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Numerical Methods for Solving the Polarized Line Transfer Equations with Partial Frequency Redistribution

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Abstract. In this paper we present a brief review on Polarized Approximate Lambda Iteration (PALI) line radiative transfer method, for solving the Hanle effect line formation problem with Angle Averaged (AA) partial frequency redistribution. The recently derived Bommier's redistribution matrix is employed in this numerical method. Few equations relevant for describing the main problem are presented. Further, we describe a simple polarization perturbative method of solving the Angle Dependent (AD) redistribution problem. Both these methods are illustrated through sample results. The general implications of the core-wing approach to partial redistribution (PRD) theory is pointed out.

1. Introduction

The resonance scattering in lines produces linearly polarized radiation. This linear polarization undergoes modification in the presence of an external weak magnetic field - a phenomenon which is known as the Hanle effect (see Stenflo, 1994; and several articles related to this topic, in the context of *Solar Polarization* presented in Stenflo & Nagendra 1996, and Nagendra & Stenflo 1999). For solar magnetic field diagnostics based on this phenomenon, one has to solve the line transfer equation, including Hanle effect redistribution matrix! There were attempts to solve this line transfer problem using direct numerical techniques (see Faurobert-Scholl 1996, and references therein). In recent years PALI methods are developed to solve the same problem, due to its well known advantages over the direct methods (see Manso Sainz & Trujillo Bueno 1999, 2001; and Nagendra et al. 1998, 1999, 2002). Recently Fluri, Nagendra & Frisch (2003) have developed an ALI method to treat the PRD with Hanle effect, based on the recent theoretical work of Bommier (1997). It handles angle-averaged redistribution functions.

It is useful to note that the Hanle effect is treated either with the density matrix approach or the scattering approach. The equivalence of these two approaches for two-level atoms with no lower-level polarization is pointed out in Frisch (1998, 1999). In this paper we have employed the scattering approach. Also, in all the computations we neglect the 'lower state polarization of the atom' (see Trujillo Bueno, 1999, 2003; and Landi Degl'Innocenti, 1999 for the inclusion of this effect in line transfer).

2. The Governing Equations

The 2-level atom non-axisymmetric polarized line radiative transfer equation (RTE) for the Stokes vector $\boldsymbol{\mathcal{I}} = (I\ Q\ U)^{\mathrm{T}}$ is given in the standard notation, by

$$\mu \frac{\partial \boldsymbol{\mathcal{I}}(\tau, x, \mathbf{n})}{\partial \tau} = (\phi(\tau, x) + \beta(\tau))\ [\boldsymbol{\mathcal{I}}(\tau, x, \mathbf{n}) - \boldsymbol{\mathcal{S}}(\tau, x, \mathbf{n})], \tag{1}$$

where the Stokes total source vector $\boldsymbol{\mathcal{S}} = (S_I,\ S_Q,\ S_U)^{\mathrm{T}}$ is of the form:

$$\boldsymbol{\mathcal{S}}(\tau, x, \mathbf{n}) = \frac{\phi(\tau, x)\boldsymbol{\mathcal{S}}_l(\tau, x, \mathbf{n}) \ + \ \beta(\tau)\boldsymbol{\mathcal{B}}_{th}(\tau)}{\phi(\tau, x) \ + \ \beta(\tau)}, \tag{2}$$

where $\boldsymbol{\mathcal{B}}_{th}(\tau) = B_{\nu_o}(\tau)\boldsymbol{\mathcal{U}}$ with $B_{\nu_o}(\tau)$ being the Planck function, and $\boldsymbol{\mathcal{U}} = (1, 0, 0)^{\mathrm{T}}$. The Stokes line source vector $\boldsymbol{\mathcal{S}}_l(\tau, x, \mathbf{n})$ in the presence of an external magnetic field B, is given by

$$\boldsymbol{\mathcal{S}}_\ell\,(\tau, x, \mathbf{n}) = \varepsilon \boldsymbol{\mathcal{B}}_{th}(\tau) + \int_{-\infty}^{+\infty} \frac{dx'}{\phi\,(\tau, x)} \oint \frac{d\Omega'}{4\pi} \boldsymbol{\mathcal{R}}\,(\tau;\ x, \mathbf{n};\ x', \mathbf{n}';\ \mathbf{B})\ \boldsymbol{\mathcal{I}}\,(\tau, x', \mathbf{n}'). \tag{3}$$

This transfer problem (as represented by Eqs. (1)-(3)), is considerably difficult to solve. Indeed the relevant PALI method is not yet developed. However, by a suitable approximation which leads to a factorized form of the redistribution matrix $\boldsymbol{\mathcal{R}}$, the problem can be greatly simplified. Bommier (1997) has already derived such a factorization. It turns out that the two-dimensional (x, x') frequency space gets divided into 5 frequency domains in that process of de-coupling. Within each domain, $\boldsymbol{\mathcal{R}}$ can be factorized into a product of scalar redistribution functions $R_{\mathrm{II,\ III}}(x, x', \Theta)$ and 'polarized phase matrices' $\hat{P}_{\mathrm{II,\ III}}$. Basically, the Angle Dependent (AD) line source vector is re-written as:

$$\boldsymbol{\mathcal{S}}_{\ell,\mathrm{AD}}(\tau, x, \mathbf{n}) = \varepsilon \boldsymbol{\mathcal{B}}_{\nu_o}(\tau) + \frac{1}{\phi(\tau, x)} \int_{-\infty}^{+\infty} dx' \oint \frac{d\Omega'}{4\pi} \Big[\hat{P}_{\mathrm{II}}(\mathbf{n}, \mathbf{n}'; \mathrm{Dn})$$

