical DH correction too much) is unrealistically high. Further details and variations of the τ formalism are discussed in Baturin *et al.* (1996) and Gabriel (1994).

We would like to repeat that such close attention to the DH theory is warranted, because it describes the main truly nonideal effect under solar conditions. Due to the relatively high temperature, the potential energy of the Coulomb interaction is small compared to the kinetic energy of particles, which allows us to believe that our calculation with DH theory makes at least asymptotic quantitative sense. But to estimate the possible error we need a physically based expression for next terms in the corresponding expansion. A potential candidate for this task is the formalism by Alastuey *et al.* (1994).

The time-honored τ-correction that has historically been incorporated in astrophysical calculations can hardly be a valid estimation. This is due to the lack of physical justification and the arbitrary choice of the parameters used. In the best case, we could consider it as an extremely exaggerated upper limit for the possible correction of the DH theory. Better physical results (though only in the framework of a specific problem) can be given by the so-called Abe expression (see Baturin *et al.*, 1996), leading to a *very small correction* to the DH theory (less than 5% anywhere). This means that simple formulations of DH theory give better results.

It is noteworthy that the OPAL equation of state has not followed the historic trend, and it is therefore close to the unmodified DH results under solar conditions. Figure 1 illustrates this. The solid line represents the relative pressure difference across a part of a solar model. The chemical composition for the *equation of state* comparison is simplified: H–He only, in number ratio 0.9:0.1. The underlying solar model (used for the $T-\rho$ track of the comparison) is "standard", that is, it contains heavy elements). The oscillatory behavior - due to interpolation - of the curves is not significant. The figure reveals the origin of the discrepancy between MHD and OPAL pressure to be due to the τ correction employed in the MHD formalism, because the discrepancy disappears when the effect of the τ correction in MHD is switched off ($\tau = 1$) (dotted line). Since OPAL seems to fare better seismologically than MHD in recent analysis (*e.g.* Christensen-Dalsgaard, *these proceedings*), it is quite likely that the τ correction has also observationally been ruled out.

4. Conclusions

Helioseismic analysis confirms the presence of Coulomb correction terms. The Debye-Hückel term for the Coulomb correction is a very good first-order approximation. However, beyond the first order contribution, things become much more subtle. Many quite reasonable equations of state appearing after 1960 that were based on the consistent free-energy minimization procedure included a way to atone the Debye-Hückel term, to prevent the overall pressure from becoming nonsensically negative. Often this shut down of the first-order term was done with the so-called τ correction, a device borrowed from the theory of electrolytes. Since the effect of the τ correction on the equation of state is rather small, it has often been adopted in many later equations of state, including MHD, without sufficient critique. Helioseismology has now reached the point where it can point to the inadequacy of the τ correction. The OPAL equation seem to fare quite better with helioseismology, because it brings in the higher-order terms in a more systematic way. Taking the τ correction out of MHD (that is, setting $\tau = 1$) will lead to new seismic comparisons between OPAL and the revised MHD. These two equations of state will then be sufficiently close to each other that the comparisons promise to shed new light on the important equation of state issues, such as the question of bound states or pressure ionization (see Christensen-Dalsgaard & Däppen, 1992, Baturin *et al.*, 1996). Although we are now dealing with very small corrections, we must keep in mind that they are not only of interest for plasma physics. Important solar physics applications (*e.g.* helium abundance determinations) rely crucially on their precise amount.

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