APPLICATION OF THE TENSOR VIRIAL EQUATIONS TO STELLAR DYNAMICS

G. SOM SUNDER

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THIS THESIS IS DEDICATED TO

MY FAMILY MEMBERS

FOR THEIR PATIENCE

AND

MY FRIENDS

FOR THEIR SPIRITS

DECLARATION

This thesis, submitted for the degree of 'Doctor of Philosophy' in Astronomy and Astrophysics, is entirely original and has not been submitted before, either in parts or in full to any University for any research degree.

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Department of Astronomy
Osmania University
Hyderabad - 500 007

and

Indian Institute of Astrophysics
Bangalore - 560 034

(G SOM SUNDER)

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I N D E X

CHAPTER 1

INTRODUCTION

1. A historical review:

The virial equations in their simplest form have been known for over two hundred years. As early as 1772, the Royal Academy of Sciences in Paris published Joseph Louis Lagrange's 'Essay on the problem of three bodies', in which he developed what is today called the Lagrange's identity, or the scalar virial equation for a system of three bodies (cf. Lagrange 1873).

For long, Lagrange's identity remained a special case germane to the three body problem. In the winter of 1842-43 Karl Gustav Jacob Jacobi generalized Lagrange's result to an N-body system (cf. Jacobi 1889). Jacobi's formulation closely parallels the present representation of the virial equation. In the same chapter he then proceeds to develop the criterion which today bears his name. The Jacobi's criterion states that a system is unstable if the total energy is positive, or in other words if the kinetic energy exceeds in magnitude the potential energy.

A little later, in 1851, Rudolf Julius Emmanuel Clausius started a long study into the mechanical nature of heat which culminated in what we know today as the first clear statement of the virial theorem: "the mean *vis viva* of the system is equal to the *virial*" (cf. Clausius 1870). The *vis viva* was the term used in the nineteenth century for what we call today the kinetic energy of the system. The term <u>virial</u> (from the Latin <u>virias</u>, the plural of <u>vis</u> and meaning forces) was coined by Clausius to represent half the mean value of the moments of the forces acting on the particles in the system $\left(\frac{1}{2}\sum_{\alpha=1}^{N} \overline{\mathbb{F}^{(\alpha)} \cdot \mathbb{X}^{(\alpha)}}\right)$ and can be identified with the average potential energy of the system.

Although the virial of Clausius has today lost much of its significance as a physical concept, the name has become attached to the theorem and its descendents.

Since these early formulations the theorem has been considerably developed in the following years. Henri Poincare (1911) used a form of the virial equation to investigate the stability of structures in different cosmological theories. Paul Ledoux (1945) developed a variational form of the virial equation to study stellar pulsations. Subrahmanyan Chandrasekhar and Enrico Fermi (1953) extended the virial theorem to include the presence of magnetic fields.

At the turn of this century Lord Rayleigh (1903) formulated a generalization of the virial theorem in which we see the beginnings of the tensor form of the theorem. This was later revived by Eugene Newman Parker (1954) and developed considerably by Chandrasekhar and coworkers. Using the virial equations and their variational forms, Chandrasekhar and coworkers investigated the structure and stability of ellipsoidal fluid masses (cf. Chandrasekhar 1969).

Today the tensorial formulation of Lagrange's identity exists not only for dynamical and thermodynamical systems but also for systems with velocity dependent forces, viscous systems, systems exhibiting macroscopic motions such as rotation, systems with magnetic fields, and even systems which require both the special and general theories of relativity for their description.

Having reviewed briefly the historical development of the virial equations, we shall, in the next section discuss their importance in stellar dynamics.

2. The virial equations in stellar dynamics:

The most direct approach to stellar dynamics is to treat the stellar system as a collection of N mass points. The equations of motion of an individual point in one such isolated cluster are

$$m^{(\alpha)} \frac{d^2 x_i^{(\alpha)}}{dt^2} = m^{(\alpha)} \frac{d u_i^{(\alpha)}}{dt} = -G\, m^{(\alpha)} \sum_{\substack{\beta=1 \\ \alpha \neq \beta}}^{N} m^{(\beta)} \frac{x_i^{(\alpha)} - x_i^{(\beta)}}{|\mathbf{x}^{(\alpha)} - \mathbf{x}^{(\beta)}|^3} \qquad (1)$$

$$(\alpha = 1, \cdots, N \; ; \; i = 1, 2, 3)$$

where the Greek superscripts distinguish the N mass points while the Latin subscripts distinguish the Cartesian components of the position and velocity vectors $\mathbf{x}$, $\mathbf{u}$.

The various conservation laws and the virial equations are just moments of the equations of motion. For example multiplying equation (1) by $x_j^{(\alpha)}$ and summing over all α we get the virial equations of the order two (cf. Chandrasekhar 1964),

$$\frac{1}{2}\frac{d^2 I_{ij}}{dt^2} = 2K_{ij} + W_{ij} , \tag{2}$$

where I_{ij}, K_{ij} and W_{ij} are respectively the moment of inertia, the kinetic energy and the potential energy tensors given by

$$I_{ij} = \sum_{\alpha=1}^{N} m^{(\alpha)} x_i^{(\alpha)} x_j^{(\alpha)} , \qquad K_{ij} = \frac{1}{2}\sum_{\alpha=1}^{N} m^{(\alpha)} u_i^{(\alpha)} u_j^{(\alpha)} ;$$

$$W_{ij} = -\frac{G}{2}\sum_{\substack{\alpha=1 \\ \alpha \neq \beta}}^{N}\sum_{\beta=1}^{N} m^{(\alpha)} m^{(\beta)} \frac{(x_i^{(\alpha)} - x_i^{(\beta)})(x_j^{(\alpha)} - x_j^{(\beta)})}{|x^{(\alpha)} - x^{(\beta)}|^3} . \tag{3}$$

Similarly, multiplying equation (1) by $u_i^{(\alpha)}$ and summing over both the superscript α and subscript i one obtains the law of energy conservation (cf. Chandrasekhar 1964)

$$K + W = E , \tag{4}$$

A complete description of the dynamics of the system is contained in the equations of motion (1). The solution of these equations are, however, cumbersome and one 'may not see the wood for the trees' in the accompanying avalanche of numerical details. It is under these circumstances that the conservation laws and the virial equations assume significance. Being moments of the equations of motion, they represent basic structural relationships that the system must satisfy at all instants of time. By studying these equations one obtains the large-scale or global properties of the system.

The common procedure in such a study is to describe the stellar system statistically by a density distribution $\rho(\mathbf{x})$ and to use equations (2) and (4) altering only the definitions of I_{ij}, K_{ij} and W_{ij} as sums to integrals thus:

$$I_{ij} = \int_V \rho(\mathbf{x})\, x_i x_j \, d\mathbf{x} \quad , \quad K_{ij} = \frac{1}{2}\int_V \rho(\mathbf{x})\, u_i u_j \, d\mathbf{x} ,$$

$$W_{ij} = -\frac{1}{2} G \iint_{V\,V} \rho(\mathbf{x})\, \rho(\mathbf{x}') \, \frac{(x_i - x_i')(x_j - x_j')}{|\mathbf{x} - \mathbf{x}'|^3} \, d\mathbf{x}\, d\mathbf{x}' , \tag{5}$$

where the integrations are effected over the entire volume V of the stellar system.

It has been pointed out by Chandrasekhar & Lee (1969) that the transition from the summation of equations (3) to the integrals of equations (5) is not strictly justified.

They however obtain the virial equations in a form similar to equation (3) by integrating the Lioville's equation for an ensemble of such systems. For a cluster in which we can neglect close interactions, a simple approach, adopted by Binney (1982) is to obtain these equations starting from the collisionless Boltzmann's equation. The procedure adopted in both these approaches is to first obtain the equations of stellar hydrodynamics (the Jeans equation)- viz. the equations of continuity and motion. The virial equations are then obtained by taking the moments of these equations.

The earliest use of the virial equations in stellar dynamics was to study the masses in clusters of galaxies from a knowledge of their sizes and velocity distributions. When the system is in steady state, equation (2) reduces to

$$2 K_{ij} = - W_{ij} . \tag{6}$$

The contracted version of this relation is

$$2 K = - W ,$$

where

$$K = \frac{1}{2} M \overline{u^2} \quad , \qquad W = - \alpha \frac{G M^2}{a} .$$

Here M, a and $\overline{u^2}$ are respectively the mass, radius and mean-square velocity of the cluster. The parameter α is a measure of the central concentration. From these relations one obtains

$$M = \frac{\overline{u^2}\, a}{2\alpha G},$$

and as $\overline{u^2}$, a and α can be got from observations, one obtains an estimate of the mass in a cluster. It is such an analysis that has led to the conclusion that the mass of a cluster of galaxies is far greater than what has been observed in visable light (cf. Limber 1961 for a review).

In recent years the tensor virial equations have been employed to study equilibrium configurations of ellipsoidal galaxies (cf. Binney 1982, also Chapter 3 of this work).

In these applications use has been made of the virial equations only for systems in or near equilibrium. However it is important to realize that systems in a state of rapid change, such as a star cluster in rapid dynamical evolution, are also subject to the time dependent form (2) of the virial equations. The tensors I_{ij} and W_{ij} are expressible in terms of the semiaxes of the system, and hence to employ equation (2) to study dynamical evolution one requires an additional system of relations that expresses the variation of K_{ij} with time.

In their study of spherical star clusters, for which a single parameter, the radius a, describes the cluster, Chandrasekhar and Elbert (1972) employ the contracted version of equation (2) coupled with the equation of energy conservation (4).

For a study of nonspherical clusters, however, we have to employ equation (2) and the tensor equivalent of equation (4), namely, the relation:

$$2\frac{d}{dt}K_{ij} = -\frac{G}{2}\sum_{\substack{\alpha=1\\ \alpha\neq\beta}}^{N}\sum_{\beta=1}^{N} m^{(\alpha)}m^{(\beta)}\frac{(x_i^{(\alpha)}-x_i^{(\beta)})(u_j^{(\alpha)}-u_j^{(\beta)})}{|\mathbb{X}^{(\alpha)}-\mathbb{X}^{(\beta)}|^3} - \frac{G}{2}\sum_{\substack{\alpha=1\\ \alpha\neq\beta}}^{N}\sum_{\beta=1}^{N} m^{(\alpha)}m^{(\beta)}\frac{(u_i^{(\alpha)}-u_i^{(\beta)})(x_j^{(\alpha)}-x_j^{(\beta)})}{|\mathbb{X}^{(\alpha)}-\mathbb{X}^{(\beta)}|^3} \quad (7)$$

which can be obtained by multiplying equation (1) by $u_j^{(\alpha)}$ and summing over all α. The contracted version of (7) is just the law of energy conservation (4). Once again the sums may be transformed into integrals. Equivalently a similar relation can be obtained as follows: One first obtains the Jeans equation of the order two by taking moments of the Boltzmann's equation. This is the relation that is known in statistical mechanics as the heat transfer equation (cf. Clemmov and Dougherty 1969). By integrating this equation over the volume V one obtains the equivalent of equation (7). We obtain these equations in a slightly more general context in Chapter 2.

The derivations of the virial equations discussed above are for isolated stellar systems. When other dynamical problems are considered these equations will have to be reformulated starting from first principles. It is, however, possible-as we shall show in chapter 2-to derive these equations from purely kinematical considerations starting with the equation of continuity in phase-space. Since the latter is valid for any collisionless system, the equations derived would also be valid for any such system.

In chapter 3 we employ the virial equations to study the evolution of a cluster characterized by a spheroidal distribution of matter and an isotropic distribution of velocities. In chapter 4 we will relax the assumption of isotropic velocity distribution. In chapter 5 we consider a stellar system which was initially in steady state but had its initial velocity distribution disturbed by an impulsive encounter with a passing perturber. The results are summarized in chapter 6.

CHAPTER 2

THE VIRIAL EQUATIONS

3. Introduction:

In this chapter we derive the various equations that we require. These equations will be derived from purely kinematical considerations starting with the equation of continuity in phase-space, and will apply to any dynamical system. Besides the virial equations, they include an expression for the rate of change of the kinetic energy tensor K_{ij}. In section 6 we develop expressions for the various tensors for an ellipsoidal mass distribution.

4. Definitions:

We consider a stellar system, a star cluster or a galaxy, consisting of a large number, N, of stars assumed to be mass points. The position, velocity and acceleration of an individual particle, measured with respect to an arbitrary (inertial or noninertial) frame of reference are denoted respectively by

$$\mathbb{X} : (x_1, x_2, x_3) \,, \quad \mathbb{u} : (u_1, u_2, u_3) \,, \quad \mathfrak{g} : (g_1, g_2, g_3) \qquad (8)$$

and the coordinate in position-velocity phase-space by

$$\mathbb{Z} : (z_1, z_2, \ldots, z_i) \equiv (x_1, x_2, x_3, u_1, u_2, u_3) \equiv (\mathbb{X}, \mathbb{u}). \qquad (9)$$

Assuming that N is large, so that the phase-space distribution is everywhere sufficiently dense, this representation for the entire system can be thought of as a continuous fluid. Hence the phase-density or distribution function $f(\mathbb{Z}, t) = f(\mathbb{X}, \mathbb{U}, t)$, defined such that the mass interior to the phase volume $(\mathbb{Z}; \mathbb{Z} + d\mathbb{Z})$ is given by (cf. Binney 1982, also Chandrasekhar 1960)

$$dM = f(\mathbb{Z}, t)\, d\mathbb{Z} = f(\mathbb{X}, \mathbb{U}, t)\, d\mathbb{X}\, d\mathbb{U}, \qquad (10)$$

satisfies the equation of continuity

$$\frac{\partial f}{\partial t} + \sum_{p=1}^{3} \frac{\partial}{\partial x_p} f u_p + \sum_{p=1}^{3} \frac{\partial}{\partial u_p} f g_p = 0 . \qquad (11)$$

The various macroscopic quantities can be defined in terms of the distribution function. Thus the density at $\mathbb{X}$ is given by

$$\rho(\mathbb{X}, t) = \int_{\mu} f(\mathbb{X}, \mathbb{U}, t)\, d\mathbb{U}, \qquad (12)$$

where the integration is effected over the entire velocity space μ . Consider some quantity with the value, per unit mass, $q(\mathbb{Z}, t)$ for a particle in state $\mathbb{Z}$. The mean value, per unit mass, for all the particles at the position $\mathbb{X}$ is given by

$$\bar{q}(\mathbb{X}, t) = \frac{1}{\rho(\mathbb{X}, t)} \int_{\mu} f(\mathbb{X}, \mathbb{U}, t)\, q(\mathbb{X}, \mathbb{U}, t)\, d\mathbb{U} \qquad (13)$$

while the value of the quantity for the entire system is

$$Q = \int_{\Gamma} f(\mathbf{z},t)\, q(\mathbf{z},t)\, d\mathbf{z} = \int_{V} \rho(\mathbf{x},t)\, q(\mathbf{x},t)\, d\mathbf{x} \qquad (14)$$

Here Γ represents the instantaneous phase-volume of the entire phase-fluid while V denotes the volume in coordinate space of the stellar system. In obtaining the last expression in equation (14) we have effected an integration over the entire velocity space μ, making use of equation (13).

Using equation (14) we can now define the various tensors that we shall use (cf. Chandrasekhar 1969).

Mass:

$$M = \int_{\Gamma} f(\mathbf{z})\, d\mathbf{z} = \int_{V} \rho(\mathbf{x})\, d\mathbf{x}. \qquad (15)$$

Moment of inertia tensors:

$$\begin{aligned} I_i &= \int_{\Gamma} f(\mathbf{z})\, x_i\, d\mathbf{z} = \int_{V} \rho(\mathbf{x})\, x_i\, d\mathbf{x}, \\ I_{ij} &= \int_{\Gamma} f(\mathbf{z})\, x_i x_j\, d\mathbf{z} = \int_{V} \rho(\mathbf{x})\, x_i x_j\, d\mathbf{x}. \end{aligned} \qquad (16)$$

Momentum tensors:

$$\begin{aligned} \mathcal{L}_i &= \int_{\Gamma} f(\mathbf{z})\, u_i\, d\mathbf{z} = \int_{V} \rho(\mathbf{x})\, u_i\, d\mathbf{x}, \\ \mathcal{L}_{i;j} &= \int_{\Gamma} f(\mathbf{z})\, u_i x_j\, d\mathbf{z} = \int_{V} \rho(\mathbf{x})\, \overline{u}_i\, x_j\, d\mathbf{x}. \end{aligned} \qquad (17)$$

Kinetic energy tensor:

$$K_{ij} = \frac{1}{2}\int_\Gamma f(\mathbf{z})\, u_i u_j \, d\mathbf{z} = \frac{1}{2}\int_V \rho(\mathbf{x})\, u_i u_j \, d\mathbf{x} . \tag{18}$$

Tensor potential:

$$\begin{aligned} \mathfrak{V}_{ij}(\mathbf{x}) &= G\int_\Gamma f(\mathbf{z}') \frac{(x_i - x_i')(x_j - x_j')}{|\mathbf{x} - \mathbf{x}'|^3}\, d\mathbf{z}' \\ &= G\int_V \rho(\mathbf{x}') \frac{(x_i - x_i')(x_j - x_j')}{|\mathbf{x} - \mathbf{x}'|^3}\, d\mathbf{x}' . \end{aligned} \tag{19}$$

Potential energy tensor:

$$\begin{aligned} W_{ij} &= -\frac{1}{2}\int_\Gamma f(\mathbf{z})\, \mathfrak{V}_{ij}(\mathbf{x})\, d\mathbf{z} \\ &= -\frac{1}{2}\int_V \rho(\mathbf{x})\, \mathfrak{V}_{ij}(\mathbf{x})\, d\mathbf{x} . \end{aligned} \tag{20}$$

The tensors I_{ij}, K_{ij}, $\mathfrak{V}_{ij}$ and W_{ij} are symmetric in the indices i and j and contract to the scalar moment of inertia I, kinetic energy K, potential $\mathfrak{V}$ and potential energy W respectively. Further, writing

$$u_i = \bar{u}_i + \tilde{u}_i , \tag{21}$$

where $\tilde{u}_i$ represents the contribution to u_i arising from the peculiar motions of the particles, we get (cf. Binney 1982)

$$K_{ij} = T_{ij} + \frac{1}{2}\Pi_{ij} , \tag{22}$$

where

$$T_{ij} = \frac{1}{2}\int_{\Gamma} f(\mathbf{z})\,\bar{u}_i\bar{u}_j\,d\mathbf{z} = \frac{1}{2}\int_V \rho(\mathbf{x})\,\bar{u}_i\bar{u}_j\,d\mathbf{x} ,$$

$$\Pi_{ij} = \int_{\Gamma} f(\mathbf{z})\,\tilde{u}_i\tilde{u}_j\,d\mathbf{z} = \int_V \rho(\mathbf{x})\,\overline{\tilde{u}_i\tilde{u}_j}\,d\mathbf{x} . \tag{23}$$

The dual vector of $\mathcal{L}_{ij}$ is the angular momentum of the system

$$\lambda_k = \sum_{ij} \epsilon_{kji}\,\mathcal{L}_{ij} . \tag{24}$$

Note that when rotating frames of reference are considered this is the angular momentum as measured in the rotating frame.