$$R_{\mathrm{II-AD}}(x, x', \Theta) \ + \ \hat{P}_{\mathrm{III}}(\mathbf{n}, \mathbf{n}'; \mathrm{Dm})\, R_{\mathrm{III-AD}}(x, x', \Theta) \Big] \boldsymbol{\mathcal{I}}(\tau, x', \mathbf{n}'), \tag{4}$$

with an exactly analogous equation for the Angle Averaged (AA) case. Dm and Dn are domain indices for scalar frequency redistribution types III and II respectively. The phase matrices $\hat{P}_{\mathrm{III}}$ and $\hat{P}_{\mathrm{II}}$ are expressed in terms of the Hanle ($\hat{P}_{\mathrm{H}}$) and Rayleigh ($\hat{P}_{\mathrm{R}}$) phase matrices, and the collisional branching ratios etc.

2.1. Frequency Domains and the Scattering Integral of PRD theory

In this Section, we discuss briefly the essential differences between the previous approaches used in polarized PRD theory, and the new approach developed by Bommier (1997), which is more realistic.

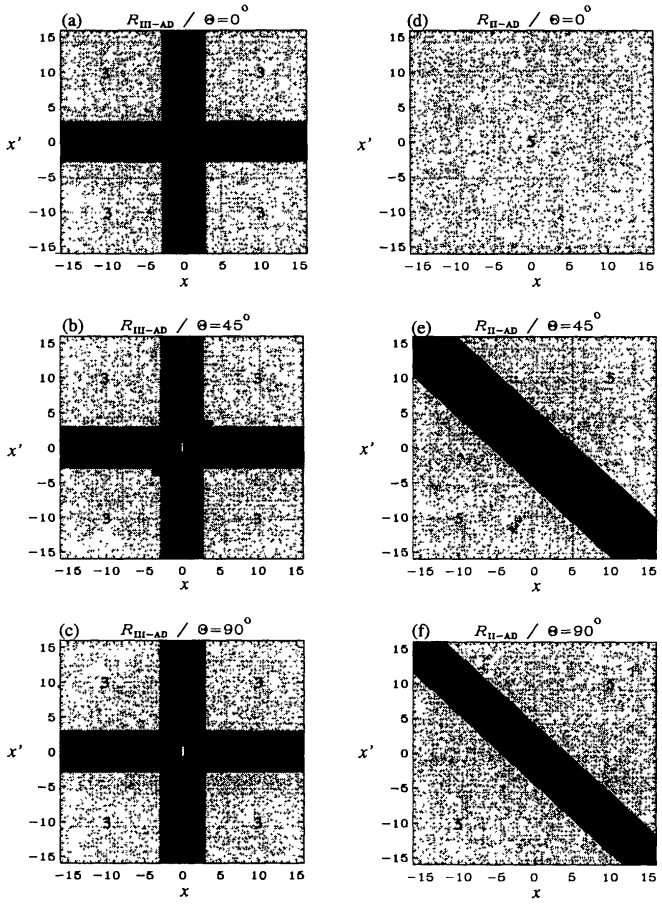

Figure 1. The frequency domains for solving the angle-dependent Hanle PRD problem. The panels (a),(b),(c) refer to $R_{\mathrm{III-AD}}(x, x', \Theta)$, and the panels (d),(e),(f) refer to $R_{\mathrm{II-AD}}(x, x', \Theta)$. The domains Dm, m=1, 2, 3 refer to $R_{\mathrm{III-AD}}$, while Dn, n=4, 5, refer to $R_{\mathrm{II-AD}}$. The dotted vertical line in panel (e) shows the 'line of integration' over incoming frequency x', for a given value of x. The damping parameter $a = 10^{-3}$ is used. The dark and light shade represent respectively, dominantly 'Hanle type scattering', and dominantly 'Rayleigh type scattering'.

Angle Dependent Frequency Domains:- The shape of the frequency domains depends on the scattering angle Θ and the incoming (x') and outgoing photon frequencies (x). The logical algorithm defining these domains are given in Bommier (1997). Here we show only the graphical representation (see Fig. 1).

Notice that the domains are mutually exclusive, and do not intersect each other. The domains are separated by sharp transition regions, which correspond to core-wing transitions in the generalized absorption profiles (Bommier 1997).

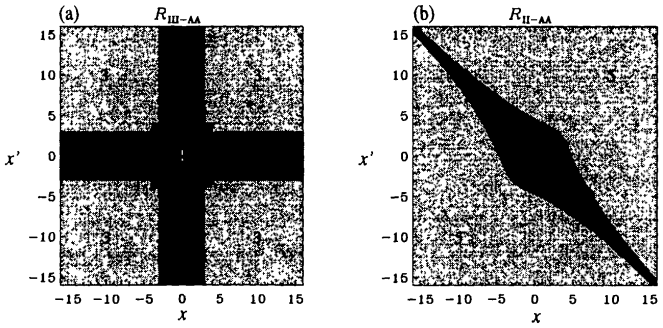

Figure 2. The angle-averaged (AA) frequency domains for solving the Hanle PRD problem. See Fig. 2 of Fluri et al. (2003) for the corresponding 'computational domains' employed in the core-wing approach, which are expected to represent the above 'physical domains'.

For large x, the contribution of domain D4 to the scattering integral is negligible because $R_{\rm II-AD}$ takes its largest value only along $x = x'$, in the diagonal band (see panel (f)). This implies that the Hanle effect mainly changes 'line core' and not the 'line wings', which is consistent with observations and the conventionally used theoretical assumption. Notice also that, for large values of x and x', the function $R_{\rm III-AD} \approx \phi(x)\ \phi(x')$. Hence for large x (wings), the main contribution to scattering integral comes from D2 - through $R_{\rm III-AD}$ type, rather than from D4 - through $R_{\rm II-AD}$ type scattering. Thus, when $R_{\rm III-AD}$ and $R_{\rm II-AD}$ type PRD contributions are equally strong in the stellar atmosphere, we should observe some 'Hanle effect in the line wings' also, contrary to the conventional theoretical assumption that Hanle effect appears only in the line core !

It is worthwhile to note that this 'line wing Hanle effect' is inherently contained in the Bommier's redistribution matrix. The only way to include it in the line transfer computations is to employ the frequency domain structure in the evaluation of the scattering integral, as described in Nagendra et al. (2002).

Angle Averaged Frequency Domains:- The Angle Averaged domain definitions are given in Bommier (1997). Fig. 2 shows the graphical representation only. Notice the similarities and differences between AD and AA domains. The AA domains are not simple angle averages of the AD domains. The AA domains for $R_{\rm III-AA}(x, x')$ and AD domains for $R_{\rm III-AD}(x, x', \Theta = 90°)$ are similar, as expected. The narrowing of the diagonal stripe in the case of $R_{\rm II-AA}$ for large frequencies, indicates a highly diminished role of $R_{\rm II-AA}$- contributed Hanle effect to polarization in the formation of line wings. Our general discussion about AD domains, holds in the AA case also.

In the case of AA domains, the nature of 'core Hanle effect' is quite different when compared to the case of AD domains. Further, in this case, the 'wing Hanle effect' is very weak, when compared to the case of AD domains. However the use of AA domains offers numerical simplicity.

2.2. The Angular Phase Matrices of the Hanle PRD Theory

The multi-polar expansion of Rayleigh and Hanle phase matrices are written as

$$\hat{P}(\mathbf{n},\mathbf{n}') \;=\; \sum_{K=0}^{K=2} W^{(K)}(J,J')\hat{P}^{(K)}(\mathbf{n},\mathbf{n}'). \tag{5}$$

$W^{(K)}(J,J')$ is de-polarization parameter. $W^{(K)}(J,J') = 1$ for a $J' = 1 \rightarrow J = 0$ transition. The zeroth-order term $\hat{P}^{(0)}$ is the isotropic scattering phase matrix (the element (1,1) is unity and the others being zero). The Hanle phase matrix $\hat{P}_{\rm H}$ depends on the field orientation $\mathbf{n}_{\rm B}$, and on the field strength parameter

$$\Gamma_{\rm B} = g_{J'}\frac{\omega_{\rm L}}{\Gamma_{\rm R}}; \quad \text{with} \quad \omega_{\rm L} = \frac{eB}{2m_e c}, \tag{6}$$

where $g_{J'}$ is the Landé factor of the upper level and $\omega_{\rm L}$ the Larmor frequency. The phase matrices introduced in the definition of line source vector are:

$$\hat{P}_{\rm III}({\rm D1}) \;=\; (\beta^{(0)}-\beta^{(2)})\hat{P}^{(0)} + \beta^{(2)}\hat{P}_{\rm H}(\mathbf{n},\mathbf{n}';\mathbf{n}_{\rm B},\Gamma_2') - \alpha\hat{P}_{\rm H}(\mathbf{n},\mathbf{n}';\mathbf{n}_{\rm B},\Gamma''), \tag{7}$$

$$\hat{P}_{\rm III}({\rm D2}) \;=\; (\beta^{(0)} - \beta^{(2)})\hat{P}^{(0)} + (\beta^{(2)} - \alpha)\hat{P}_{\rm H}(\mathbf{n},\mathbf{n}';\mathbf{n}_{\rm B},\Gamma_2'), \tag{8}$$

$$\hat{P}_{\rm III}({\rm D3}) = (\beta^{(0)}-\beta^{(2)})\hat{P}^{(0)}+\frac{\beta^{(2)}-\alpha}{\beta^{(2)}}\left[(\beta^{(2)} - \alpha)\hat{P}_{\rm H}(\mathbf{n},\mathbf{n}';\mathbf{n}_{\rm B},\Gamma_2') + \alpha\hat{P}_{\rm R}(\mathbf{n},\mathbf{n}')\right], \tag{9}$$

and

$$\hat{P}_{\rm II}({\rm D4}) \;=\; \alpha\hat{P}_{\rm H}(\mathbf{n},\mathbf{n}';\mathbf{n}_{\rm B},\Gamma''), \tag{10}$$

$$\hat{P}_{\rm II}({\rm D5}) \;=\; \alpha\hat{P}_{\rm R}(\mathbf{n},\mathbf{n}'). \tag{11}$$

In the above equations, $\hat{P}_{\rm R}$ is the non-magnetic Rayleigh phase matrix. All the symbols have the same meaning as in Nagendra et al. (2002).

2.3. Magnetic Field Modified Collision Rates and Branching Ratios

The modified magnetic field strength parameters Γ_K' and Γ'' are given by

$$\Gamma_2' = \beta^{(2)}\Gamma_{\rm B}; \quad \text{and} \quad \Gamma'' = \alpha\Gamma_{\rm B}, \tag{12}$$

where α and $\beta^{(2)}$ are collisional branching ratios. Following Bommier (1997),

$$\alpha = \frac{\Gamma_{\rm R}}{\Gamma_{\rm R}+\Gamma_{\rm I}+\Gamma_{\rm E}}; \qquad \beta^{(K)} = \frac{\Gamma_{\rm R}}{\Gamma_{\rm R}+\Gamma_{\rm I}+D^{(K)}}, \tag{13}$$

where $K = 0$ or 2. The quantity $\Gamma_{\rm R}$ is the radiative de-excitation rate, and $\Gamma_{\rm I}$ is the rate of inelastic collisions between radiating and perturbing atoms which causes destruction of the upper excited state. and leads to transitions.