Beside these tensors we shall also require the moments of the acceleration field:

$$U_i = \int_\Gamma f(\mathbb{z})\, g_i \, d\mathbb{z} = \int_V \rho(\mathbb{x})\, \bar{g}_i \, d\mathbb{x} \,,$$

$$U_{i;j} = \int_\Gamma f(\mathbb{z})\, g_i\, x_j \, d\mathbb{x} = \int_V \rho(\mathbb{x})\, \bar{g}_i\, x_j \, d\mathbb{x} \,, \qquad (25)$$

$$H_{i;j} = \int_\Gamma f(\mathbb{z})\, g_i\, u_j \, d\mathbb{z} = \int_V \rho(\mathbb{x})\, \overline{g_i u_j} \, d\mathbb{x} \,.$$

It is also easily seen that (cf. Chandrasekhar 1969)

$$\int_\Gamma f(\mathbb{z}) \frac{\partial V}{\partial x_i} \, d\mathbb{z} = 0 \,,$$

$$\int_\Gamma f(\mathbb{z}) \frac{\partial V}{\partial x_i} x_j \, d\mathbb{z} = W_{ij} \,. \qquad (26)$$

5. The macroscopic equations:

The virial equations and the other macroscopic equations are most easily obtained with the help of the following theorem:

Theorem 1: If $q(\mathbb{z},t)$ is an attribute of an element of the phase fluid then

$$\frac{d}{dt} \int_\Gamma f(\mathbb{z},t)\, q(\mathbb{z},t)\, d\mathbb{z} = \int_\Gamma f(\mathbb{z},t) \frac{dq}{dt} \, d\mathbb{z} \,, \qquad (27)$$

where

$$\frac{dq}{dt} = \frac{\partial q}{\partial t} + \sum_{p=1}^{3} u_p \frac{\partial q}{\partial x_p} + \sum_{p=1}^{3} g_p \frac{\partial q}{\partial u_p} , \qquad (28)$$

is the total time derivative as we follow the element in its motion through phase-space, and the integration is carried out over Γ , the instantaneous phase volume of the phase fluid.

The proof is analogous to that for the corresponding theorem in hydrodynamics and follows from the equation of continuity (11) which expresses the constancy of each element of the phase fluid in its motion through phase space:

$$\frac{d}{dt} \int_{\Gamma} f(\mathbb{Z}, t)\, d\mathbb{Z} = \frac{dM}{dt} = 0. \qquad (29)$$

By substituting various expressions for q in equation (27) we obtain the various macroscopic equations.

Rate of change of the moments of inertia: With $q = x_i$ and $q = x_i x_j$ equation (27) yields

$$\frac{dI_i}{dt} = \int_{\Gamma} f(\mathbb{Z})\, u_i \, d\mathbb{Z} = \mathcal{L}_i , \qquad (30)$$

$$\frac{dI_{ij}}{dt} = \int_{\Gamma} f(\mathbb{Z}) (u_i x_j + x_i u_j)\, d\mathbb{Z} = \mathcal{L}_{i;j} + \mathcal{L}_{j;i} . \qquad (31)$$

Rate of change of momentum: With $q = u_i$ and $q = u_i x_j$ in equation (27) we obtain

$$\frac{d\mathcal{L}_i}{dt} = \int_\Gamma f(\mathbb{Z})\, g_i \, d\mathbb{Z} = U_i \,, \tag{32}$$

$$\frac{d\mathcal{L}_{ij}}{dt} = \int_\Gamma f(\mathbb{Z})\,(u_i u_j + g_i x_j)\, d\mathbb{Z} = 2 K_{ij} + U_{i;j} \,. \tag{33}$$

The virial equations: Combining equation (30) with equation (32) and equation (31) with equation (33) we get the virial equations of the orders one and two respectively:

$$\frac{d^2 I_i}{dt^2} = U_i \,, \tag{34}$$

$$\frac{d^2 I_{ij}}{dt^2} = 4 K_{ij} + (U_{i;j} + U_{j;i}) \,. \tag{35}$$

For an isolated stellar system with respect to an inertial frame of reference, the acceleration is

$$g_i = \frac{\partial \mathfrak{V}}{\partial x_i} \,, \tag{36}$$

and hence from the definitions (25) using the relations (26) we obtain

$$U_i = 0 \,, \qquad U_{i;j} = W_{ij} \,. \tag{37}$$

In this case the virial equations (34) and (35) take their familiar forms

$$\frac{d^2 I_i}{dt^2} = 0 , \tag{38}$$

$$\frac{1}{2} \frac{d^2 I_{ij}}{dt^2} = 2 K_{ij} + W_{ij} . \tag{39}$$

Virial equations of higher orders may be derived in a similar fashion (cf. Appendix 2).

Rate of change of angular momentum: Multiplying equation (35) by ϵ_{kji} and summing over all i and j we get the expression for the rate of change of angular momentum

$$\frac{d}{dt} \lambda_k = \frac{d}{dt} \left(\sum_{i,j=1}^{3} \epsilon_{kji} J_{i;j} \right) = \sum_{i,j=1}^{3} \epsilon_{kji} U_{i;j} . \tag{40}$$

Since W_{ij} is symmetric in the indices i and j, for an isolated stellar system equation (40) is the law of conservation of angular momentum

$$\frac{d}{dt} \lambda_k = 0 . \tag{41}$$

Rate of change of the kinetic energy tensor: Setting in equation (27) we get

$$2 \frac{dK_{ij}}{dt} = \int_{\Gamma} f(Z)(g_i u_j + u_i g_j) dZ = H_{i;j} + H_{j;i} . \tag{42}$$

For an isolated non rotating stellar system using equation (36) we obtain

$$2\frac{dK_{ij}}{dt} = \left[-\frac{G}{2}\iint_{\Gamma\Gamma} f(\mathbb{Z})f(\mathbb{Z}')\frac{(x_i-x_i')(u_j-u_j')}{|\mathbb{X}-\mathbb{X}'|^3}\,d\mathbb{Z}\,d\mathbb{Z}' \right. \tag{43}$$
$$\left. -\frac{G}{2}\iint_{\Gamma\Gamma} f(\mathbb{Z})f(\mathbb{Z}')\frac{(u_i-u_i')(x_j-x_j')}{|\mathbb{X}-\mathbb{X}'|^3}\,d\mathbb{Z}\,d\mathbb{Z} \right].$$

This relation may be compared with equation (7).

Rate of change of the potential energy: This is most easily obtained using the following theorem:

Theorem 2: If $q(\mathbb{Z}, \mathbb{Z}', t)$ is a two particle function of the particles at $\mathbb{Z}$ and $\mathbb{Z}'$ of the phase fluid then

$$\frac{d}{dt}\iint_{\Gamma\Gamma} f(\mathbb{Z})f(\mathbb{Z}')q(\mathbb{Z},\mathbb{Z}',t)\,d\mathbb{Z}\,d\mathbb{Z} = \iint_{\Gamma\Gamma} f(\mathbb{Z})f(\mathbb{Z}')\frac{dq}{dt}\,d\mathbb{Z}\,d\mathbb{Z}' \tag{44}$$

where

$$\frac{dq}{dt} = \frac{\partial q}{\partial t} + \sum_{p=1}^{3} u_p\frac{\partial q}{\partial x_p} + \sum_{p=1}^{3} g_p\frac{\partial q}{\partial u_p} + \sum_{p=1}^{3} u_p'\frac{\partial q}{\partial x_p'} + \sum_{p=1}^{3} g_p'\frac{\partial q}{\partial u_p'}, \tag{45}$$

is the time derivative as we follow the two elements in their motion through phase-space. The proof is similar to that for theorem 1.

Setting $q = \dfrac{-G}{2}\dfrac{1}{|\mathbb{X}-\mathbb{X}'|}$ we obtain

$$\frac{dW}{dt} = \sum_{p=1}^{3}\frac{G}{2}\iint_{\Gamma\Gamma} f(\mathbb{Z})f(\mathbb{Z}')\frac{(x_p-x_p')(u_p-u_p')}{|\mathbb{X}-\mathbb{X}'|^3}\,d\mathbb{Z}\,d\mathbb{Z}'. \tag{46}$$

Combining this relation with the contracted version of equation (43) we obtain the energy conservation relation for an isolated stellar system (cf. equation 4)

$$\frac{d}{dt}(K+W) = 0, \tag{47}$$

or equivalently

$$K + W = E, \text{ a constant.} \tag{48}$$

In this work we confine ourselves to isolated non rotating stellar systems. However the general expressions obtained above apply to any stellar system and would be of use when more general dynamical problems are considered.

The equations derived above can also be obtained by taking the moments of the hydrodynamical (or Jean's equations (cf. Appendix 1).

6. The tensors for an ellipsoidal system:

The definitions and relations of sections 4 and 5 apply to any stellar system. Of particular interest are ellipsoidal configurations with density distributions of the form

$$\rho(x) = \rho_c (1-m^2)^{\nu}, \tag{49}$$

with

$$m^2 = \sum_{p=1}^{3} \frac{x_p^2}{a_p^2} . \tag{50}$$

Here x_i are the coordinates in a Cartesian frame of reference fixed at the centre of the ellipsoidal with the axes rested along the body axes; a_i are the semiaxes of the ellipsoid while ρ_c is the central density; ν is a nonnegative parameter, increasing values of which indicate increasing central concentration. The case $\nu = 0$ corresponds to the case of homogeneous ellipsoids. Density distributions of the form (49) have been used, for example, by Perek (1962) and Kerr & de Vaucouleurs (1956) to describe the distribution of matter in ellipsoidal galaxies.

The equidensity strata of density distributions of the form (49) are similar to an concentric with the bounding ellipsoid. We can therefore use the results of Roberts (1962). For the function $F(m^2)$ we have

$$F(m^2) = \int_{m^2}^{1} \rho(m^2)\, dm^2 = \frac{\rho_c}{\nu+1} (1 - m^2)^{\nu+1} , \tag{51}$$

and hence

$$M = \pi a_1 a_2 a_3 \rho_c \left[\sqrt{\pi} \; \frac{\Gamma(\nu+1)}{\Gamma(\nu+\frac{5}{2})} \right], \tag{52}$$

$$I_i = 0, \tag{53}$$

$$I_{ij} = \frac{1}{5} M a_i^2 \delta_{ij} \; \phi(\nu) , \tag{54}$$

$$W_{ij} = -\frac{3}{10} \frac{G M^2}{a_1 a_2 a_3} a_i^2 A_i \delta_{ij} \psi(\nu) , \tag{55}$$

where

$$\phi(\nu) = \frac{5}{2\nu+5} , \tag{56}$$

$$\psi(\nu) = \frac{5}{3(\nu+1)^2 \sqrt{\pi}} \left[\frac{\Gamma(\nu+\frac{5}{2})}{\Gamma(\nu+1)} \right]^2 \frac{\Gamma(2\nu+3)}{\Gamma(2\nu+\frac{7}{2})} , \tag{57}$$

and the A_i are the index symbols defined by Chandrasekhar (1969)

$$A_i = a_1 a_2 a_3 \int_0^\infty \frac{du}{\Delta} , \tag{58}$$

with $\Delta^2 = (a_1^2+u)(a_2^2+u)(a_3^2+u)$.

The functions $\phi(\nu)$ and $\psi(\nu)$ are measures of the difference of the moment of inertia and the potential energy tensors of a distribution with the density parameter ν , from that of a homogeneous ellipsoid with the same mass and semiaxes. These functions are listed in table 1 for a few values of ν . For a spherical configuration with $a_1 = a_2 = a_3 = a$ equation (55) contracts to yield

$$W = -\frac{3}{5}\,\frac{GM^2}{a}\,\psi(\nu). \tag{59}$$

On comparing this with the expression

$$W = -\frac{3}{5-n}\,\frac{GM^2}{a}\,, \tag{60}$$

for a polytropic sphere of index n (cf. Chandrasekhar 1939) we see that spheres with indices ν = 1, 4, 12 and 55 have potential energies in fair agreement with polytropic spheres of indices n = 1, 2, 3 and 4 respectively.

For a spheroidal system, axes $a_1 = a_2$ and a_3 the index symbols are given by (cf. Chandrasekhar 1969)

$$\begin{aligned} A_1 = A_2 &= \frac{\sqrt{1-y}}{y}\, S(y) - \frac{1-y}{y}\,, \\ A_3 &= \frac{2}{y} - \frac{2\sqrt{1-y}}{y}\, S(y)\,, \end{aligned} \tag{61}$$

where

$$y = 1 - a_3^2/a_1^2 \tag{62}$$

and

$$S(y) = \begin{cases} \dfrac{\sin^{-1}\sqrt{y}}{\sqrt{y}} & y > 0 \\ 1 & y = 0 \\ \dfrac{\sinh^{-1}\sqrt{-y}}{\sqrt{-y}} & y < 0 \end{cases} \tag{63}$$

Table 1

ν	$\phi(\nu)$	$\psi(\nu)$	$\psi(\nu)\sqrt{\phi(\nu)}$
0	1.000	1.00	1.00
1	0.714	1.19	1.0055
2	0.556	1.36	1.014
4	0.385	1.65	1.024
12	0.172	2.50	1.037
55	0.0435	5.03	1.050
G	$5/2$	$3/2$	

Appendix 1 - Hydrodynamical equations

As mentioned in the introduction the various macroscopic equations of section 5 are moments of the hydrodynamical (or Jean's) equations. In fact if we consider an arbitrary phase volume γ , the more general statement of theorem 1 is

$$\frac{d}{dt}\int_{\gamma} f(z)\, q(z)\, dz = \int_{\gamma} f(z)\, \frac{dq}{dt}\, dz . \tag{A1}$$

Choosing $\gamma = \gamma^{*}$, the phase volume occupied by the stars within an arbitrary volume V* we obtain

$$\frac{d}{dt}\int_{\gamma^*} f\, q\, dz = \iint_{V^*\mu} f\, \frac{dq}{dt}\, dx\, du = \int_{V^*} \rho\, \overline{\frac{dq}{dt}}\, dx . \tag{A2}$$

Further since $f \to 0$ as $u_i \to \infty$ we also obtain

$$\frac{d}{dt}\int_{V^*} f\, q\, dx = \iint_{V^*\mu}\left\{\frac{\partial}{\partial t} f q + \sum_{p=1}^{3}\frac{\partial}{\partial x_p} f q u_p + \sum_{p=1}^{3}\frac{\partial}{\partial u_p} f q \dot{q}_p\right\} dx\, du$$
$$= \int_{V^*}\left\{\frac{\partial}{\partial t}\rho \bar{q} + \sum_{p=1}^{3}\frac{\partial}{\partial x_p}\rho\, \overline{q u_p}\right\} dx. \tag{A3}$$

On comparing equation (A3) with equation (A2) and noting that V* is arbitrary we obtain the general expression for the hydrodynamical equations:

$$\frac{\partial}{\partial t}\rho \bar{q} + \sum_{p=1}^{3}\frac{\partial}{\partial x_p}\rho\, \overline{q u_p} = \rho\, \overline{\frac{dq}{dt}}. \tag{A4}$$

A similar expression may be found in works on plasma physics (cf. Delcroix 1965, equation 9.14) Equation (27) is merely the integral of equation (A4) over the volume V.

With q = 1 and u_i equation (A4) yields the equations of continuity and motion

$$\frac{\partial}{\partial t}\rho + \sum_{p=1}^{3}\frac{\partial}{\partial x_p}\rho \bar{u}_p = 0, \tag{A5}$$

$$\rho\left[\frac{\partial \bar{u}_i}{\partial t} + \sum_{p=1}^{3}\bar{u}_p\frac{\partial \bar{u}_i}{\partial x_p}\right] = \rho \bar{g}_i - \sum_{p=1}^{3}\frac{\partial}{\partial x_p}\rho\, \overline{\tilde{u}_i\tilde{u}_p}. \tag{A6}$$

Equations (30) and (31) are the moments of equation (A5) while equations (32) and (33) are the moments of equation (A6) (cf. Binney 1982).

Setting $q = u_i u_j$ in equation (A4) results in

$$\frac{\partial}{\partial t}\rho\,\overline{u_i u_j} + \sum_{p=1}^{3}\frac{\partial}{\partial x_p}\rho\,\overline{u_i u_j u_p} = \rho\left(\overline{g_i u_j} + \overline{u_i g_j}\right), \tag{A7}$$

which can be rearranged using equations (A5) and (A6) in the form

$$\rho\left[\frac{\partial}{\partial t}\overline{\tilde{u}_i \tilde{u}_j} + \sum_{p=1}^{3}\bar{u}_p\frac{\partial}{\partial x_p}\overline{\tilde{u}_i\tilde{u}_j}\right] + \sum_{p=1}^{3}\frac{\partial}{\partial x_p}\rho\,\overline{\tilde{u}_i\tilde{u}_j\tilde{u}_p}$$

$$+\sum_{p=1}^{3}\frac{\partial\bar{u}_i}{\partial x_p}\rho\,\overline{\tilde{u}_j\tilde{u}_p} + \sum_{p=1}^{3}\frac{\partial\bar{u}_j}{\partial x_p}\rho\,\overline{\tilde{u}_i\tilde{u}_p} = \rho\left[\overline{\tilde{g}_i\tilde{u}_j} + \overline{\tilde{u}_i\tilde{g}_j}\right]. \tag{A8}$$

Equation (A8) is the equation of heat transfer in plasma physics (cf. Clemmow and Dougherty, 1969, equation 11-19). Equation (42) is the integral of equation (A7) over the entire volume $\int dV$

Appendix 2: Virial equations of higher orders

The virial equations of higher orders can be obtained in a fashion analogous to that adopted in section 5. Setting $q = x_i x_j x_k$ and $q = x_i x_j x_k x_l$ in equation (27) we obtain

$$\frac{d}{dt} I_{ijk} = \mathcal{L}_{i;jk} + \mathcal{L}_{j;ki} + \mathcal{L}_{k;ij} ,$$

$$\frac{d}{dt} I_{ijkl} = \mathcal{L}_{i;jkl} + \mathcal{L}_{j;kli} + \mathcal{L}_{k;lij} + \mathcal{L}_{l;ijk} , \tag{A9}$$

where

$$I_{ijk} = \int_{\Gamma} f x_i x_j x_k \, dZ , \qquad I_{ijkl} = \int_{\Gamma} f x_i x_j x_k x_l \, dZ ,$$

$$\mathcal{L}_{i;jk} = \int_{\Gamma} f u_i x_j x_k \, dZ , \qquad \mathcal{L}_{i;jkl} = \int_{\Gamma} f u_i x_j x_k x_l \, dZ , \tag{A10}$$

are the moment of inertia and momentum tensors of the orders three and four.

Similarly, with $q = u_i x_j x_k$ and $q = u_i x_j x_k x_l$ in equation (27), and using the definitions

$$K_{ij;k} = \frac{1}{2} \int_{\Gamma} f u_i u_j x_k \, dZ , \qquad K_{ij;kl} = \frac{1}{2} \int_{\Gamma} f u_i u_j x_k x_l \, dZ ,$$

$$U_{i;jk} = \int_{\Gamma} f g_i x_j x_k \, dZ , \qquad U_{i;jkl} = \int_{V} f g_i x_j x_k x_l \, dZ , \tag{A11}$$

we obtain the relations

$$\frac{d}{dt} \mathcal{L}_{i;jk} = 2 K_{ij;k} + 2 K_{ik;j} + U_{i;jk} .$$

$$\frac{d}{dt} \mathcal{L}_{i;jkl} = 2 K_{ij;kl} + 2 K_{ik;lj} + 2 K_{il;jk} + U_{i;jkl} . \tag{A12}$$

which when combined with equations (A9) yield the virial equations of the orders three and four

$$\frac{d^2}{dt^2} I_{ijk} = 4K_{ij;k} + 4K_{jk;i} + 4K_{ki;j} + U_{i;jk} + U_{j;ki} + U_{k;ij} ,$$

$$\frac{d^2}{dt^2} I_{ijkl} = 4K_{ij;kl} + 4K_{jk;li} + 4K_{kl;ij} + 4K_{li;ij}$$

$$+ 4K_{ik;jl} + 4K_{jl;ki} + U_{i;jkl} + U_{j;kli} + U_{k;lij} + U_{l;ijk} . \qquad \text{(A13)}$$

For an isolated stellar system, observed from an inertial frame of reference, on using equation 36, these equations 36, these equations take the familiar forms

$$\frac{1}{2}\frac{d^2}{dt^2} I_{ijk} = 2K_{ij;k} + 2K_{jk;i} + 2K_{ki;j} + W_{ij;k} + W_{jk;i} + W_{ki;j} ,$$

$$\frac{1}{2}\frac{d^2}{dt^2} I_{ijkl} = 2K_{ij;kl} + 2K_{jk;li} + 2K_{kl;ij} + 2K_{li;jk}$$

$$+ 2K_{ik;jl} + 2K_{jl;ik} + W_{ij;kl} + W_{jk;li} \qquad \text{(A14)}$$

$$+ W_{kl;ij} + W_{li;jk} + W_{ik;jl} + W_{jl;ik} .$$

CHAPTER 3

ISOLATED SPHEROIDAL STELLAR SYSTEMS

I. ISOTROPIC VELOCITY DISTRIBUTIONS

7. Introduction:

Chandrasekhar and Elbert (1972 $\equiv$ Paper I) used the tensor virial equations to study the dynamical evolution of isolated homogeneous spheroidal clusters with isotropic velocity distributions. In this chapter we reexamine the dynamical evolution of such isolated stellar systems. In section 9 we discuss the case of a homogeneous spheroid and in section 10 a heterogeneous spheroid.

8. The basic equations:

We consider an isolated cluster which is initially spheroidal (either oblate or prolate) with semiaxes a_i and a given total energy E. The system is governed by the virial equations (equation 39)

$$\frac{1}{2}\frac{d^2 I_{11}}{dt^2} = 2K_{11} + W_{11} ,$$

$$\frac{1}{2}\frac{d^2 I_{33}}{dt^2} = 2K_{33} + W_{33} , \tag{64}$$

From equations (64) we obtain the equilibrium condition (cf. Binney 1978)

$$2K_{11} - 2K_{33} = |W_{11}| - |W_{33}| , \tag{65}$$

or

$$(2T_{11} - 2T_{33}) + (\Pi_{11} - \Pi_{33}) = |W_{11}| - |W_{33}|. \quad (66)$$

For a spheroidal system $|W_{11}| - |W_{33}| \neq 0$ and hence for configurations in equilibrium

$$K_{11} \neq K_{33} \quad (67)$$

It may also be noted from equation (66) that either $(T_{11} - T_{33})$ or $(\Pi_{11} - \Pi_{33})$ or both may be nonzero. It was using this result that Binney (1978) concluded that elliptical galaxies may be supported either by rotation or by velocity anisotropy.

We now consider a spheroidal cluster in which the velocity distribution always remains isotropic so that (cf Paper I and Som Sunder & Kochhar 1984)

$$K_{11} = K_{22} = K_{33} = \frac{1}{3} K, \quad (68)$$

and the expression (48) for the total energy takes the form

$$E = K + 2W_{11} + W_{33} = \text{constant} \quad (69)$$

The virial equations (64) become

$$\frac{1}{2}\frac{d^2 I_{11}}{dt^2} = \frac{2}{3}E - \frac{1}{3}W_{11} - \frac{2}{3}W_{33} ,$$

$$\frac{1}{2}\frac{d^2 I_{33}}{dt^2} = \frac{2}{3}E - \frac{4}{3}W_{11} + \frac{1}{3}W_{33} . \tag{70}$$

One can see from equation (65) that an isotropic velocity distribution is not consistent with the cluster being aspherical and in equilibrium. We now discuss the dynamical evolution of such a cluster.

9. Homogeneous spheroids:

We first consider homogeneous spheroids. Choosing our units of mass, length and time such that

$$M = 1 , \quad a_1 = 1 \quad (\text{at } t = 0) , \quad G = 1 , \tag{71}$$

we obtain from equations (54) and (55) the expressions

$$I_{11} = \frac{1}{5} a_1^2 , \qquad I_{33} = \frac{1}{5} a_3^2 ,$$

$$W_{11} = -\frac{3}{10}\frac{1}{a_3} A_1 , \qquad W_{33} = -\frac{3}{10}\frac{a_3}{a_1^2} A_3 . \tag{72}$$

Introducing these expressions into equations (69) one obtains

$$\frac{d^2 a_1^2}{dt^2} = 4Q + \frac{1}{a_1}\left(\frac{a_1}{a_3} A_1 + 2\frac{a_3}{a_1} A_3\right), \tag{73}$$

$$\frac{d^2 a_3^2}{dt^2} = 4Q + \frac{1}{a_1}\left(4\frac{a_1}{a_3} A_1 - \frac{a_3}{a_1} A_3\right), \tag{74}$$

where

$$Q = E/|W_0| \qquad \text{and} \qquad W_0 = -\frac{3}{5}. \tag{75}$$

W_0 is the potential energy of a sphere of unit radius. Introducing the expressions (61) for A_1 and A_3, equations (73) and (74) reduce to the pair of equations

$$\frac{d^2 a_1^2}{dt^2} = 4Q + \frac{1}{a_1} f_1(y), \tag{76}$$

$$a_1^2 \frac{d^2 y}{dt^2} + 2\frac{da_1^2}{dt}\frac{dy}{dt} = -4Qy + \frac{1}{a_1} f_2(y), \tag{77}$$

where (cf Paper I)

$$f_1(y) = \frac{3}{y}\sqrt{1-y} + \frac{4y-3}{y} S(y),$$

$$f_2(y) = \left(\frac{4}{y} - 3\right)\sqrt{1-y} - \frac{4y^2-9y+9}{y} S(y). \tag{78}$$

The quantities a_1, Q and t defined here are related to z, Q and t of Paper I (equations 46, 47 & 50) by

$$z = a_1^2, \quad Q_{\text{Paper I}} = \frac{1}{4Q}, \quad t_{\text{Paper I}} = \sqrt{4|Q|}\, t. \tag{79}$$

A comparison of equations (75) and (76) with the corresponding equations (48) and (49) of Paper I shows that the r.h.s. of the latter should be multiplied by $E/|E|$. Thus equations (48) and (49) of Paper I can be used only when E is positive. They do not apply to systems with negative energies.

The equations (75) and (76) have been solved with the boundary conditions

$$a_1 = 1, \quad y = y_0 \qquad \frac{da_1}{dt} = \frac{da_3}{dt} = \frac{dy}{dt} = 0. \tag{80}$$

The results are discussed below.

Results:

In the case of spherical systems, $y = 0$, equation (76) is identically satisfied and equation (73) reduces to

$$\frac{d^2 a_1^2}{dt^2} = 4Q + \frac{2}{a_1}, \tag{81}$$

which is identical with equation (26) of Paper I for spherical systems except for a factor 2 multiplying the r.h.s., which results from a slightly different unit of time employed in this case. As was shown in Paper I the solutions to equation (81) are for

$$Q=0: \quad t = \pm \frac{1}{\sqrt{2}} \left\{ \frac{2}{3} (a_1 + 2) \sqrt{a_1 - 1} \right\}, \tag{82}$$

$$Q>0: \quad t = \pm \frac{1}{\sqrt{2}} \left\{ \frac{1}{Q} \sqrt{Q a_1^2 + a_1 - (Q+1)} - \frac{1}{2 Q^{3/2}} \cosh^{-1}\left(\frac{2 Q a_1 + 1}{2Q+1} \right) \right\}, \tag{83}$$

$$Q<0: \quad t = \pm \frac{1}{\sqrt{2}} \left\{ \frac{1}{(-Q)} \sqrt{Q a_1^2 + a_1 - (Q+1)} + \frac{1}{2(-Q)^{3/2}} \cos^{-1}\left(\frac{2 Q a_1 + 1}{2Q+1} \right) \right\} \tag{84}$$

$$Q=-0.5: \quad a_1 = 1 \tag{85}$$

From these relations it is clear that systems with positive energy ($Q \geqslant 0$) expand and are eventually dispersed. Those with negative energies ($-1 < Q < 0, Q \neq -0.5$) oscillate with a period $\pi / [\sqrt{2}\, (-Q)^{3/2}]$ and amplitude $|2 + \frac{1}{Q}|$. The case Q=-0.5 is that of equilibrium configurations, while Q = -1 is the case when the initial kinetic energy is zero and in this case the system collapses to a point after a time $\pi/2\sqrt{2}$.

Turning now to spheroidal systems, the results of Paper I show that systems with positive energies expand and are eventually dispersed. The behaviour of a_1, is almost identical with that given by equations (82) and (83). The initial oblateness or prolateness is reduced as the system expands. Some oblate systems may even become prolate as a consequence.

We have integrated equations (76) and (77) for several negative values of Q. The results are illustrated in figures 1-3. Figure 1 shows the behaviour of a_1 with time. For a given Q the curves are almost identical on the scale of the graph and correspond to the solutions given by equations (84) and (85). The period of oscillation is $\pi / [\sqrt{2}\ (-Q)^{3/2}]$ and the amplitude is $|2 + 1/Q|$.

Figure 2 illustrates the behaviour of y for Q = -0.25 and -0.50. In both these cases y oscillates with approximately the period $\pi / [\sqrt{2}\ (-Q)^{3/2}]$. Initially oblate systems $y > 0$ become prolate $(y < 0)$ and then oblate again. Similarly initially prolate systems become oblate and then prolate in a time $\sim \pi / [\sqrt{2}\ (-Q)^{3/2}]$.

The physical significance of these results is as follows: The assumed isotropic velocity distribution corresponds in the equilibrium state to a spherical distribution of matter. Since the velocity distribution is constrained by assumption to remain isotropic the mass distribution tends to sphericity. Since the equations do not allow for any damping, the system oscillates between oblate (prolate) and prolate (oblate) shapes.

In this context it is significant that Kormendy (1984) suggests that bar like (prolate) structures in spiral galaxies may evolve to lens shaped (oblate) forms.

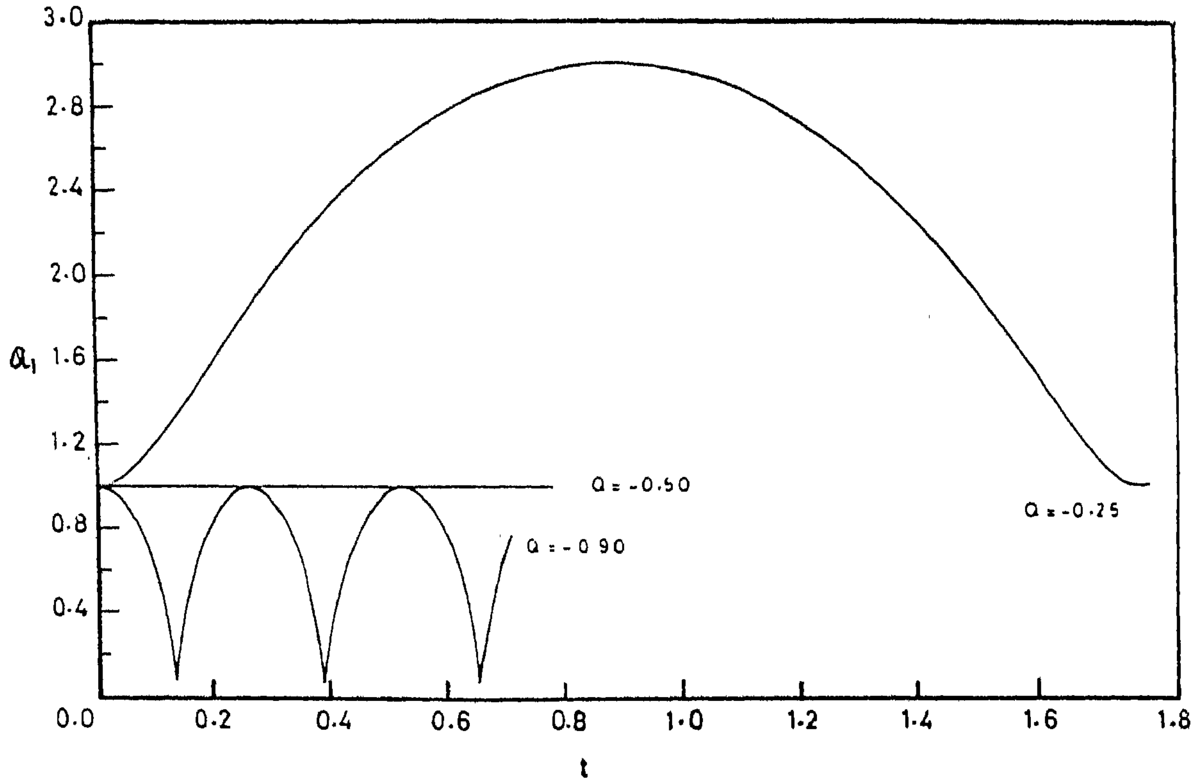

Figure 1. The evolution of homogeneous spheroidal clusters. The ordinate is the semiaxis a_1 of the circular equatorial section and the abscissa measures time in the units given by equation 71. The curves are labelled by the values of Q to which they belong. The curves are identical on the scale of the figure for the various values of γ_0 .

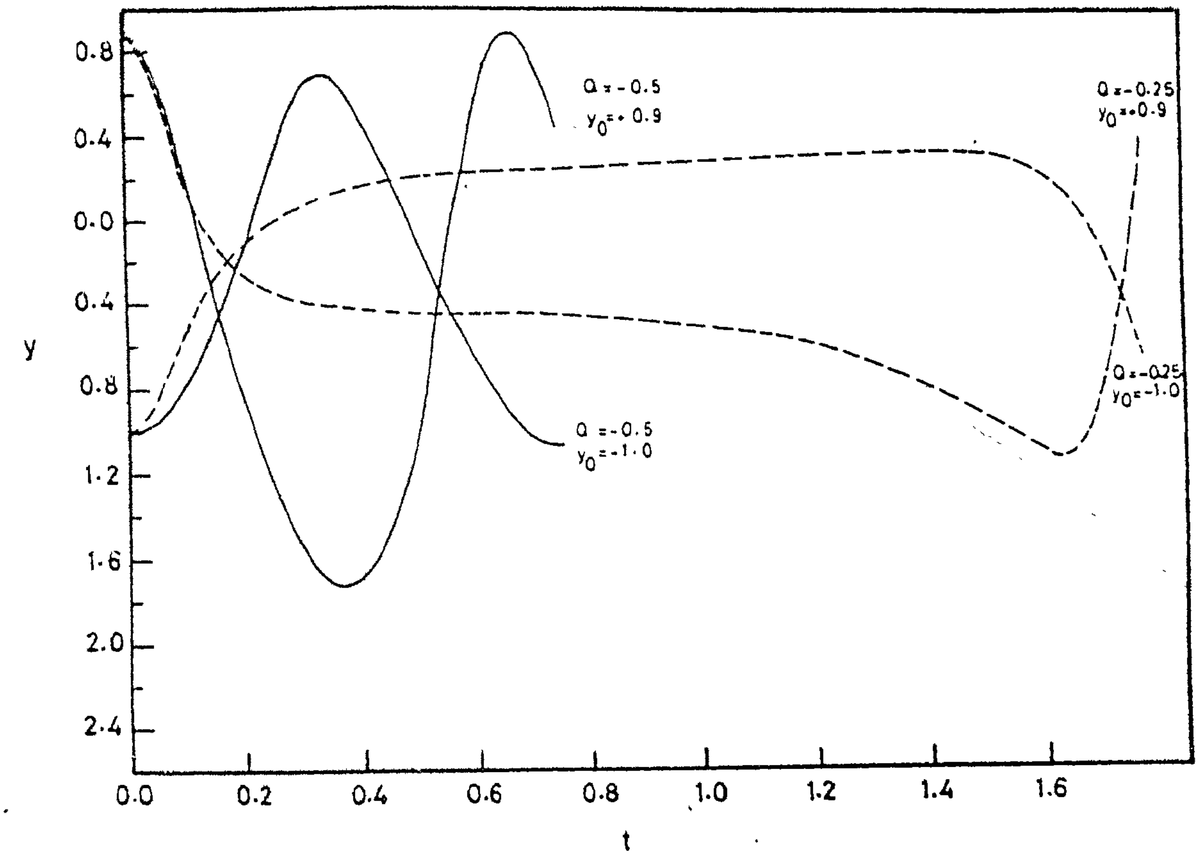

Figure 2. The evolution of homogeneous spheroidal clusters. The ordinate y is a measure of the eccentricity (e^2) positive for oblate spheroids and negative for prolate spheroids. The abscissa measures time in the same units as figure 1. The curves are labelled by the values of Q and y_0 to which they belong.