The total rate of elastic collisions $\Gamma_{\rm E}$ causes collisional broadening of the upper level. $\Gamma_{\rm E}$ produces frequency non-coherence in atomic and hence also in the laboratory frame. The major source of $\Gamma_{\rm E}$ are neutral hydrogen atoms. $D^{(K)}$ are the K-multi-pole de-polarizing collision rates. We know that $D^{(0)} = 0$, and we assume that $D^{(2)}$ is a simple function of $\Gamma_{\rm E}$, such as $D^{(2)} = {\rm c}\,\Gamma_{\rm E}$, with

$0 \leq c \leq 1$. We employ $c = 0.5$ in this paper. The $D^{(K)}$ are most effective causes of decreasing the linear polarization in spectral line cores, because they destroy atomic alignment (Faurobert 1996, and reference to the earlier work therein; Nagendra 1994, 1995). The thermalization parameter ε is defined as $\varepsilon = \Gamma_I/(\Gamma_R + \Gamma_I)$, where the stimulated emission is neglected. The Voigt damping parameter a is defined through the relation

$$a = \frac{\Gamma_R + \Gamma_E}{4\pi\Delta\nu_D}, \quad \text{where} \quad \Delta\nu_D = \frac{\nu_0}{c}\sqrt{\frac{2kT_e}{M_a}}. \tag{14}$$

3. The GPALI Method to Solve the Hanle PRD Problem using Bommier's Redistribution Matrices

In this Section we describe a PALI method which is suitable only for the Angle Averaged version of Bommier (1997) redistribution matrices. The ability to Fourier expand the intensity $\boldsymbol{\mathcal{I}}$, source vector $\boldsymbol{\mathcal{S}}$ with respect to φ, and to expand $\hat{P}_H$ w.r.t φ and φ', allows one to derive "$\varphi-$ independent reduced transfer problem in Fourier space". The transformation is described in Faurobert-Scholl (1991). The relevant PALI method for Hanle scattering is described in Nagendra et al. (1998) for CRD case, and subsequently extended to the 'standard PRD case' in Nagendra et al. (1999). The required generalization for the present situation is detailed in Fluri et al. (2003). We essentially solve the Hanle-PRD problem for the 6-component reduced intensity in the Fourier space, - and later transform back to real space - to recover the non-axisymmetry (or φ - dependence). Only the main equations are given below. The domain based core-wing approach, is the new feature of this GPALI method.

3.1. The Basic Equations of the Reduced Transfer Problem

The RTE for 6-component axi-symmetric intensity vector $\mathbf{I} = (I_I\ I_Q\ I_{\pm1}\ I_{\pm2})$ is:

$$\mu\frac{\partial \mathbf{I}(\tau,x,\mu)}{\partial\tau} = (\phi(\tau,x) + \beta(\tau))[\mathbf{I}(\tau,x,\mu) - \mathbf{S}(\tau,x)]. \tag{15}$$

The total source vector in the 6-component representation is

$$\mathbf{S}_x = \frac{\phi_x \mathbf{S}_{\ell,x} + \beta B_{\nu_0}\mathbf{U}}{\phi_x+\beta}, \qquad \mathbf{U} = (1,0,0,0,0,0)^T. \tag{16}$$

In this case, the 6-component line source vector takes the form

$$\mathbf{S}_{\ell,x} = \frac{1}{\phi_x}\int_{-\infty}^{+\infty}\left[\hat{M}_{II}R_{II} + \hat{M}_{III}R_{III}\right]\mathbf{J}_{x'}\,dx' + \epsilon B_{\nu_0}\mathbf{U}. \tag{17}$$

The (6×6)-matrices $\hat{M}_{III}$ and $\hat{M}_{II}$ in Fourier space are abbreviations for

$$\begin{aligned}\hat{M}_{III} &= b_{III,1}\hat{W}^2_{III,1}\hat{H}_{B,III,1} + b_{III,2}\hat{W}^2_{III,2}\hat{H}_{B,III,2}\ ,\\ \hat{M}_{II} &= b_{II}\hat{W}^2_{II}\hat{H}_{B,II}\ ,\end{aligned} \tag{18}$$

where the b's are magnetic field modified collisional branching factors involving the $\Gamma_{\rm B}$. The (6×6)-matrices $\hat{W}$ are magnetic field modified $W_2(J, J')$ de-polarization matrices. The (6×6)-matrices $\hat{H}_{\rm B}$ are collisionally modified $\hat{H}_{\rm B}$ matrices (the magnetic matrices). The reduced mean intensity vector $\mathbf{J}_{x'}$ is:

$$\mathbf{J}_{x'} \;=\; \frac{1}{2}\int_{-1}^{+1} \hat{B}_0^{\rm T}\,(\mu')\,\hat{B}_0\,(\mu')\,\mathbf{I}\,(\tau, x', \mu')\,d\mu'\;, \tag{19}$$

where 'T' stands for the transpose of matrix $\hat{B}_0$. The (14×6)-matrix $\hat{B}_0$ is defined in Fluri et al. (2003). It resembles the $\hat{B}(\mu)$ matrix in Nagendra et al. (1998) except for a minor difference. Other quantities are not repeated here as they are listed in the above reference. In the conventional PRD theory (see eg. Nagendra et al. 1999), b and W^2 are single scalars – indeed only one type of $\hat{H}_{\rm B}$ matrix exists. Clearly, the collisions and weak magnetic field together result in a coupling between angular and frequency redistribution, although in each frequency domain, there is no coupling between frequency and angular redistribution.