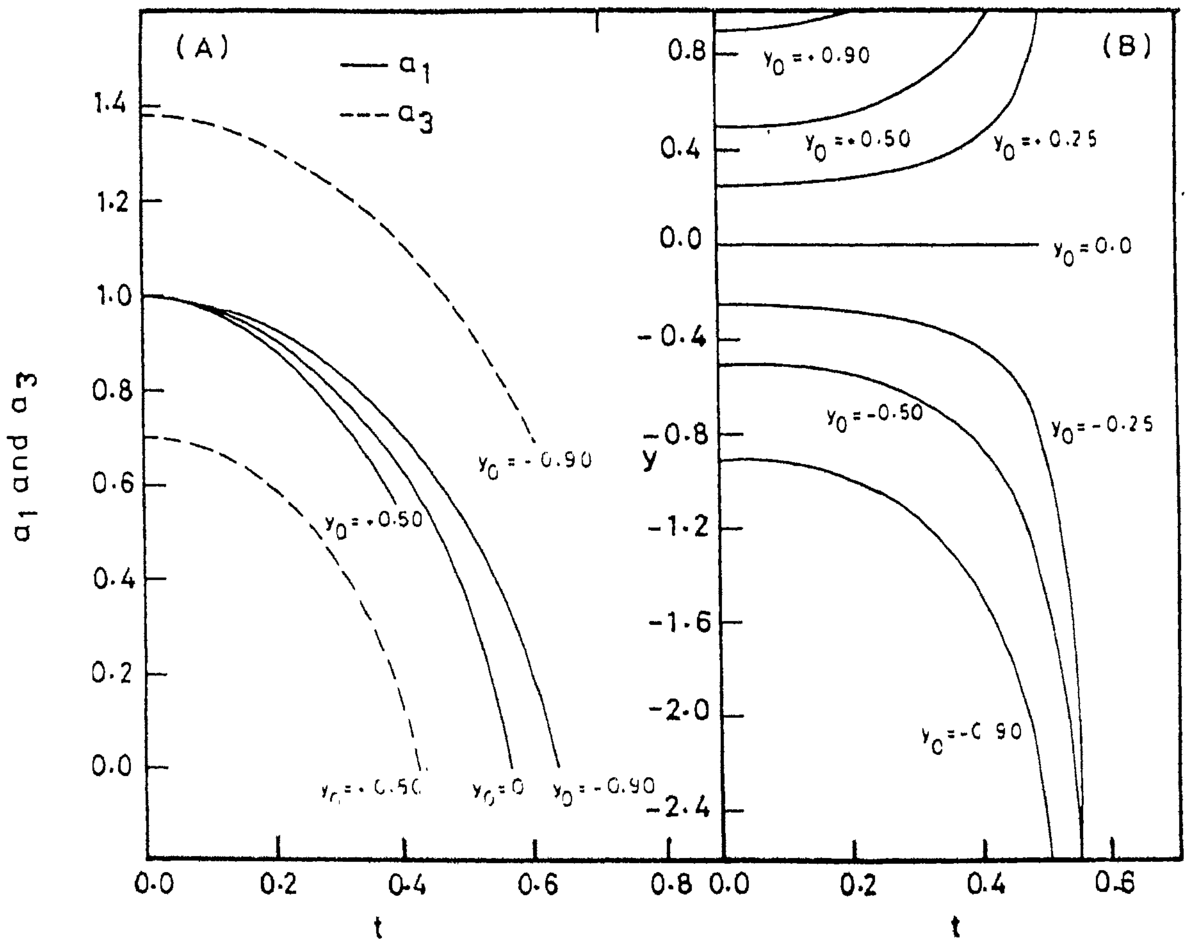

Figure 3. The evolution of homogeneous spheroids with zero initial kinetic energy (A) depicts the behaviour of the semiaxis a_1 and a_3 while (B) the behaviour of the eccentricity as the system evolves. The curves are labelled by the values of y_0 to which they belong.

Figure 3A depicts the behaviour of the two axes a_1 and a_3 for systems with zero initial kinetic energy. It is seen that the smaller of the two axes a_1 and a_3 always collapses to a point. The behaviour of y for such systems (figure 3B) shows that the initial oblateness or prolateness is enhanced as the collapse progresses. This result is in agreement with those of Lin et al (1965).

10. Heterogeneous ellipsoids:

We now consider the dynamical evolution of a heterogeneous spheroid with the density distribution of the form (cf. equation 49)

$$\rho(x) = \rho_c \left[1 - \sum_{p=1}^{3} \frac{x_p^2}{a_p^2} \right]^{\nu} \qquad (86)$$

We further assume that the form of the density distribution continues to be maintained as the system evolves i.e. ν is independent of time.

We choose our units of mass, length and time such that

$$M = 1 , \quad a_1 = 1 \ (\text{at } t = 0) , \quad G \frac{\psi(\nu)}{\phi(\nu)} = 1 . \qquad (87)$$

The relations for the moment of inertia and potential energy tensors are (using equations (54) and (55))

$$I_{11} = \frac{1}{5} a_1^2 \phi(\nu), \quad I_{33} = \frac{1}{5} a_3^2 \phi(\nu), \tag{88}$$

$$W_{11} = -\frac{3}{10} \frac{1}{a_1} A_1 \phi(\nu), \quad W_{33} = -\frac{3}{10} \frac{a_3}{a_1^2} A_3 \phi(\nu). \tag{89}$$

substituting these expressions into equations (64) and making use of the fact that ν and hence $\phi(\nu)$ is (by assumption) independent of time, we have

$$\frac{d^2 a_1^2}{d t^2} = 4Q + \frac{1}{a_1} f_1(y), \tag{90}$$

$$a_1^2 \frac{d^2 y}{d t^2} + 2 \frac{d a_1^2}{d t} \frac{d y}{d t} = -4Qy + \frac{1}{a_1} f_2(y), \tag{91}$$

where the functions $f_1(y)$ and $f_2(y)$ are defined by the relations (78) and Q is given by

$$Q = E / |W_0| \tag{92}$$

with $W_0 = -\frac{3}{5} \psi(\nu)$

W_0 is the potential energy of a sphere of unit radius with the density parameter ν. Equations (90) and (91) are identical in form to equations (76) and (77) and hence the conclusions of section 9 also apply to heterogeneous spheroids with isotropic velocity distributions.

Thus the dynamical evolution of heterogeneous spheres is also given by equations (82) to (85). Similarly heterogeneous spheroids with positive energies also expand and the behaviour of a_1 is given approximately by equations (82) and (83). This expansion is accompanied by a decrease in the asphericity. Also, heterogeneous spheroids with negative energies oscillate both in size and eccentricity with a period $\approx \pi / [\sqrt{2}\ (-Q)^{3/2}]$. The behaviour of is given approximately by the expressions (84). Those initially possessing zero kinetic energy collapse. The collapse is accompanied by an increase in the asphericity so that an oblate spheroid tends towards a flattened disc while a prolate spheroid tends towards a spindle shape.

The time scales however are different for different central concentration as a consequence of the presence of the parameter ν in the units of time chosen (equation 87) so as to compare systems with different central concentrations we represent by $t_0^{(\nu)}$ the unit of time for a density parameter ν. Also denoting by $Q^{(\nu)}$ and $P^{(\nu)}$ the values of Q and the period P for a system with the density parameter ν, and since

$$t_0^{(\nu)} = \sqrt{\frac{\phi(\nu)}{\psi(\nu)}}\ t^{(0)}, \quad Q^{(\nu)} = \frac{Q^{(0)}}{\psi(\nu)} = \frac{5}{3}\ \frac{E}{\psi(\nu)} \tag{93}$$

we have for the period

$$P^{(\nu)} = P^{(0)} \psi(\nu) \sqrt{\phi(\nu)} . \tag{94}$$

From table (1) it is seen that $\psi(\nu)\sqrt{\phi(\nu)}$ varies very little over a wide range of ν and hence we conclude that systems with the same negative value of E oscillate both in size and eccentricity with approximately the same period whatsoever their central concentration.

The amplitude of the oscillation is given by

$$\left| 2 + \frac{1}{Q^{(\nu)}} \right| = \left| 2 + \frac{3}{5} \frac{\Psi(\nu)}{E} \right| . \tag{95}$$

As can be seen from this equation when $Q^{(\nu)} < -\frac{1}{2}$ the amplitude decreases with increasing ν , while it increases with increasing ν when $Q^{(\nu)} > -\frac{1}{2}$.

CHAPTER 4

ISOLATED SPHEROIDAL STELLAR SYSTEMS
II. ANISOTROPIC VELOCITY DISTRIBUTIONS

11. Introduction

In chapter 3 we studied the dynamical evolution of an isolated stellar system in which the velocity distribution always remains isotropic. Velocity isotropy as observed in gas clouds is the result of close interactions between the particles constituting the system. As was seen in Chapter 2 stellar systems are better approximated as collisionless, in which case, an anisotropic mass distribution will tend to make the velocity distribution also anisotropic. In this chapter we study the dynamical evolution of a cluster with mass and velocity distributions which are both anisotropic.

12. The basic equations:

For an isolated stellar system the virial equations of the order two in an inertial frame of reference (equation 39) are

$$\frac{1}{2}\frac{d^2 I_{ij}}{dt^2} = 2K_{ij} + W_{ij}. \qquad (96)$$

The equations for the rate of change of the kinetic energy tensor (equation 42) are

$$2\frac{dK_{ij}}{dt} = H_{i;j} + H_{j;i}\,. \tag{97}$$

We consider ellipsoidal systems with density distributions of the form (cf. equation 49)

$$\rho(\mathbf{x}) = \rho_c\left[1 - \sum_{p=1}^{3}\frac{x_p^2}{a_p^2}\right]^{\nu}. \tag{98}$$

Here ν is a positive parameter and the coordinate axis is aligned along the body axis of the ellipsoid. We further assume that the form of the system is non-rotating so that this coordinate system is inertial.

With a view to evaluating the expression on the r.h.s. of equation (97) we assume that the mean velocity is a linear function of the coordinates:

$$\bar{u}_i = \sum_{p=1}^{3} q_{ip}\,x_p\,, \tag{99}$$

(cf. Chandrasekhar 1969, p.36). Inserting this expression into the definitions for $\mathcal{L}_{i;p}$ (equation 17) and using the fact that I_{ip} is diagonal in the chosen frame of reference we obtain

$$\mathcal{L}_{i;j} = \int_V \rho\,\bar{u}_i\,x_j\,d\mathbf{x} = \sum_{p=1}^{3} q_{ip}\,I_{pj} = q_{ij}\,I_{jj}\,. \tag{100}$$

From the relation (equation 31)

$$\frac{dI_{ij}}{dt} = \mathcal{L}_{i;j} + \mathcal{L}_{j;i} \tag{101}$$

and from the definition for the angular momentum (equation 24) we obtain

$$\begin{aligned} \mathcal{L}_{i;j} + \mathcal{L}_{j;i} &= \frac{dI_{jj}}{dt}\,\delta_{ij}\,, \\ \mathcal{L}_{i;j} - \mathcal{L}_{j;i} &= \sum_{k=1}^{3} \epsilon_{kji}\,\lambda_k\,. \end{aligned} \tag{102}$$

Hence making use of equation (100) we have

$$q_{ij} = \frac{\mathcal{L}_{i;j}}{I_{jj}} = \frac{1}{2I_{jj}}\,\frac{dI_{jj}}{dt}\,\delta_{ij} + \sum_{k=1}^{3} \epsilon_{kji}\,\frac{\lambda_k}{2I_{jj}}\,. \tag{103}$$

Substituting this into equation (99) we see that

$$u_i = \frac{1}{2I_{ii}}\,\frac{dI_{ii}}{dt}\,x_i + \sum_{j,k=1}^{3} \epsilon_{kji}\,\frac{\lambda_k}{2I_{jj}}\,x_j\,. \tag{104}$$

It will be noted that equation (104) is identical with equation 44 of chapter 4 in Chandrasekhar (1969) for a homogeneous ellipsoid in which the form is nonrotating.

Substituting equation (99) into the expression defining $H_{i;j}$ (equation 25) and making use of the fact that the acceleration is given by (equation 36)

$$g_i = \frac{\partial V}{\partial x_i} \tag{105}$$

we obtain, using equation (26), since W_{ij} is diagonal

$$H_{i;j} = \int_V \rho(x) \frac{\partial V}{\partial x_i} \bar{u}_j \, dx = \sum_{p=1}^{3} q_{jp} W_{pi} = q_{ji} W_{ii} \tag{106}$$

Since I_{ij} and W_{ij} are always diagonal so is K_{ij} and therefore we have from equation (97) on using equation (106)

$$\begin{aligned} H_{i;j} + H_{j;i} &= q_{ji} W_{ii} + q_{ij} W_{jj} \\ &= \frac{1}{2} \sum_{k=1}^{3} \epsilon_{kji} \lambda_k \left(\frac{W_{ii}}{I_{ii}} - \frac{W_{jj}}{I_{jj}} \right) \\ &= 0 \qquad (i \neq j) . \end{aligned} \tag{107}$$

From equation (107) it follows that

$$\text{if} \quad a_i \neq a_j , \; \lambda_k = 0 \qquad (i \neq j \neq k). \tag{108}$$

Thus under the assumption that the mean velocity is a linear function of the coordinates, no streaming motions can exist in a nonrotating collisionless triaxial ellipsoidal stellar system. For a spheroidal system axes $a_1 = a_2$ and a_3 however, $\lambda_1 = \lambda_2 = 0$, but λ_3 may be nonzero. Under these circumstances equation (103) can be rewritten as

$$q_{ij} = \frac{1}{2 I_{jj}} \frac{d I_{jj}}{dt} \delta_{ij} + \epsilon_{3ji} \frac{\lambda_3}{2 I_{jj}} \tag{109}$$

Using equations (103) and (106) the diagonal terms of equation (97) yield

$$\frac{dK_{ii}}{dt} = H_{i;i} = q_{ii} W_{ii} = \frac{\partial W_{ii}}{\partial I_{ii}} \frac{dI_{ii}}{dt} . \tag{110}$$

This expression can be integrated using equation (96), and results in

$$K_{ii} = \frac{1}{8 I_{ii}} \left(\frac{dI_{ii}}{dt} \right)^2 + \frac{\chi_i}{2 I_{ii}} , \tag{111}$$

where the χ_i are constants of integration.

From the definition (23) of T_{ii} and equation (99) and (109) we get for a spheroidal system

$$T_{ii} = \frac{1}{8 I_{ii}} \left(\frac{dI_{ii}}{dt} \right)^2 + \frac{\lambda_3^2}{2 I_{11}} (1 - \delta_{i3}) \tag{112}$$

and therefore

$$K_{ii} = T_{ii} + \frac{1}{2} \Pi_{ii}$$

$$= \frac{1}{8 I_{ii}} \left(\frac{dI_{ii}}{dt} \right)^2 + \frac{\lambda_3^2}{2 I_{11}} (1 - \delta_{i3}) + \frac{1}{2} \Pi_{ii} . \tag{113}$$

The first term represents the kinetic energy of expansion or contraction; the second, the kinetic energy of streaming motions; while the third is the energy of random motions.

A comparison of equations (111) and (113) yields

$$\frac{1}{4}\lambda_3^2(1-\delta_{i3}) + \Pi_{ii} I_{ii} = \chi_i \text{, a constant.} \tag{114}$$

Since by the law of angular momentum conservation (equation 40) λ_3 is a constant, we have

$$\Pi_{ii} I_{ii} = \chi_i - \frac{1}{4}\lambda_3^2(1-\delta_{i3}) \text{, a constant.} \tag{115}$$

When no net rotations exist in the system i.e. λ_3 is also zero, equation (115) reduces to

$$\Pi_{ii} I_{ii} = \chi_i \text{, a constant.} \tag{116}$$

Just as the virial equations can be obtained by taking the moments of the equations of continuity and momentum transfer, equation (116) may also be obtained by integrating (after introducing the approximation (99)) the equation

$$\begin{aligned} &\rho\left[\frac{\partial}{\partial t}\overline{\tilde{u}_i\tilde{u}_j} + \sum_{p=1}^{3}\bar{u}_p\frac{\partial}{\partial x_p}\overline{\tilde{u}_i\tilde{u}_j}\right] + \sum_{p=1}^{3}\frac{\partial}{\partial x_p}\rho\,\overline{\tilde{u}_i\tilde{u}_j\tilde{u}_p} \\ &+ \sum_{p=1}^{3}\frac{\partial\bar{u}_i}{\partial x_p}\rho\,\overline{\tilde{u}_j\tilde{u}_p} + \sum_{p=1}^{3}\frac{\partial\bar{u}_j}{\partial x_p}\rho\,\overline{\tilde{u}_i\tilde{u}_p} = \rho\left(\overline{\tilde{g}_i\tilde{u}_p} + \overline{\tilde{u}_i\tilde{g}_p}\right) \end{aligned} \tag{117}$$

over the volume V. Equation (117) is the 'heat transfer equation', and is obtained by taking the second moment in velocity space of the Boltzmann's equation (see chapter 2 appendix 1). In the case of spherical stellar systems the scalar equivalent of equation (116)

$$\Pi I = \text{constant}, \tag{118}$$

follows from the law of energy conservation.

Introducing equation (111) into equation (96) we obtain

$$\frac{d^2 I_{ii}}{dt^2} = \frac{1}{2 I_{ii}}\left(\frac{d I_{ii}}{dt}\right)^2 + 2 W_{ii} + 2 \frac{X_i}{I_{ii}} . \tag{119}$$

We use this equation to study the dynamical evolution of spheroidal stellar systems. The constancy of the expression for the total energy (equation 48)

$$E = \sum_{i=1}^{3}\left[\frac{1}{8 I_{ii}}\left(\frac{d I_{ii}}{dt}\right)^2 + \frac{X_i}{2 I_{ii}}\right] + W, \tag{120}$$

will serve as a check on the numerical integration technique.

For a homogeneous spheroid with axes a_1, a_2 and a_3 choosing our units of mass length and time such that

$$M = 1, \qquad a_1 = 1 \quad (\text{at } t = 0), \qquad G = 1, \tag{121}$$

we get the expressions (cf. equations (54) and (55) with $\nu = 0$)

$$I_{11} = I_{22} = \frac{1}{5} a_1^2 , \qquad I_{33} = \frac{1}{5} a_3^2$$

$$W_{11} = W_{22} = -\frac{3}{10} \frac{1}{a_3} A_1 , \quad W_{33} = -\frac{3}{10} \frac{a_3^2}{a_1^2} A_3 \tag{122}$$

Introducing these expressions into equation (119) we obtain

$$\frac{d^2 a_1}{dt^2} = \frac{c_1}{2 a_1^3} - \frac{3}{2} \frac{1}{a_1 a_3} A_1 ,$$

$$\frac{d^2 a_3}{dt^2} = \frac{c_3}{2 a_3^3} - \frac{3}{2} \frac{1}{a_1^2} A_3 , \tag{123}$$

and from equation (120)

$$E = \frac{1}{10} \left[2 \left(\frac{da_1}{dt} \right)^2 + \left(\frac{da_3}{dt} \right)^2 + 2 \frac{c_1}{2 a_1^2} + \frac{c_3}{2 a_3^2} \right] - \frac{3}{5} \frac{1}{a_1} S(y) , \tag{124}$$

where

$$c_i = 25 \chi_i . \tag{125}$$

13. Systems with zero initial kinetic energy:

In this section we investigate systems that have zero kinetic energy initially, that is

$$\text{at} \quad t = 0: \quad K_{11} = K_{22} = K_{33} = 0 . \tag{126}$$

From equation (113) we then obtain, at $t = 0$

$$\frac{dI_{11}}{dt} = \frac{dI_{22}}{dt} = 0 \qquad \text{or} \qquad \frac{da_1}{dt} = \frac{da_3}{dt} = 0 \tag{127}$$

and also

$$\lambda_3 = 0 , \qquad \Pi_{11} = \Pi_{22} = \Pi_{33} = 0 \tag{128}$$

Also from (114) we conclude that (128) remains valid for all t. Thus the problem considered is the evolution of a pressureless nonrotating system and is identical with the one considered by Lin et al. (1965 ≡ Paper II). From (116) we also obtain

$$\chi_i = 0 \quad \text{or} \quad c_i = 0 , \tag{129}$$

and hence equations (119) reduce to

$$\begin{aligned} \frac{d^2 a_1}{dt^2} &= -\frac{3}{2} \frac{1}{a_1 a_3} A_1 , \\ \frac{d^2 a_3}{dt^2} &= -\frac{3}{2} \frac{1}{a_1^2} A_3 . \end{aligned} \tag{130}$$

These relations it will be noted are identical with those of Paper II except for multiplying factors.

Results:

The results of numerical integration of equations (130) for various initial values of initial eccentricity

$$y = y_0 \text{ (assigned)} \quad \text{at} \quad t = 0, \tag{131}$$

are presented in figures 4 and 5.

In the spherical case these results are identical with the analytical solution (equation 84 with $Q = -1$)

$$t = \frac{1}{\sqrt{2}} \left[\sqrt{a_1(1-a_1)} + \frac{1}{2} \cos^{-1}(2a_1 - 1) \right] \tag{132}$$

In the spheroidal case, proceeding in a fashion similar to that in Paper II we can obtain an approximate series solution:

$$a_1 = \cos^2\theta$$

$$\begin{aligned} t &= \frac{1}{A_0} \left\{ \left[\theta + \frac{1}{2} \sin 2\theta \right] - \frac{A_2}{A_0} \left[\theta - \frac{1}{2} \sin 2\theta \right] \right\} \\ y &= 1 - \left\{ E^{(0)} + E^{(2)} (\tan\theta)^2 \left[1 + \frac{E^{(4)}}{E^{(2)}} (\tan\theta)^2 \right] \right\}^2 \end{aligned} \tag{133}$$

where the various coefficients are given by the following sequence of relations

$$A_1^{(1)} = \frac{\sqrt{1-y_0}}{y_0} S(y_0) - \frac{1-y_0}{y_0} ,$$

$$E^{(0)} = \sqrt{1-y_0} ,$$

$$\alpha^{(0)} = \left[\frac{3}{E^{(0)}} A_1^{(1)} \right]^{1/2} ,$$

$$E^{(2)} = \frac{3}{[\alpha^{(0)}]^2} \left[3 A_1^{(1)} - 2 \right] , \qquad (134)$$

$$A_1^{(2)} = - E^{(2)} \frac{\sqrt{1-y_0}}{y_0^2} \left[2 - \frac{2-3y_0}{\sqrt{1-y_0}} S(y_0) \right] ,$$

$$\frac{\alpha^{(2)}}{\alpha^{(0)}} = \frac{3}{4 E^{(0)} [\alpha^{(0)}]^2} \left[A_1^{(2)} - A_1^{(1)} \frac{E^{(2)}}{E^{(0)}} \right] ,$$

$$\frac{E^{(4)}}{E^{(2)}} = \left\{ [S(y_0)]^2 + S(y_0)\sqrt{1-y_0} - 2 \right\} \Big/ \left\{ 3 \left[S(y_0) - \sqrt{1-y_0} \right]^2 \right\} ,$$

The values of these coefficients for a few values of y_0 are listed in table 2.

Table 2

Values for the coefficients in the expansion (133)

y_0	$\alpha^{(0)}$	$\frac{\alpha^{(2)}}{\alpha^{(0)}}$	$E^{(0)}$	$\dot{E}^{(2)}$	$\frac{E^{(4)}}{E^{(2)}}$
0.9	1.826	0.200	0.316	-0.851	0.050
0.5	1.556	0.063	0.707	-0.356	0.039
0.0	1.414	0.0	1.0	0.0	0.033
-0.5	1.327	-0.035	1.225	0.267	0.030
-0.9	1.276	-0.062	1.378	0.447	0.025

It is found that the series solution (equation 133) is in very good agreement with the results of numerical integration.

It is seen from figure 4 that the smaller axis always collapses to a point. In the case of spherical systems the collapse occurs in time $\pi/2\sqrt{2}$. The collapse time is less greater than this value for oblate spheroids while it is greater less than this value for prolate ones. The collapse is accompanied by an enhancement in the asphericity of the system (Figure 5). Thus as a result an oblate spheroid approaches a thin disc whereas a prolate one approaches a spindle shape. These results are in qualitative agreement with those of chapter 3 but disagree in quantitative details.

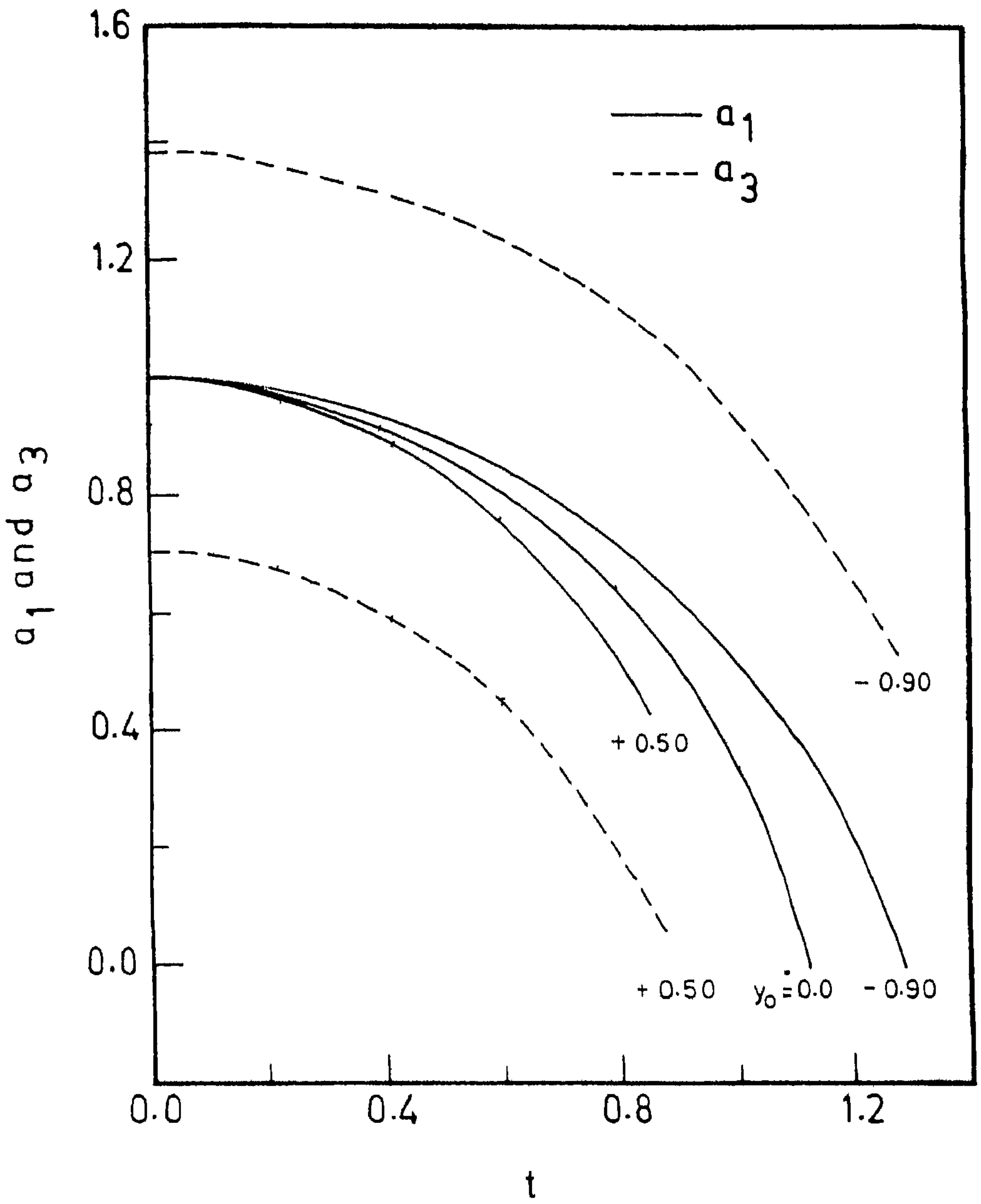

Figure 4. The evolution of homogeneous spheroidal cluster with anisotropic velocity distribution and zero initial kinetic energy. The figure depicts the behaviour of the two semiaxes a_1 and a_3 with respect to time measured in the units given by equation (121). The curves are labelled by the values of y_0 to which they belong.

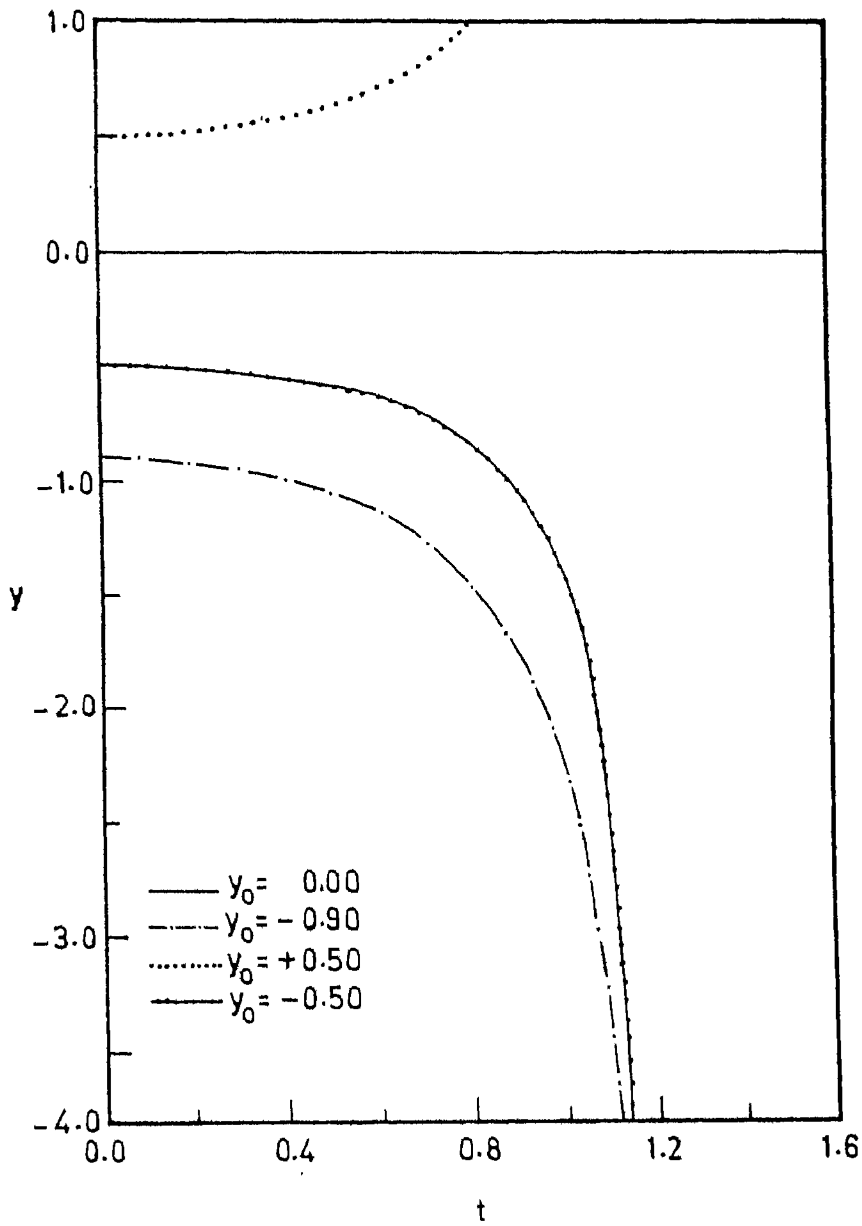

Figure 5. The evolution of homogeneous spheroidal clusters with anisotropic velocity distribution and zero initial kinetic energy. The ordinate y is a measure of the eccentricity (e^2) positive for oblate spheroids and negative for prolate ones. The abscissa measures time in the units given by equation (121). The curves are labelled by the values of y_0 to which they belong.

Both kinematic and dynamic effects are involved in the above process (cf. Binney 1980). The kinematic effect is that if during a collapse both short and long axes are shrinking at the same pace, the short axis will go to zero before the long one, thus causing the flattening to increase without limit. The dynamical effect, which Binney calls "self-tidal distortion", is due to the fact that the shortest axis is subject to the greatest acceleration towards the centre. As a result it collapses faster than the longer axis leading to even further enhancements in asphericity.

However if the initial pressure is even slightly nonzero, it follows from equation (116) that even for a collisionless system the pressure increases as the system contracts and this in turn will stop the collapse. We study such systems in the next section.

14 Systems with negative total energies

We now study systems with a given initial eccentricity y_0 and a negative total energy E. We further assume that the velocity distribution is <u>initially</u> (locally) isotropic so that

$$K_{11} = K_{22} = K_{33} \quad \text{at } t = 0 . \tag{135}$$

Note that the assumption is made only at the initial instant and not throughout the evolution as in chapter 3. We further assume that the initial kinetic energy of mass motions is zero i.e.

$$\text{at } t=0 \quad T_{11} = T_{22} = T_{33} = 0 \tag{136}$$

$$\text{and } \Lambda_3 = 0 , \tag{137}$$

so that the entire kinetic energy is in the form of pressure energy initially. By the law of conservation of angular momentum (equation 41) Λ_3 remains zero as the system evolves while (112) yields

$$\frac{da_1}{dt} = \frac{da_3}{dt} = 0 \quad \text{at} \quad t = 0. \tag{138}$$

For a system with total energy E and initial eccentricity y_0 we obtain as using the relations (111) (120) (135) and (138)

$$C_1 = 4\left[Q + S(y_0)\right],$$
$$C_3 = (1 - y_0)\, C_1 , \tag{139}$$

where

$$Q = E/|W_0| \qquad \text{with} \quad W_0 = -\frac{3}{5} . \tag{140}$$

W_0 is the potential energy of a sphere of unit radius.

Equations (123) have been integrated for Q = -0.25, -0.50 and -0.75 for various values of y_0 making use of the relations (139), and the results are presented below.

Results:

Consider first the case of Q = -0.75 (figures 6 & 7). For all the cases of y_0 considered $K_{11} < |W_{11}|$ initially and hence the a_1 axis contracts. This contraction is accompanied by an increase in the pressure, which causes the system to expand once again. These oscillations have a period $P \approx \pi/[\sqrt{2}\,(-Q)^{3/2}] \approx 3.5$. The amplitude of the oscillation increases with decreasing y_0. In the spherical case the behaviour is identical with that given by the analytical solution (equation 84). The eccentricity y (figure 7) also decreases initially. At about ½P however there is a steep rise and a fall after which the system slowly returns to $y \approx y_0$. The system does not however in general return to the exact initial conditions after a cycle; the only exception is when the system is spherical, $y_0 = 0$.

In the case when Q = -0.25 (figures 8 & 9), $K_{11} > |W_{11}|$ initially for all the cases considered and the a_1 axis expands and then contracts in a period $P \approx \pi/[\sqrt{2}\,(-Q)^{3/2}] \approx 18$ In this case the amplitude increases with increasing y_0. The eccentricity y once again shows a double-wave behaviour over the same period.

The case when Q =-0.50 (figures 10, 11) is that of equilibrium when $y_0 = 0$. When $y_0 \neq 0$ however the Q_1 axis oscillates with a period $P \approx \pi/[\sqrt{2}\,(-Q)^{3/2}] \approx 6.5$ while the eccentricity shows a double-wave behaviour, over approximately the same period.

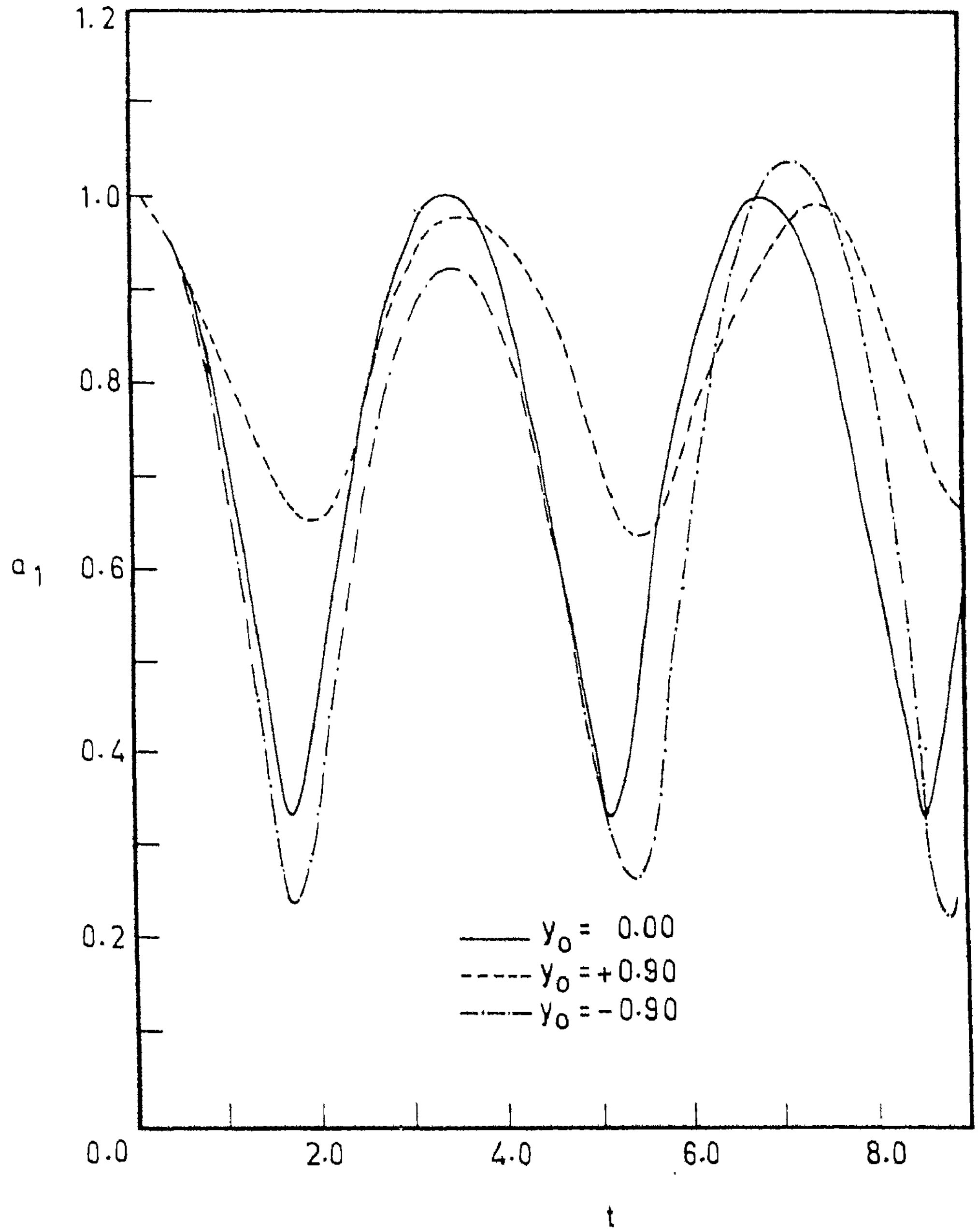

Figure 6. The evolution of homogeneous spheroidal clusters with anisotropic velocity distribution and negative total energies. Case Q = -0.75. The figure depicts the behaviour of the radius a_1 of the equitorial section with respect to time measured in the units given by equation (121). The curves are labelled by the values of y_0 to which they belong.