3.2. The Basic Equations of the GPALI Method

This method represents a generalization of the conventional core-wing treatment of PRD (Nagendra et al. 1999), to include collisions and the frequency domains in the definition of scattering integrals. It is called Generalized PALI method (GPALI), and is based on the core-wing approach of Paletou & Auer (1995). We can write expression for the line source vector corrections $\delta\mathbf{S}_{\ell,x}^{(n)}$ as

$$\delta\mathbf{S}_{\ell,x}^{(n)} - \frac{1}{\phi_x}\int_{-\infty}^{+\infty}\Big(\hat{M}_{\rm II}\,R_{\rm II} + \hat{M}_{\rm III}\,R_{\rm III}\Big)\,p_{x'}\,\hat{\Lambda}^*_{x'}\Big[\delta\mathbf{S}_{\ell,x'}^{(n)}\Big]\;dx' \;=\; \mathbf{r}_x^{(n)}, \tag{20}$$

where $p_x = \phi_x/(\phi_x + \beta)$, and the frequency dependent residual vector is

$$\mathbf{r}_x^{(n)} \;=\; \mathbf{S}_{\ell,x}^{(n)}({\rm FS}) - \mathbf{S}_{\ell,x}^{(n)}\;. \tag{21}$$

The line source vector $\mathbf{S}_{\ell,x}^{(n)}({\rm FS})$ is explicitly computed using the equation

$$\mathbf{S}_{\ell,x}^{(n)}({\rm FS}) \;=\; \frac{1}{\phi_x}\int_{-\infty}^{+\infty}\Big(\hat{M}_{\rm II}\,R_{\rm II} + \hat{M}_{\rm III}\,R_{\rm III}\Big)\,\mathbf{J}_{x'}^{(n)}\;dx' + \;\epsilon B_{\nu_0}\mathbf{U}, \tag{22}$$

where the mean intensity $\mathbf{J}_x^{(n)} = \hat{\Lambda}_x[\mathbf{S}_x^{(n)}]$ is computed using the short characteristic Formal Solver (FS). It is important to recognize that the $\hat{M}$ matrix depends on the branching factors b, de-polarization matrices $\hat{W}$, and the magnetic matrices $\hat{H}_{\rm B}$, all of which 'differ from domain to domain'. In other words, $\hat{M}$ matrices depend on (x, x') in a step-wise manner, and hence can not easily be taken out of the frequency integral ! This exactly is the reason for numerical complexity of the GPALI numerical scheme.

3.3. Iterative Corrections to the Line Source Vector

When the concept of core-wing separation is applied to the $R_{\rm II}$ and $R_{\rm III}$ PRD functions (see Fluri et al. 2003), we obtain different expressions for corrections in the line core and wing. The iterative corrections $\delta\mathbf{S}_{\ell,\rm x}$ in the line core frequencies $|x| \leq v_{\rm c}(a)$ are written as:

$$\delta\mathbf{S}_{\ell,x}^{(n)} = \left(\hat{M}_{\rm D1} + \hat{M}_{\rm D4}\right) \Delta\mathbf{T}_{\rm core}^{(n)} + \mathbf{r}_x^{(n)} \quad , \tag{23}$$

where the frequency independent operator $\Delta\mathbf{T}_{\rm core}^{(n)}$ (notice the advantages accrued by the frequency independence of this operator!) is given by

$$\Delta\mathbf{T}_{\rm core}^{(n)} = \left[\hat{I} - \int_{-v_{\rm c}(a)}^{+v_{\rm c}(a)} \phi_{x'}\, p_{x'}\, \hat{\Lambda}_{x'}^{*} dx' \cdot \left(\hat{M}_{\rm D1} + \hat{M}_{\rm D4}\right)\right]^{-1} \int_{-v_{\rm c}(a)}^{+v_{\rm c}(a)} \phi_{x'}\, p_{x'}\, \hat{\Lambda}_{x'}^{*}\left[\mathbf{r}_{x'}^{(n)}\right] dx', \tag{24}$$

with $\hat{I}$ being a (6×6) identity operator. The corrections $\delta\mathbf{S}_{\ell,\rm x}$ in the wing frequencies $|x| > v_{\rm c}(a)$ are written as:

$$\delta\mathbf{S}_{\ell,x}^{(n)} = \left[\hat{I} - \alpha_x\, p_x\, \hat{M}_{\rm D5}\, \hat{\Lambda}_x^{*}\right]^{-1} \left[\left(\hat{M}_{\rm D2} + (1-\alpha_x)\, \hat{M}_{\rm D5}\right) \Delta\mathbf{T}_{\rm core}^{(n)} + \mathbf{r}_x^{(n)}\right]. \tag{25}$$

In the iterative cycle, the vector $\Delta\mathbf{T}_{\rm core}^{(n)}$ is computed first, and then substituted into the core and wing source vector correction equations to obtain $\delta\mathbf{S}_{\ell,x}^{(n)}$ at all the frequencies within the line profile. A comparison with a simple vertical domain based core-wing approach (Nagendra et al. 1999) shows that instead of $\hat{M}$ matrices (or their sums), we have only a single $\hat{H}_{\rm B}$ matrix in the simple case.