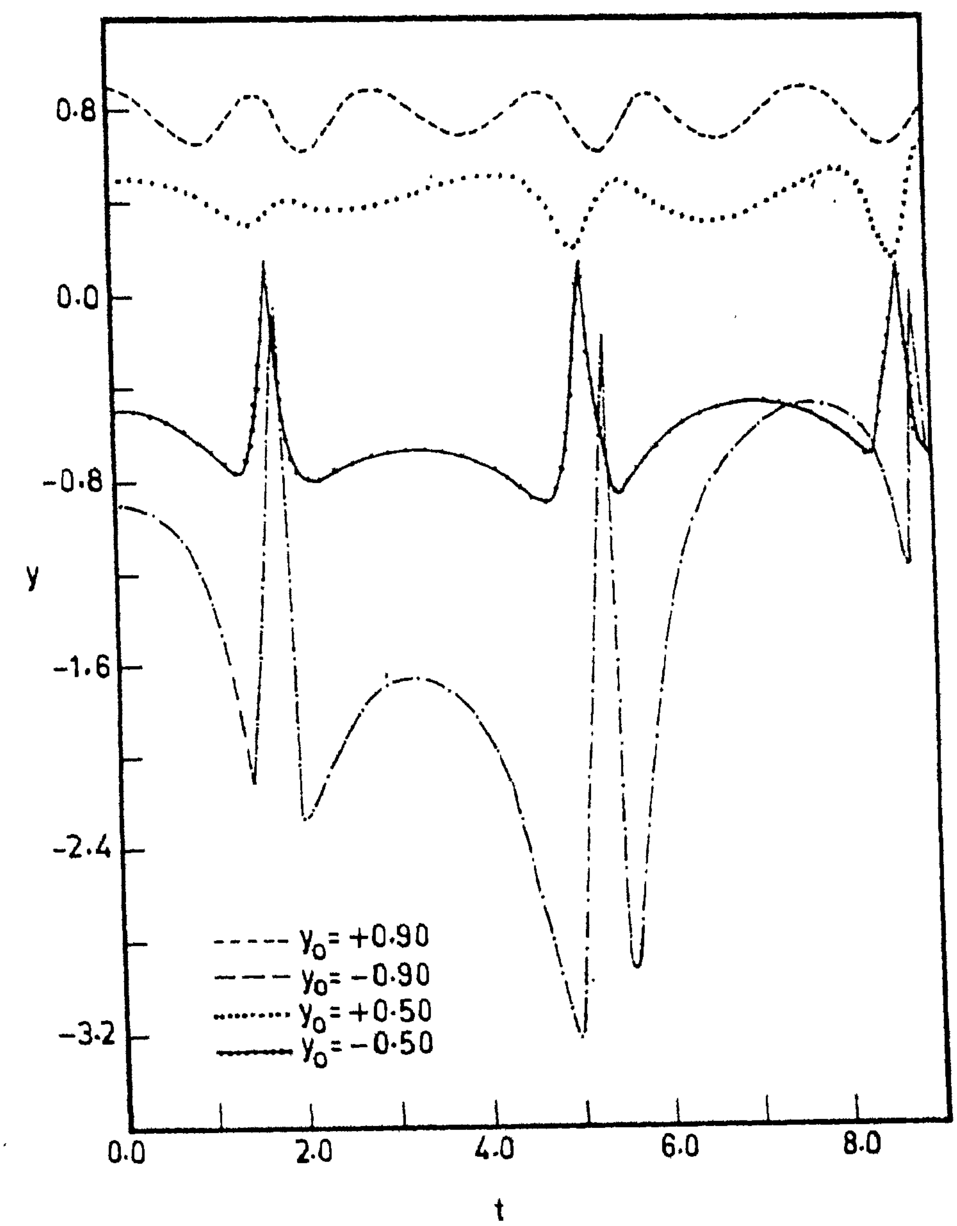

Figure 7. The evolution of homogeneous spheroidal clusters with anisotropic velocity distribution and negative total energy. Case Q =-0.75. The ordinate y is a measure of the eccentricity, positive for oblate spheroids and negative for prolate ones. The abscissa measures time in the units given by equation (121). The curves are labelled by the values of y_0 to which they belong.

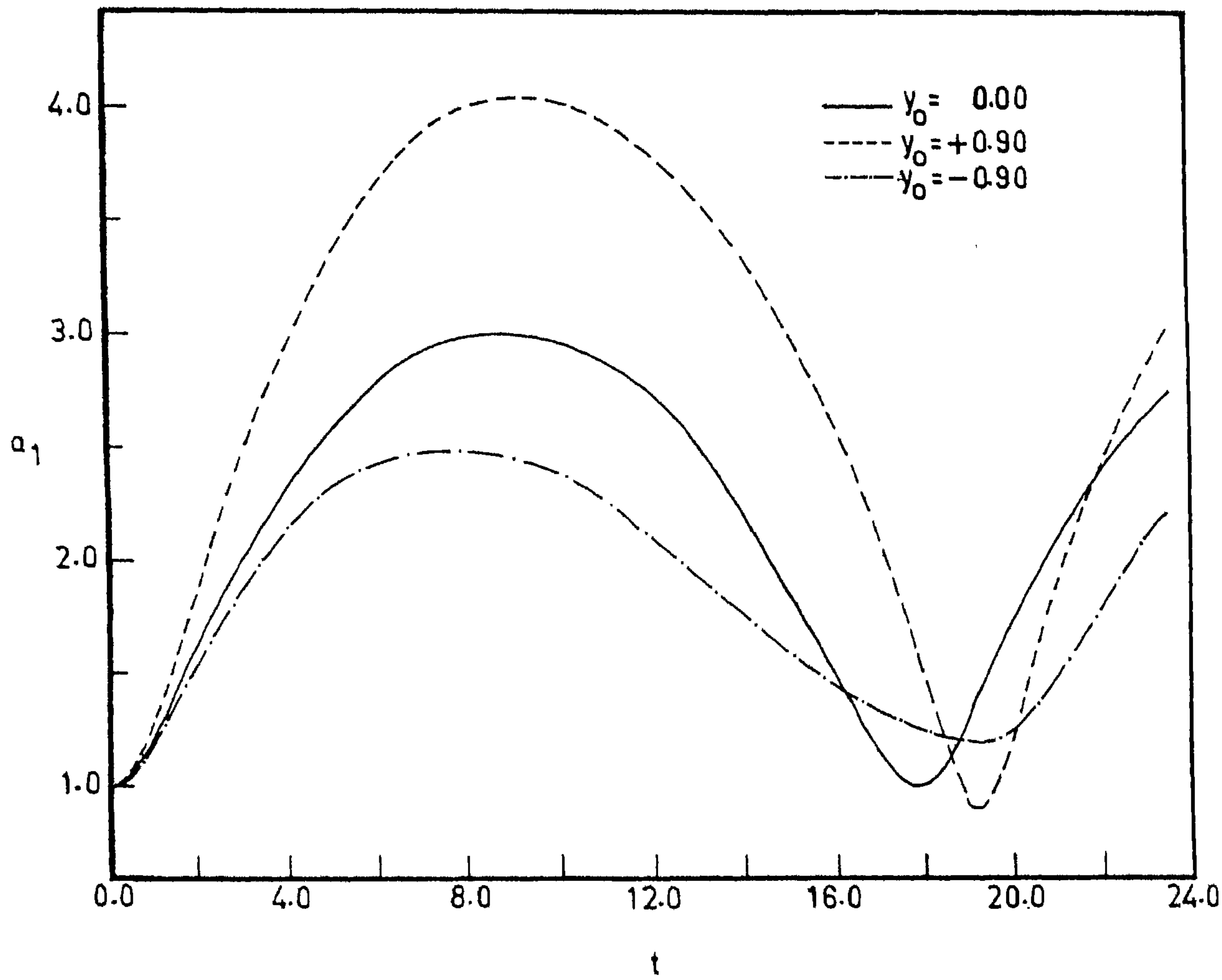

Figure 8. The evolution of homogeneous spheroidal clusters with anisotropic velocity distribution and negative total energy. Case Q = −0.25. The curves depict the behaviour of the semiaxis a_1 with time in the same units as figure 6.

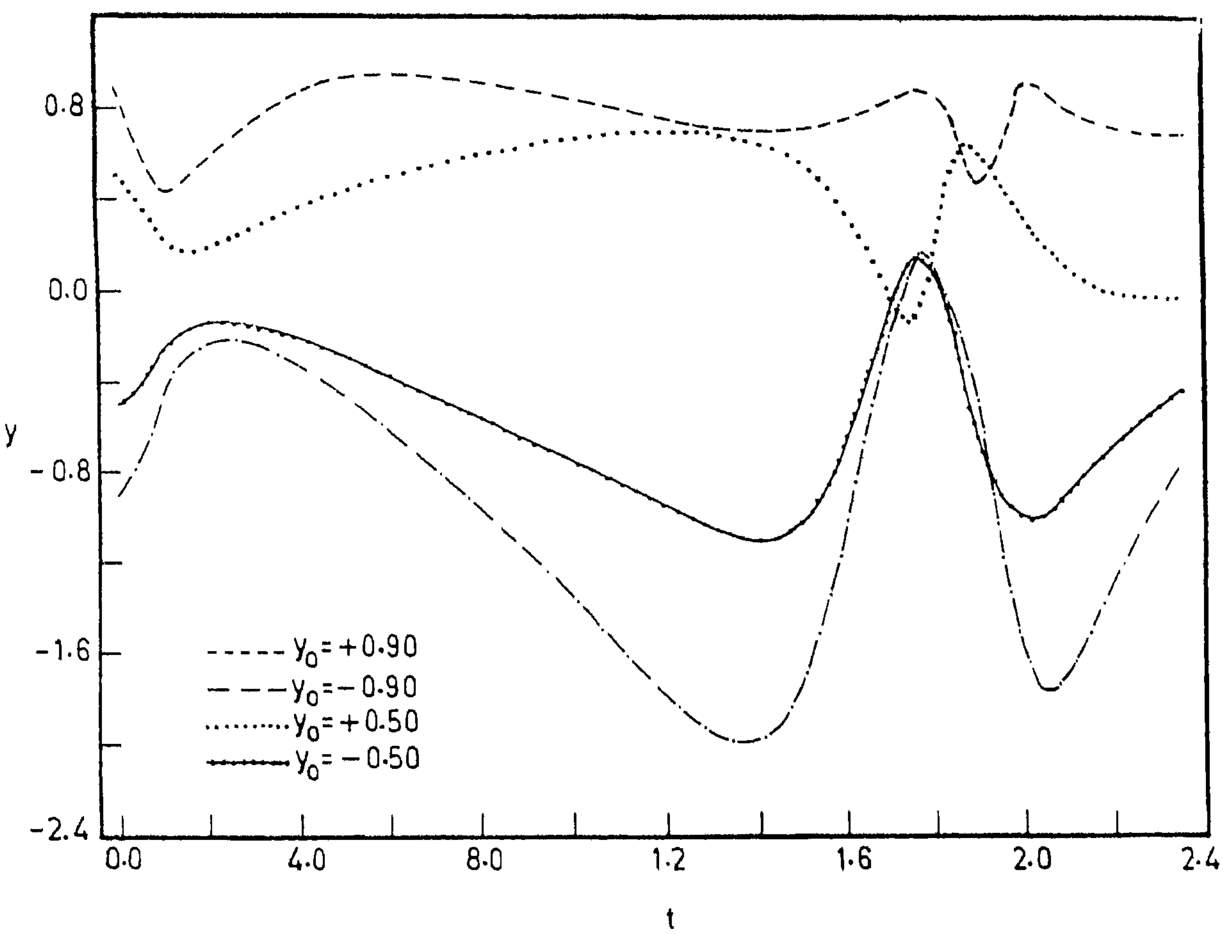

Figure 9. The evolution of homogeneous spheroidal clusters with anisotropic velocity distribution and negative total energies. Case Q = -0.25. The curves depict the behaviour of the eccentricity y with time in the same units as figure 7.

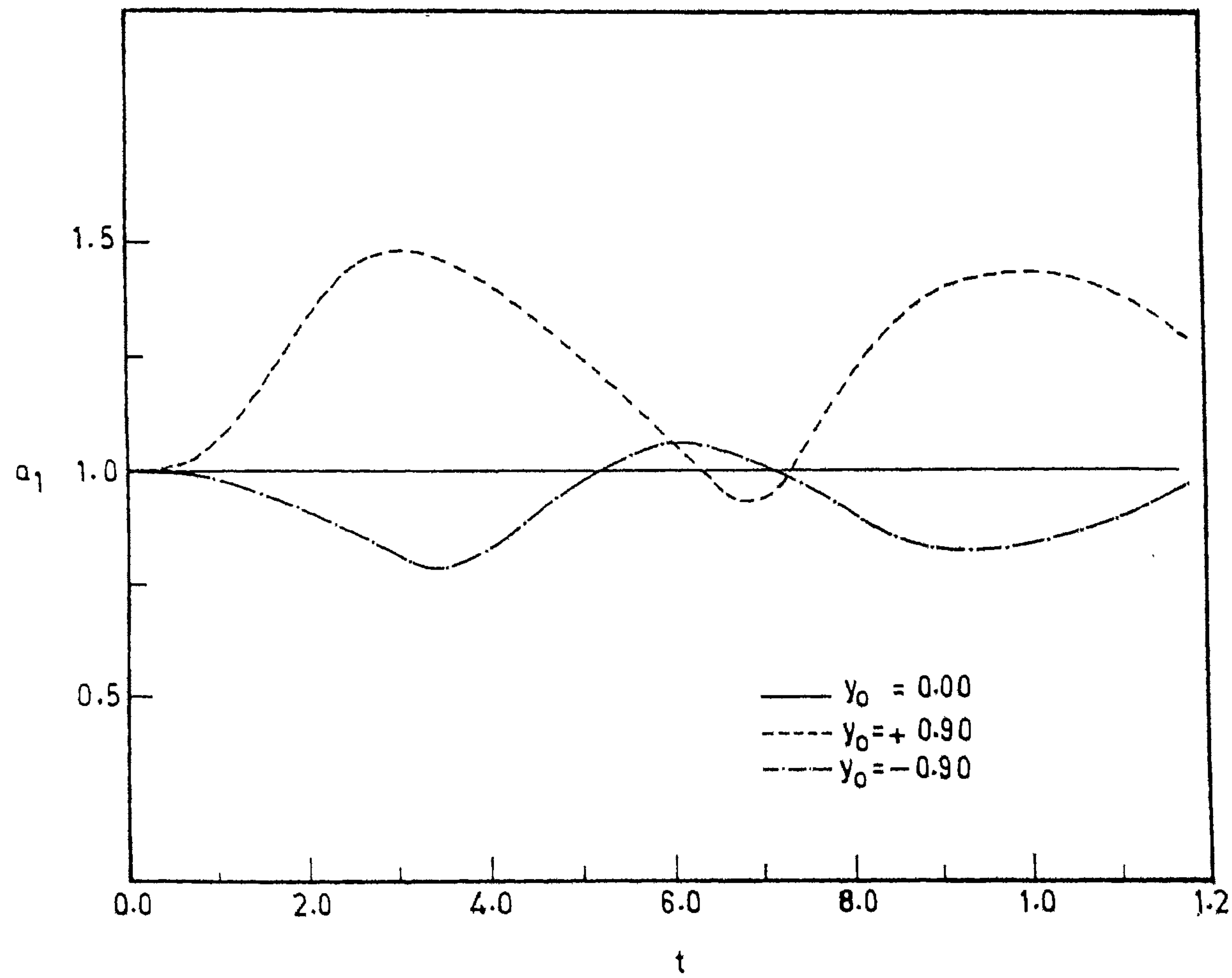

Figure 10. The evolution of homogeneous spheroidal clusters with anisotropic velocity distribution and negative total energies. Case Q = −0.50. The curves depict the behaviour of the semiaxis a_1 with time in the same units as figure 6.