Notice the two great advantages of the core-wing separation scheme:– (1) in the original integral equation for the correction (Eq. 20), different frequency points are strongly coupled through the redistribution functions. Such a coupling does not exist in the above algebraic equations for $\delta\mathbf{S}_{\ell,\rm x}^{(n)}$. Just one inversion of a (6×6) matrix is required for each frequency point x, - done only once outside the GPALI loop !. (2) Similarly, the simple core integral, and one inversion of a (6×6) matrix in the expression for $\Delta\mathbf{T}_{\rm core}^{(n)}$ is also performed only once outside the iteration loop. The above two advantages contribute to a sharp decrease in the CPU time required for each iteration step of PALI. The convergence criterion for GPALI method is same as in the simple PRD case solved with PALI.

Fig. 3 shows the convergence behavior of GPALI. The panel (a) shows the usual pattern of rapid convergence. The convergence is slower for the small values of ε (indicative of non-locality - the NLTE), and for the semi-infinite atmospheres (role of multiple scattering - randomization of errors), and non-uniformly illuminated slab (high non-linearity of source function gradient) etc. See Faurobert et al. (1997) for a discussion of these effects on PALI convergence. The slow diffusion in frequency and space in the case of $R_{\rm II-AA}(x, x')$ is often responsible for slower convergence (demonstrated in Nagendra et al. 1999). The panel (b) in Fig. 3 shows a rather extreme case to test an iterative method! It is a pure scattering case without true absorption and emission. The characteristic

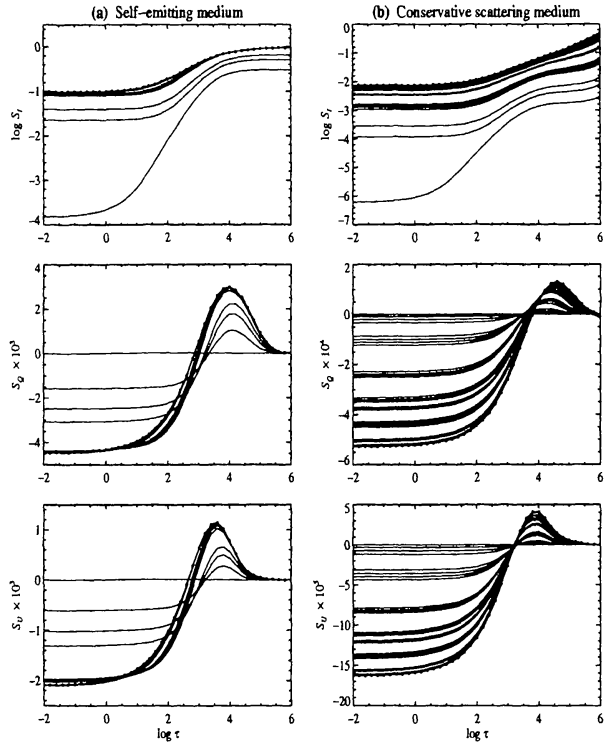

Figure 3. The Figure shows the convergence behavior of Stokes source functions for frequency $x = 3$, in the direction $\mu = 0.11$, and $\varphi = 0°$. The dots identify the converged solution. Panel (a):– A self-emitting slab model with the parameters $(T, a, \epsilon, \beta, \Gamma_E/\Gamma_R) = (2 \cdot 10^6, 10^{-3}, 10^{-4}, 0, 10^{-1})$, and the magnetic field represented by $(\Gamma_B, \theta_B, \varphi_B)$=$(1, 30°, 0°)$. The convergence behavior in this case is typical of all the standard PALI methods. Panel (b):– The results for an irradiated pure conservative scattering slab model with the parameters $(T, a, \epsilon, \beta, \Gamma_E/\Gamma_R) = (2 \cdot 10^6, 10^{-3}, 0, 0, 10^{-1})$ and the magnetic field parameters given by $(\Gamma_B, \theta_B, \varphi_B)$=$(1, 30°, 0°)$. The convergence is relatively slow in this case because of highly non-local nature of the radiation field, which makes it a difficult benchmark.

scale of photon thermalization is larger than the physical dimensions of the medium, in this particular case. This fact, together with the dominance of $R_{\mathrm{II-AA}}(x, x')$, as we have chosen $\Gamma_E/\Gamma_R = 0.1$, causes a slow convergence – nevertheless notice the uniform and stable convergence of the GPALI method !

Tests like this one, and the comparison with solutions computed by direct methods (Nagendra et al. 1999; Fluri et al. 2003; and Nagendra 2003), convince us of the two interesting characteristics of PALI methods:- (a) their ability to handle non-local nature of diffuse radiation field through 'operator splitting', with the scalar intensity always serving as the master mode to drive the polarized iteration cycle, (b) the felicity with which the formal solution (FS) step accurately handles the 'seemingly discontinuous representation' of the highly sensitive redistribution integral, which employ the idea of core-wing separation, or complex frequency domains of Bommier's PRD theory. It is not out of con-

text, to mention that the concept of core-wing separation in ALI (Paletou & Auer 1995), is 'generic' to all types of PRD functions (R_i, i = I, II, III, V), and a very useful practical tool in the polarized line transfer work. The above observations are amply demonstrated through the performance of PALI and GPALI methods, which were used on a variety of theoretical problems.

4. The Perturbation Method to Solve the Hanle PRD Problem

The exact theory of Angle Dependent (AD) collisional Hanle effect with PRD is quite complex. The required PALI method is not yet developed. Hence we employ a perturbative method to solve the line transfer problem, and explore the new physical effects involved. The idea is to treat polarization as a perturbation to the intensity. In the AD-problem, the frequency domains are angle (Θ) dependent, and also we have to use angle dependent scalar redistribution functions $R_{\rm II,III}(x, x', \Theta)$. The perturbation method works only when the polarization is weak, which is often the case for the Hanle effect. We use a 'short characteristic formal solver' (FS) in the perturbative improvement of the previous solution. The numerical method, and the special care required to handle the AD scattering integral, are described in Nagendra et al. (2002).