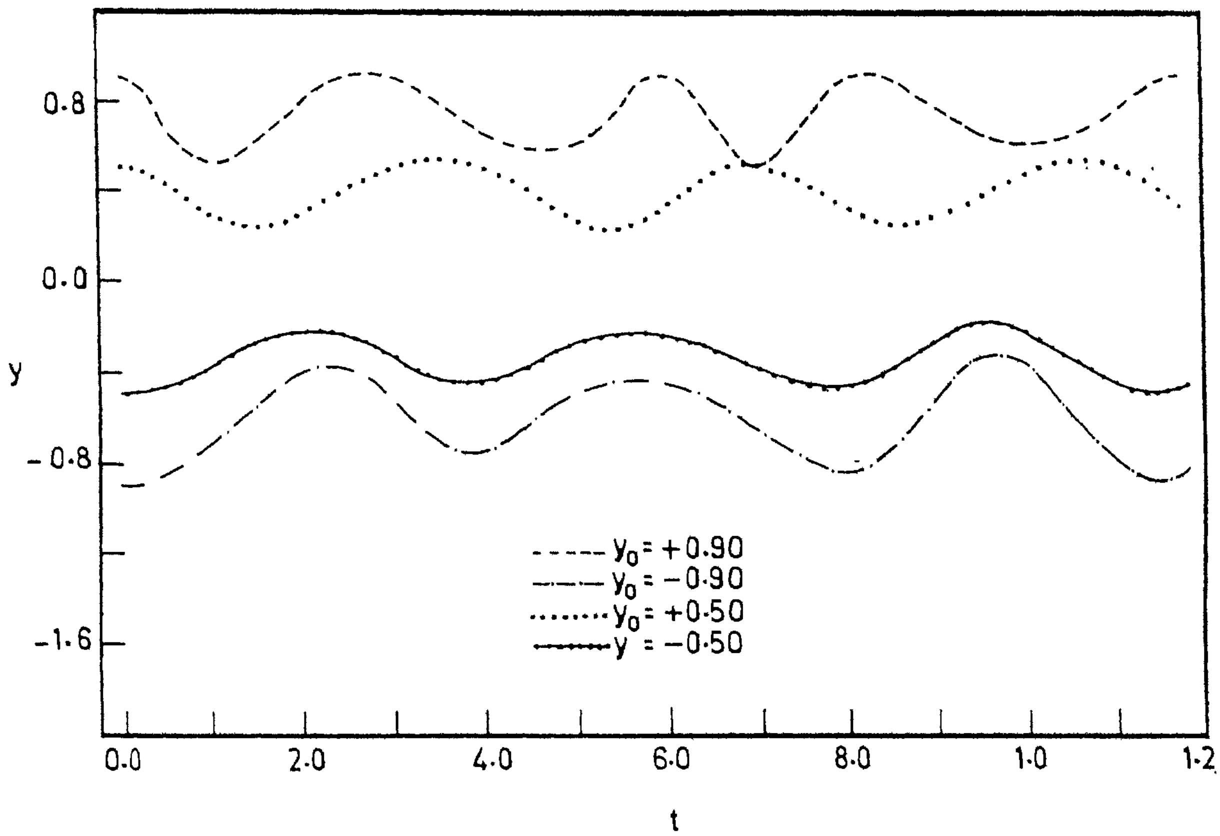

Figure 11. The evolution of homogeneous spheroidal clusters with anisotropic velocity distribution and negative total energies. Case $Q = -0.50$. The curves depict the behaviour of the eccentricity y with time in the same units as figure 7.

In general when the total energy is negative, i.e. $Q<0$ the a_1 axis executes finite amplitude oscillation of period $P \approx \pi / [\sqrt{2}\ (-Q)^{3/2}]$. The eccentricity also shows a double-wave behaviour over this period. These results should be compared with those of chapter 3, where the velocity distribution has been assumed to be isotropic. In the anisotropic case the amplitude of the oscillation in a_1 depends upon y_0 , whereas in the isotropic case it is independent of y_0 . Secondly, y shows a double-wave behaviour. Also, though y oscillates, the system does not go from oblate to prolate or from prolate to oblate as in the isotropic velocity case.

15. Systems with positive total energies:

We now consider systems with a given initial eccentricity y_0 and a positive total energy E . Equations (123) have been integrated for $Q = E/|W_0| = 0$, 0.5 and 1.0 and the results are illustrated in figures (12) and (13).

It is seen from figure 12 that systems with positive energies expand and are eventually dispersed, systems with larger energies expanding faster. When $y_0 = 0$, the behaviour is identical with the analytical solution (equations 82 and 83). For the same Q, the more oblate the system (i.e. the greater the y_0) the faster it expands. The expansion is initially accompanied by a rapid drop in the eccentricity (figure 13),

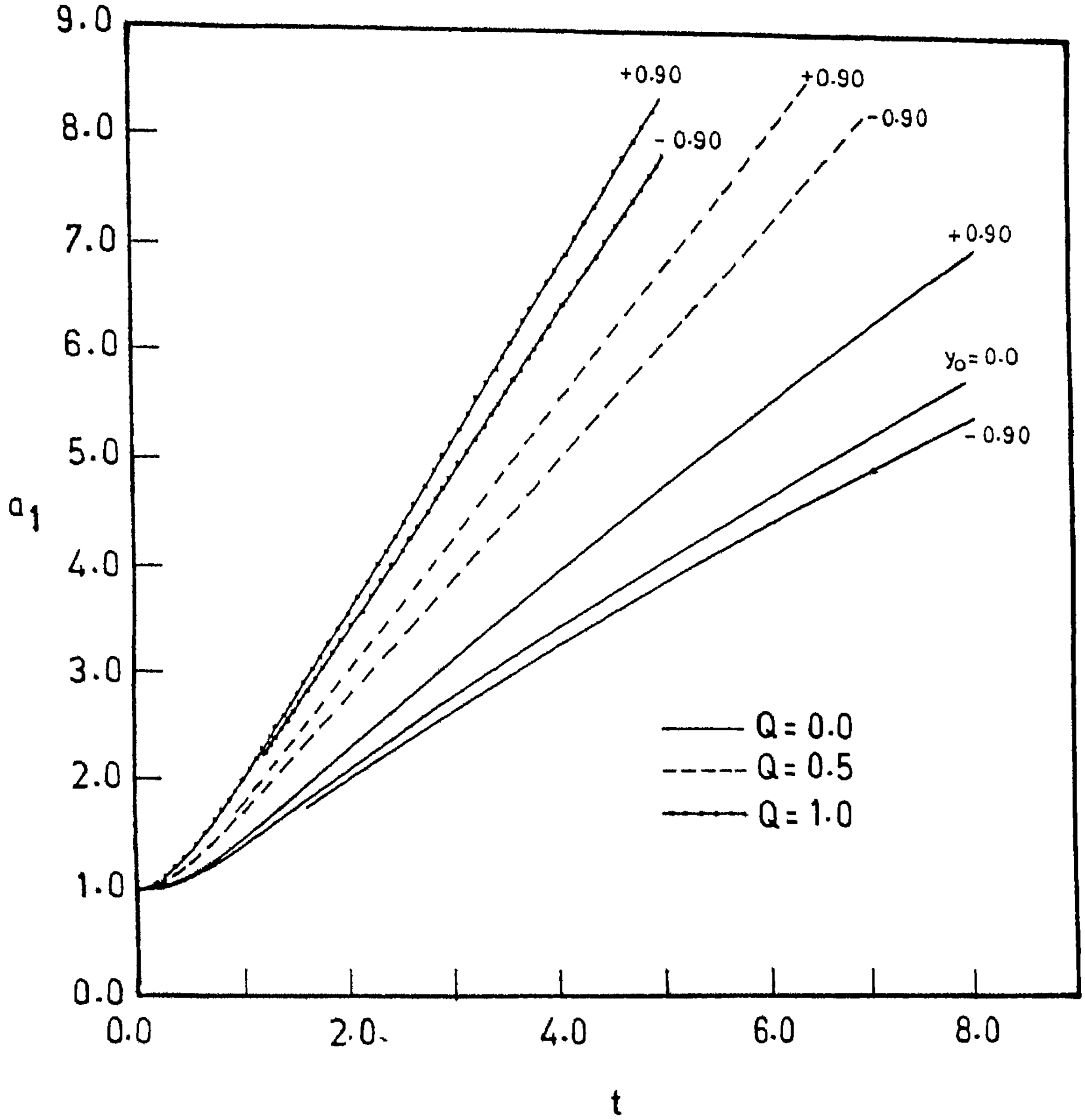

Figure 12. The evolution of homogeneous spheroidal clusters with anisotropic velocity distribution and positive total energies. The curves depict the behaviour of the semiaxis a_1 with time in the same units as Figure 6. The curves are labelled by the values of Q and y_0 to which they belong.

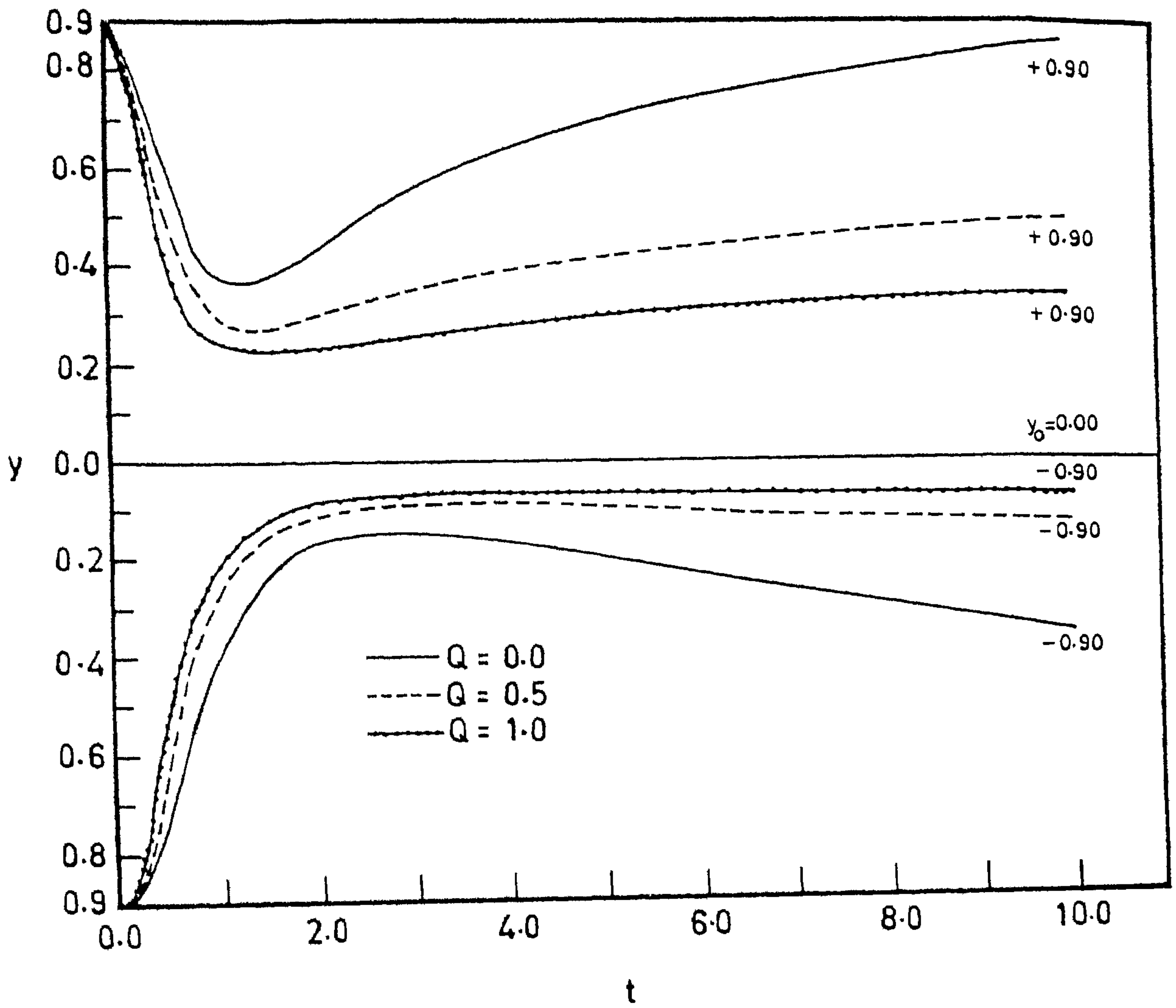

Figure 13. The evolution of homogeneous spheroidal clusters with anisotropic velocity distribution and positive total energies. The curves depict the behaviour of the eccentricity y with time in the same units as Figure 6. The curves are labelled by the values of Q and y_0 to which they belong.

i.e. both oblate and prolate system become more spherical. The eccentricity then tends to a finite limit. The results are in general agreement with those of Paper I and chapter 3. However in the case of anisotropic velocity distributions, an oblate (prolate) system remains oblate (prolate), unlike in the isotropic case where the eccentricity may change sign (Paper I).

16. Heterogeneous spheroids:

We now consider heterogeneous spheroids with density distribution of the form (equation 49)

$$\rho(\mathbf{x}) = \rho_c \left[1 - \sum_{p=1}^{3} \frac{x_p^2}{a_p^2} \right]^{\nu}. \tag{141}$$

We assume that the form of the distribution function continues to be maintained as the system evolves, i.e. ν is independent of time. Choosing our units of mass, length and time such that (cf. equation 87)

$$M = 1, \quad a_1 = 1 \ (\text{at } t = 0), \quad G \frac{\Psi(\nu)}{\phi(\nu)} = 1, \tag{142}$$

we obtain (cf. equations 54 and 55)

$$I_{11} = \frac{1}{5} a_1^2 \, \phi(\nu), \qquad I_{33} = \frac{1}{5} a_3^2 \, \phi(\nu),$$

$$W_{11} = -\frac{3}{10} \frac{1}{a_1} \phi(\nu) A_1 \qquad W_{33} = -\frac{3}{10} \frac{a_3}{a_1^2} \phi(\nu) A_3. \tag{143}$$

Substituting these expressions in (119) and making use of the fact that ν and hence $\phi(\nu)$ is independent of time we obtain

$$
\begin{aligned}
\frac{d^2 a_1}{dt^2} &= \frac{C_1}{2a_1^3} - \frac{3}{2}\,\frac{1}{a_1 a_3}\,A_1 , \\
\frac{d^2 a_3}{dt^2} &= \frac{C_3}{2a_3^3} - \frac{3}{2}\,\frac{1}{a_1^2}\,A_3 ,
\end{aligned}
\tag{144}
$$

where

$$
C_i = \frac{5\alpha\lambda_i}{[\phi(\nu)]^2} . \tag{145}
$$

Equations (144) are identical with equations (123) and hence the results of the earlier sections also apply to heterogeneous spheroids.

CHAPTER 5

TIDAL INTERACTIONS BETWEEN STELLAR SYSTEMS IN HYPERBOLIC ENCOUNTERS

17. Introduction:

Spitzer (1958) studied the changes in the energy of a galactic star cluster in a hyperbolic encounter with a passing interstellar gas cloud assuming that the stars in the cluster remain stationary during the encounter (the impulsive approximation). Since then this approximation has been used by several workers to study energy changes in spherical galaxies due to a passing perturber (cf. Alladin & Narasimhan 1982 for a review). Gerhard & Fall (1983) using a version of the tensor virial equation for a plane stratified gravitational system, and the impulsive approximation, studied the energy changes on disc galaxies due to tidal interactions.

In this chapter we use tensor virial equations to obtain the change that arise in the energy of a spheroidal stellar system as a result of an impulsive tidal interaction. We also use these equations to obtain the angular momentum transfered to the system as a result of such an encounter.

18 Basic equations:

We consider a spheroidal stellar system with the axes $a_1 = a_2$ and a_3, of mass M and density distribution of the form (cf. equation 49)

$$\rho(\mathbf{x}) = \rho_c \left[1 - \sum_{p=1}^{3} \frac{x_p^2}{a_p^2} \right]^{\nu}, \qquad (146)$$

where ν is nonnegative parameter, and the coordinates x_i are with respect to a cartesian frame of reference with origin at the centre of the spheroid and oriented along the body axes.

We assume that the system is initially in steady state so that

$$\frac{d^2 I_{ij}^{(i)}}{dt^2} = \frac{dI_{ij}^{(i)}}{dt} = 0, \qquad (147)$$

where the superscripts (i) imply that they are the initial values of the quantities i.e. before the encounter. From the equations (31) and (39) it follows that

$$\begin{aligned} \mathcal{L}_{i;j}^{(i)} + \mathcal{L}_{j;i}^{(i)} &= 0, \\ K_{ij}^{(i)} &= -\frac{1}{2} W_{ij}^{(i)}. \end{aligned} \qquad (148)$$

We now consider a perturber of mass M' rushing past the system in a straight line with a velocity $\mathbb{V}$ and a pericentric distance $\mathbb{p}$. Without loss of generality we can choose our x_1- axis such that $\mathbb{p}$ lies in the $x_1 - x_3$ plane. We denote by θ the polar angle of $\mathbb{p}$, while ψ is the angle $\mathbb{V}$ makes with respect to the $x_1 - x_3$ plane (figure 14). It follows that

$$\hat{\mathbb{p}} = \begin{pmatrix} \sin\theta \\ 0 \\ \cos\theta \end{pmatrix}, \qquad \hat{\mathbb{V}} = \begin{pmatrix} -\cos\psi\cos\theta \\ \sin\psi \\ \cos\psi\sin\theta \end{pmatrix}, \qquad (149)$$

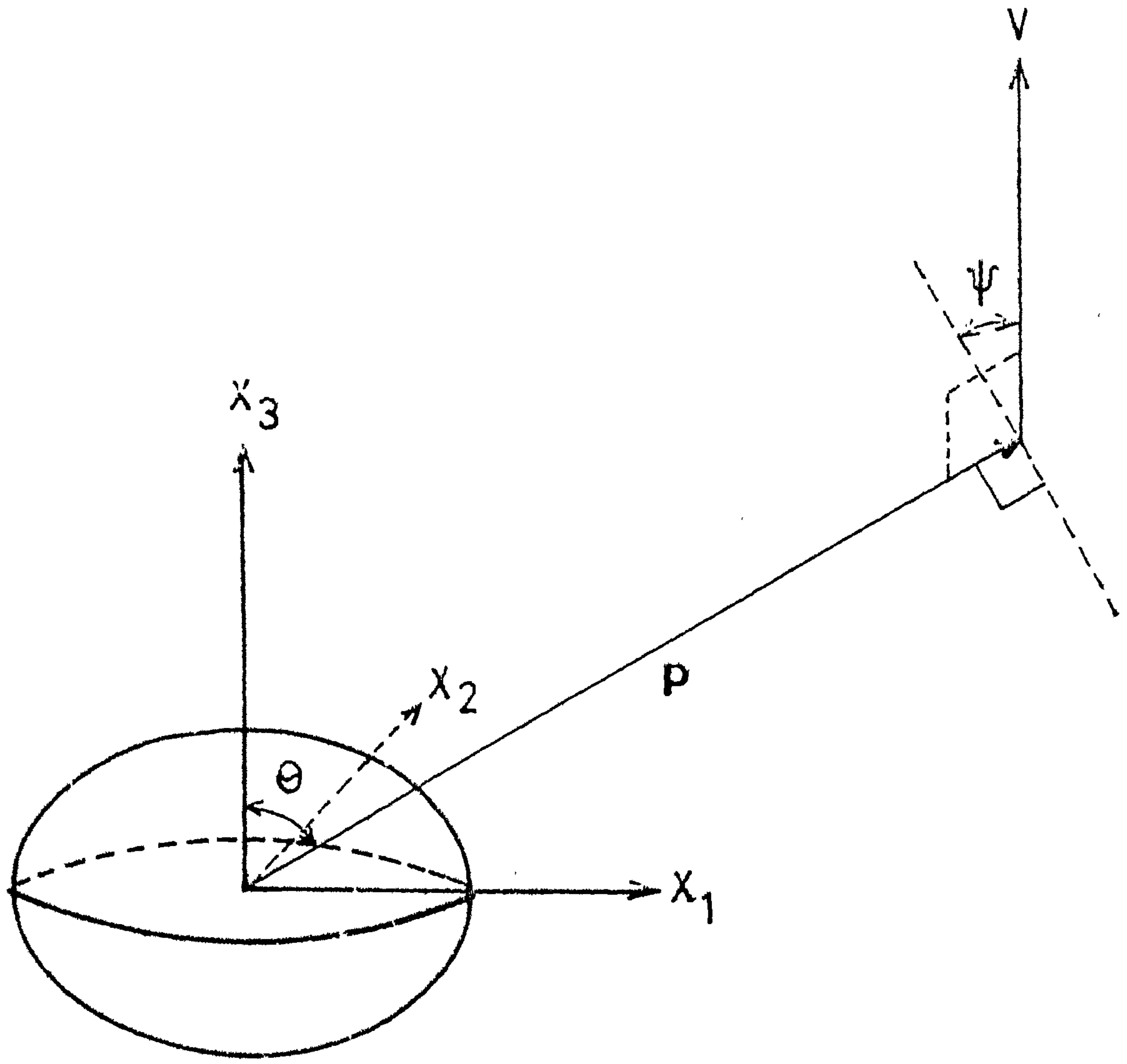

Figure 14. Coordinate system used to study encounter. The equitorial plane of the spheroid is the $X_1 - X_2$ plane while $\mathbb{p}$ lies in the $X_1 - X_3$ plane and has an azimuthal angle θ. The velocity vector $\mathbb{v}$ is perpendicular to $\mathbb{p}$ and makes an angle ψ with the $X_1 - X_3$ plane.

where $\hat{\mathbb{p}}$ and $\hat{\mathbb{V}}$ are the unit vectors in the directions of $\mathbb{p}$ and $\mathbb{V}$ respectively. Note that

$$\mathbb{p} \cdot \mathbb{V} = 0 \tag{150}$$

as is required by the assumption that the perturber moves along a straight line.