4.1. The Main Steps of the Perturbation Method

Without elaborating on the theoretical and numerical aspects of this method, we give below only the main steps of the perturbation method for solving the Angle Dependent Hanle PRD problem. The method proceeds in an iterative way, the solution of each step serving as starting point for the next one. In the first two steps, the solution is calcultated iteratively. In the first step we solve an unpolarized PRD problem for isotropic scattering by a scalar ALI method. In the second step we solve a polarized PRD resonance scattering problem, assuming an angle-averaged redistribution function, using a PALI method. In the third step we solve the Hanle problem with an angle-averaged frequency redistribution function, the angle-averaged domains of Fig. 2 and a perturbation method. Finally, in the last and fourth step, we consider the Hanle problem with an angle-dependent frequency redistribution function, the domains of Fig. 1 and calculate the solution with a perturbation method as in step 3.
For guiding the above mentioned 4-step perturbative cycle, we apply the same convergence testing procedure as in the case of standard PALI method.

4.2. An Example: Angle Dependent Hanle Redistribution, and the Effect of Elastic Collisions

To illustrate the perturbation method, we take a simple example of the formation of a moderately strong resonance line, computed using the AD partial frequency redistribution theory of Bommier (1997).

Fig. 4 shows the effect of elastic collisions $\Gamma_{\rm E}$ and de-polarizing collisions $D^{(2)}$ on emergent (I, Q, U) line profiles in the AA and AD cases. The atmospheric parameters used for computing these profiles are: $[T,\ a,\ \varepsilon,\ \beta,\ B_\nu] = [2\times 10^4,\ 10^{-3},\ 10^{-3},\ 0,\ 1]$. The magnetic field is characterized by: $[\Gamma_{\rm B},\ \theta_{\rm B},\ \varphi_{\rm B}] = [1,\ 30^\circ,\ 0^\circ]$. The collisions are characterized by: $(\Gamma_{\rm E}/\Gamma_{\rm R}) = 0,\ 0.1,\ 1,\ 10,\ 100$

and $D^{(2)} = 0.5\Gamma_E$. While $(\Gamma_E/\Gamma_R) = 0$ (pure R_{II} type redistribution) represents chromospheric layers, with negligible role for elastic collisions, $(\Gamma_E/\Gamma_R) = 100$ (pure R_{III} type redistribution) represents base of the photospheric layers, where the elastic collisions dominate over other types of collision. We briefly interpret the results of Fig. 4 as follows (see Nagendra et al. 2002 for further explanation):

The Stokes I Profile: The panels (a) for AA and (d) for AD in Fig. 4 show the strong impact of elastic collisions on the I-profile as Γ_E/Γ_R is increased in magnitude. In the line core we can assume $R_{III} \approx R_{II}$. Hence the redistribution factor $\gamma_{coh} R_{II} + (1-\gamma_{coh}) R_{III}$ in the scattering integral becomes independent of the coherence fraction $\gamma_{coh} = \alpha/(1-\varepsilon)$ with $\alpha = \Gamma_R/(\Gamma_R + \Gamma_I + \Gamma_E)$, as seen in the Fig. 4. Hence the line core is insensitive to this elastic scattering parameter. In the line wing, the profiles go from pure R_{II} to pure R_{III} case respectively, as the value of Γ_E/Γ_R goes from 0 to 100. Clearly a line broadening takes place in the wings of the I profile, when elastic collision strength is increased.

The Stokes Q Profile: The panel (b) for AA, and panel (e) for AD show that, in terms of the sensitivity of Stokes Q parameter, both are equally sensitive to (Γ_E/Γ_R). The sensitivity w.r.t the de-polarizing collision rate $D^{(2)}$ is also nearly similar, for both AA and AD cases. Generally Stokes Q is relatively less affected by the angle-frequency coupling in the scattering event. As in the case of I, there is a broadening of the Q profile wings, and a decrease in magnitude of Q in the line core, w.r.t an increase in (Γ_E/Γ_R). As the ratio (Γ_E/Γ_R) is increased from 0 to 1, R_{III} becomes more and more dominant compared to R_{II} leading to an increased line wing ($x > 5$) anisotropy, and a consequent increase in the Stokes Q parameter. It is connected with the asymptotic behavior of the respective source vectors for these two redistribution functions (Frisch 1980).

The Stokes U Profile: The panel (c) for the AA case shows a decrease of Stokes U at all the line frequencies, and there is no broadening, as Γ_E/Γ_R is varied from 0 to 100. Panel (f) for the AD case shows a monotonic decrease of Stokes U in the line core with increasing values of Γ_E/Γ_R. An interesting feature in the AD case, is the weak but extended wings of Stokes U. It arises due to the φ - dependence of the redistribution function, plus the collisional broadening of I profile, which controls U. This shallow U profile, or the 'wing-Hanle effect' (see for eg. the dashed line in the panel (f) in Fig. 4), in addition to the well known 'core-Hanle effect' in the line core, is naturally contained in the Hanle PRD theory of Bommier (1997).

Recently Bianda et al. (2003) have discovered a persistent appearance of Stokes U in the far wings of Ca I 4227 Å line. They propose a possible mechanism by which a transfer of Hanle angular coherence from line core to the line wing may happen (when $\Gamma_E \approx \Gamma_R$), through the following sequence:- (a) First there is radiative excitation at a frequency point in the line core region. (b) Atom relaxes in the excited state (radiative life time), during which (classically), the Hanle precession of the excited oscillator continues, 'until an elastic collision occurs'. (c) The elastic collision shifts the atom to a wing frequency point - without largely destroying the atomic polarization of the upper state. (d) And finally the emission of photon occurs from the line wing, causing a de-polarization and a rotation of the plane of linear polarization in the wing frequencies.

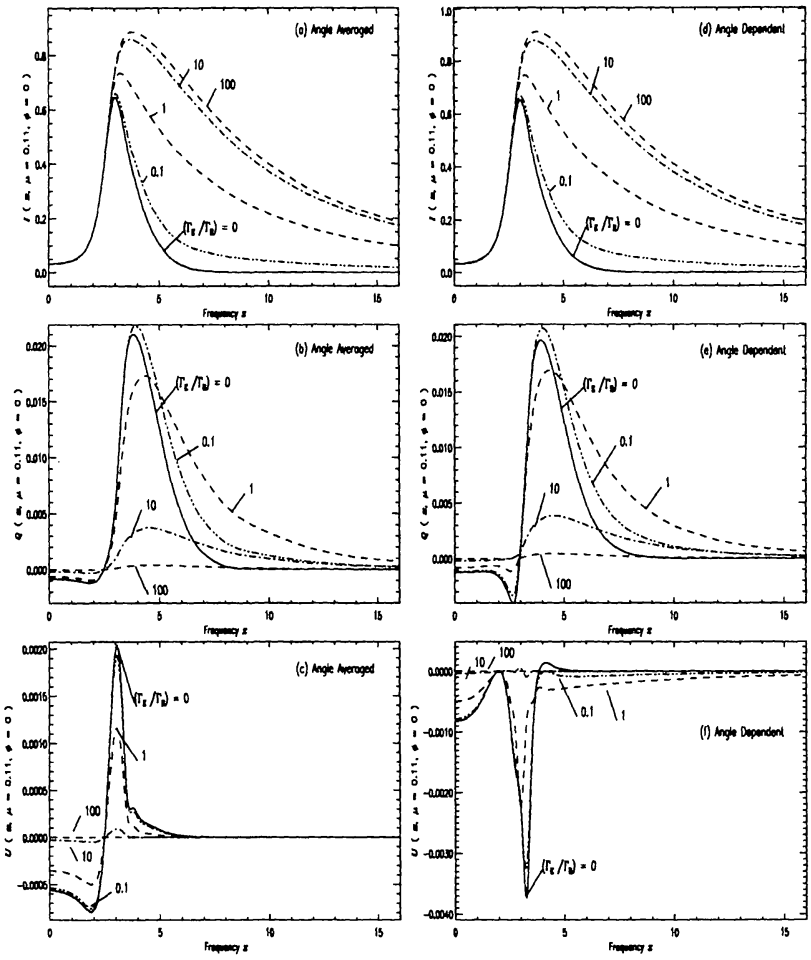

Figure 4. The effect of elastic collisions on emergent line polarization under the AA (left panels), and the AD (right panels) approximations. The collisions are characterized by the ratio $(\Gamma_E/\Gamma_R) = 0, 0.1, 1, 10, 100$. The curves are identified by this collision strength ratio. Notice that the Stokes U under the AD case is more sensitive to the elastic collisions, than under the AA case. See Sect. 4 for details.

Although it is premature to ascribe the 'Angle Dependent (AD) Hanle partial redistribution' as a possible explanation of this so called 'wing Hanle effect', it is at least clear from our calculations, that the use of AD Hanle PRD theory would be necessary, instead of AA theory, in analyzing Stokes U data, whenever one intends to perform a 'fine analysis' using Hanle line transfer theory.

5. Conclusions

The GPALI method provides a fast method to solve the complex Hanle scattering PRD line transfer problems. Until now, we have addressed only problems involving AA redistribution functions, using this iterative method. The core-

wing separation approach is a powerful tool in adopting any new PRD related physical problem to an existing PALI numerical code.

On the one hand, we have been able to use the Fourier azimuthal expansion of the angular phase matrices, as we always worked with the AA redistribution functions. Such a Fourier expansion transforms the Stokes vector transfer equation to an 'azimuth independent reduced transfer equation', which is simpler to solve by the existing PALI methods. On the other hand, the corresponding PALI method involving Angle Dependent (AD) redistribution functions is not yet formulated, because, whereas in the AA case we are able to decipher the basis on which some level of the angle and frequency de-coupling can be achieved (as done in Fluri et al. 2003), it is not clear whether a similar approach can be used in the AD case. Thus, for the present, there is no other way but to employ a 'polarization perturbation method' to solve the AD problem (see Nagendra et al. 2002). This method has a drawback from the practical point of view, as it is highly time consuming (about 1000 times slower than GPALI method).

The application of Bommier's (1997) redistribution matrices seems to be essential, for any serious modeling work connected with solar line polarization. It makes substantial difference to the way in which we interpret solar polarimetric data based on the Hanle effect.

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Discussion

J. TRUJILLO BUENO: Over the last few years you have been developing a code for the numerical solution of the two-level atom Hanle effect problem in one-dimensional plane-parallel geometry, neglecting ground level polarization, but taking into account PRD effects according to Bommier's (1997) formalism. These assumptions are more than sufficient for modelling the very interesting Ca I 4227 Å line that Bianda et al. (2003) have observed extensively, including the disk-center observations of moderately magnetized regions regions for the forward scattering case suggested in Trujillo Bueno (2001). It would be very interesting if you could apply your code for the interpretation of such observations.

NAGENDRA: Yes, I agree that some of the new PRD effects seen in these calculations would be confirmed only when we compare with high quality Hanle effect observations of the Sun. However, the use of a realistic atmosphere is necessary for such modeling computations, while comparing with observed data.