Assuming that $\mathbb{p}$ is large so that the tidal forces on a star can be approximated to the first order in X/p where $\mathbb{X}$ is the position vector of the star, the change in velocity of the star as a result of the encounter is (cf. Gerhard & Fall 1983)

$$\Delta u_i = \left(\frac{2GM'}{Vp^2}\right)\left[\sum_{p=1}^{3}(2\hat{p}_i\hat{p}_p + \hat{V}_i\hat{V}_p)x_p - x_i\right] \tag{151}$$

In the impulsive approximation, the stars remain motionless during the encounter and hence the moment of inertia and potential energy tensors remain unchanged, i.e.

$$I_{ij}^{(o)} = I_{ij}^{(i)}$$

$$W_{ij}^{(o)} = W_{ij}^{(i)}$$

where the superscript (o) is used to indicate the values of the quantities immediately after the encounter. As a result of the change in velocity (equation 151) the momentum and kinetic energy tensors change however. The change in the momentum tensor is given by

$$\Delta \mathcal{L}_{i;j} = \mathcal{L}^{(0)}_{i;j} - \mathcal{L}^{(1)}_{i;j} = \int_{\Gamma} f(\mathbb{Z})\, \Delta u_i\, x_j\, d\mathbb{Z}$$

$$= \left(\frac{2GM'}{Vp^2}\right)\left[\sum_{p=1}^{3} (2\hat{p}_i \hat{p}_p + \hat{v}_i \hat{v}_p) I_{pj} - I_{ij}\right]$$

$$= \left(\frac{2GM'}{Vp^2}\right)\left[\frac{1}{5} M a_i^2 (1 - y\,\delta_{j3})\,\phi(\nu)\right]\left[(2\hat{p}_i \hat{p}_j + \hat{v}_i \hat{v}_j) - \delta_{ij}\right] \quad (152)$$

on making use of equation (151), and the expression (cf. equation 54)

$$I_{ij} = \frac{1}{5} M a_i^2 (1 - y\,\delta_{i3})\ \phi(\nu)\ \delta_{ij} \quad (153)$$

The value of the kinetic energy tensor after the encounter is

$$K'_{ij} = \frac{1}{2}\int f\, (u_i^{(1)} + \Delta u_i)(u_j^{(1)} + \Delta u_j)\ d\mathbb{Z}$$

$$= K^{i}_{ij} + \frac{1}{2}\int_{\Gamma} f\,(u_i^{(1)} \Delta u_j + u_j^{(1)} \Delta u_i)\, d\mathbb{Z} + \frac{1}{2}\int f\, \Delta u_i\, \Delta u_j\, d\mathbb{Z} \quad (154)$$

Hence the change in the kinetic energy tensor, on using equation (151), becomes

$$\Delta K_{ij} = \frac{1}{2}\left(\frac{2GM'}{Vp^2}\right)\left[\sum_{p=1}^{3}(2\hat{p}_j \hat{p}_p + \hat{v}_j \hat{v}_p)\mathcal{L}^{(1)}_{i;p} + \sum_{p=1}^{3}(2\hat{p}_i \hat{p}_p + \hat{v}_i \hat{v}_p)\mathcal{L}^{(1)}_{j;p} - (\mathcal{L}^{(1)}_{i;j} + \mathcal{L}^{(1)}_{j;i})\right]$$

$$+ \frac{1}{2}\left(\frac{2GM'}{Vp^2}\right)^2\left[\sum_{pq=1}^{3}(2\hat{p}_i \hat{p}_p + \hat{v}_i \hat{v}_p)(2\hat{p}_j \hat{p}_q + \hat{v}_j \hat{v}_q) I^{(1)}_{pq} \right. \quad (155)$$

$$\left. - \sum_{p=1}^{3}(2\hat{p}_i \hat{p}_p + \hat{v}_i \hat{v}_p) I^{(1)}_{pj} - \sum_{q=1}^{3}(2\hat{p}_j \hat{p}_p + \hat{v}_j \hat{v}_p) I^{(1)}_{pi} + I^{(1)}_{ij}\right]$$

Contracting this expression and making use of the expression (148), (150) and (153) it follow that the change in the total kinetic energy is

$$\Delta K = \frac{1}{2}\left(\frac{2GM'}{Vp^2}\right)^2 \left[I^{(0)} - \sum_{p=1}^{3} \hat{\nu}_p \hat{\nu}_p I^{(0)}_{pp} \right]$$

$$= \frac{1}{2}\left(\frac{2GM'}{Vp^2}\right)^2 \left(\frac{1}{5} M a_1^2 \; \phi(\nu)\right)\left[2 - y\left(1 - \hat{\nu}_3^2\right)\right] \tag{156}$$

Since there is no change in the potential energy as a result of the encounter (equation 152) the total change in energy $\Delta E = \Delta K$ and hence using equation (156) we obtain

$$\Delta E = \frac{1}{2}\left(\frac{2GM'}{Vp^2}\right)^2 \left(\frac{2}{5} M a_1^2 \; \phi(\nu)\right)\left[1 - \frac{1}{2} y\left(1 - \cos^2\psi \sin^2\theta\right)\right] \tag{157}$$

For a spherical system with the density distribution of the form given by equation (146), the mean square radius is given by

$$r_c^2 = \frac{1}{M} \int_V \rho(x) \, |x|^2 \, dx = \frac{3}{5} a_1^2 \, \phi(\nu) \tag{158}$$

and in this case equation (157) reduces to the formula of Spitzer (1958 equation 9)

$$\Delta E = \frac{1}{2} M \left(\frac{2GM'}{Vp^2}\right)^2 \frac{2}{3} r_c^2 \tag{159}$$

From equation (152) and the definition of the angular momentum (equation 24), we obtain, for the change in angular momentum as a result of the encounter, the expression

$$\Delta\lambda_k = \lambda_k^{(2)} - \lambda_k^{(1)} = \sum_{i,j=1}^{3} \epsilon_{kji}\, \Delta \mathcal{L}_{ij}$$

$$= \left(\frac{2GM'}{Vp^2}\right)\left(\frac{1}{5} M a_1^2\, \phi(\nu)\right) y \left[\sum_{i=1}^{3} \epsilon_{ki3} \left(2\hat{p}_i \hat{p}_3 + \hat{v}_i \hat{v}_3\right)\right] \qquad (160)$$

or explicitly, using equation (149),

$$\Delta\lambda_1 = \left(\frac{2GM'}{Vp^2}\right)\left(\frac{1}{5} M a_1^2\, \phi(\nu)\right) y\, f_1(\psi,\theta)$$

$$\Delta\lambda_2 = \left(\frac{2GM'}{Vp^2}\right)\left(\frac{1}{5} M a_1^2\, \phi(\nu)\right) y\, f_2(\psi,\theta) \qquad (161)$$

$$\Delta\lambda_3 = 0$$

with

$$f_1(\psi,\theta) = \sin\psi \cos\psi \sin\theta$$

$$f_2(\psi,\theta) = -\cos\theta \sin\theta \left(1 + \sin^2\psi\right) \qquad (162)$$

The net angular momentum transfered to the system as a result of the encounter is

$$\Delta\lambda = \left(\frac{2GM'}{Vp^2}\right)\left(\frac{1}{5} M a_1^2\, \phi(\nu)\right) |y| \left(f_1^2 + f_2^2\right)^{1/2} \qquad (163)$$

The angular momentum vector lies on the equitorial (the $x_1 - x_2$) plane and its azimuthal angle is given by

$$\Phi = \tan^{-1}\left(\frac{f_2}{f_1}\right) \qquad (164)$$

19 Changes in energy and angular momentum:

From equation (157) it is seen that the energy transfered, ΔE, varies

(1) directly as the mass M of the system,

(2) directly as the square of the equitorial radius a_1,

(3) directly as the square of the mass M' of the perturber,

(4) inversely as the square of the speed V of the perturber,

(5) inversely as the fourth power of the impact parameter p,

(6) inversely as the density parameter ν, i.e. decreases with increasing central concentration.

These results are identical with those of Spitzer (1958) for a spherical system.

From equation (162) it follows that the angular momentum transfered, $\Delta\lambda$, varies

(1) directly as the mass M of the system,

(2) directly as the square of the equitorial radius a_1,

(3) directly as the mass of the perturber M',

(4) inversely as the speed V of the perturber,

(5) inversely as the square of the impact parameter p,

(6) inversely as the density parameter ν, i.e. decreases with increasing central concentration.

The angular depedence of . ΔE is given by the function

$$1 - \frac{1}{2} y \left[1 - \cos^2\psi \sin^2\theta \right] \tag{165}$$

and that of $\Delta\lambda$ by the function

$$\sin\theta \left[\sin^2\psi \cos^2\psi + \cos^2\theta (1 + \sin^2\psi)^2 \right]^{1/2} \tag{166}$$

These functions are graphically illustrated in figures (15) and (16) and a few important cases we listed in table 3.

It is seen that ΔE has the maximum value when the collision is perpendicular to the equatorial plane $(\psi = 0^\circ, \theta = 90^\circ)$ and this value is independent of the eccentricity of the ellipsoid. For other orientations of the collision ΔE is less than this value for oblate spheroids and greater than this value for prolate ones. Thus the energy transferred is greater for a sphere than for a disc, and greater for a spindle shaped galaxy than for a sphere.

No angular momentum is transferred when the perturber mover perpendicular to the equatorial plane. Encounters with other orientations produce in the stellar system a rotation about an axis lying in the equatorial plane. If the resultant rotation is differential it would result in the formation of warps in disc shaped galaxies.

Note that no rotations in the equatorial plane of the system are produced as a result of an impulsive encounter. However N-body simulations by Miller (1984) indicate that such rotations may also result from galactic interactions.

Table 3

Change in energy and angular momentum in an impulsive encounter

Orientation of $\mathbb{V}$	Orientation of $\mathbb{p}$	Change in Energy ΔE	Change in angular momentum: Magnitude $\Delta\lambda$	Direction: Oblate	Direction: Prolate
$\mathbb{V}$ coplanar with $\mathbb{p}$ & a_3 $(\psi = 0^\circ)$	$\mathbb{p} \parallel a_3$ $(\theta = 0^\circ)$	$1 - \frac{1}{2} y$ $\begin{cases} \text{Min (Oblate)} \\ \text{Max (Prolate)} \end{cases}$	0		
	$\mathbb{p} \perp a_3$ $(\theta = 90^\circ)$	1 $\begin{cases} \text{Max (Oblate)} \\ \text{Min (Prolate)} \end{cases}$	0		
	$\mathbb{p}$ at 45° to a_3	$1 - \frac{1}{4} y$	$\frac{1}{2}$ (max)	$-x_2$	$+x_2$
	Other	$1 - \frac{1}{2} y \cos^2\theta$	$\frac{1}{2} \sin 2\theta$	$-x_2$	$+x_2$
$\mathbb{V} \perp \mathbb{p}$ & a_3 $(\psi = 90^\circ)$	$\mathbb{p} \parallel a_3$ $(\theta = 0^\circ)$	$1 - \frac{1}{2} y$, constant.	0	—	—
	$\mathbb{p} \perp a_3$ $(\theta = 90^\circ)$		0	—	—
	$\mathbb{p}$ at 45° to a_3		1 (max)	$-x_2$	$+x_2$
	Other		$\sin 2\theta$	$-x_2$	$+x_2$

Table 3 (continued)

Orientation of $\mathbb{v}$	Orientation of $\mathbb{p}$	Change in Energy ΔE	Change in angular momentum Magnitude $\Delta\lambda$	Direction Oblate	Direction Prolate
$\mathbb{v}$ at 45° to $\mathbb{p}-a_3$ plane	$\mathbb{p} \parallel a_3\ (\theta = 0^\circ)$	$1 - \frac{1}{2} y$	0	—	—
	$\mathbb{p} \perp a_3\ (\theta = 90^\circ)$	$1 - \frac{1}{4} y$	$\frac{1}{2}$	About axis in x_1-x_3 plane	Opposite to that in oblate case
	$\mathbb{p}$ at 45° to a_3	$1 - \frac{3}{8} y$	$\frac{\sqrt{11}}{4}$		
	Other	$1 - \frac{1}{2} y \left[1 - \frac{1}{2}\sin^2\theta\right]$	$\frac{1}{2}\sin\theta\left[1 + 9\cos^2\theta\right]^{1/2}$		
Other	$\mathbb{p} \parallel a_3\ (\theta = 0^\circ)$	$1 - \frac{1}{2} y$	0	About axis in x_1-x_3 plane	Opposite to that in oblate case
	$\mathbb{p} \perp a_3\ (\theta = 90^\circ)$	$1 - \frac{1}{2} y \sin^2\psi$	$\frac{1}{2}\sin 2\psi$		
	$\mathbb{p}$ at 45° to a_3	$1 - \frac{1}{2} y \left[1 - \frac{1}{2}\cos^2\psi\right]$	$\frac{1}{2}\left[\frac{1}{4}(\sin 2\psi)^2 + \frac{1}{2}(1+\sin^2\psi)^2\right]^{1/2}$		
	Other	$1 - \frac{1}{2} y \left[1 - \cos^2\psi \sin^2\theta\right]$	$\sin\theta\left[\sin^2\psi\cos^2\psi + \cos^2\theta(1+\sin^2\psi)^2\right]^{1/2}$		

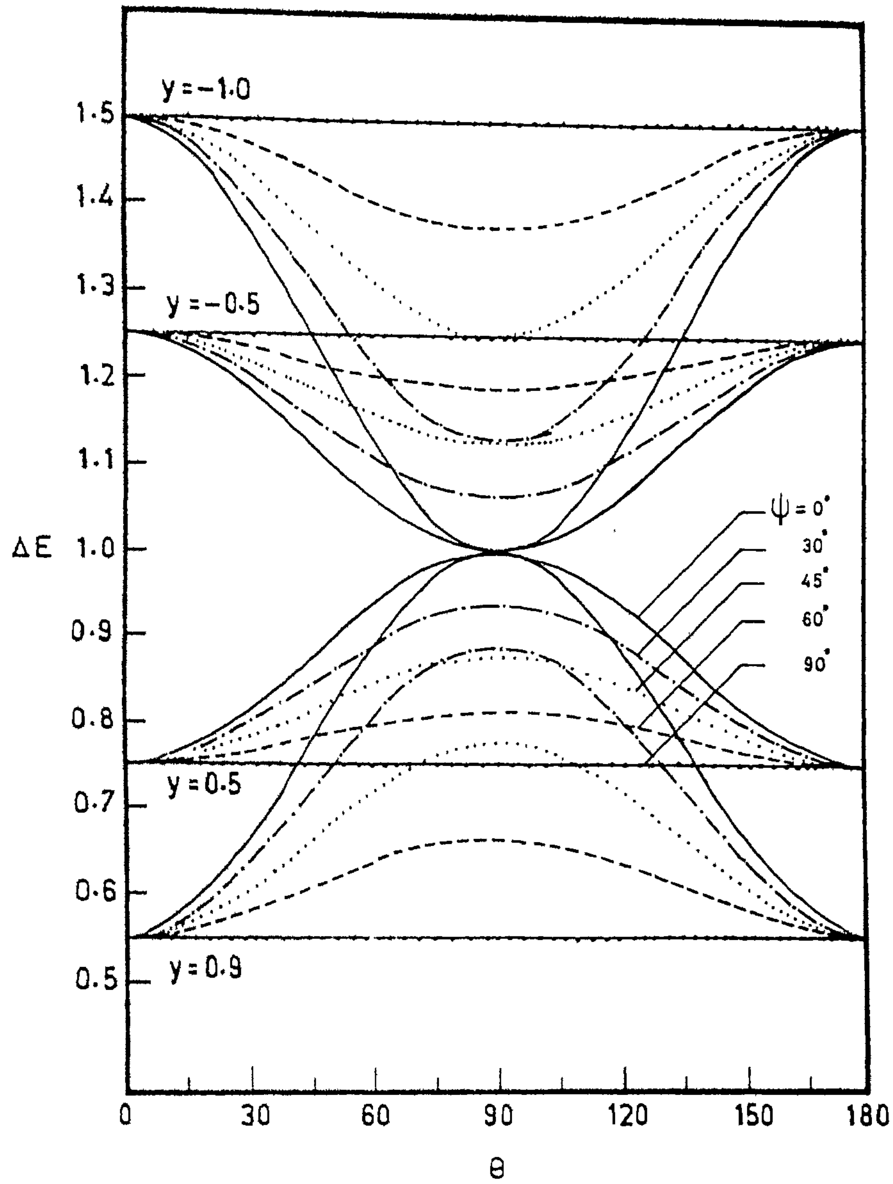

Figure 15. The variation of ΔE with orientation of p and ψ. The ordinate change in energy given by equation 165, while the abscissa is the angle Θ defined in figure 14. The curves are labelled by the angle ψ (defined in figure 14) and the y_0 to which they belong.

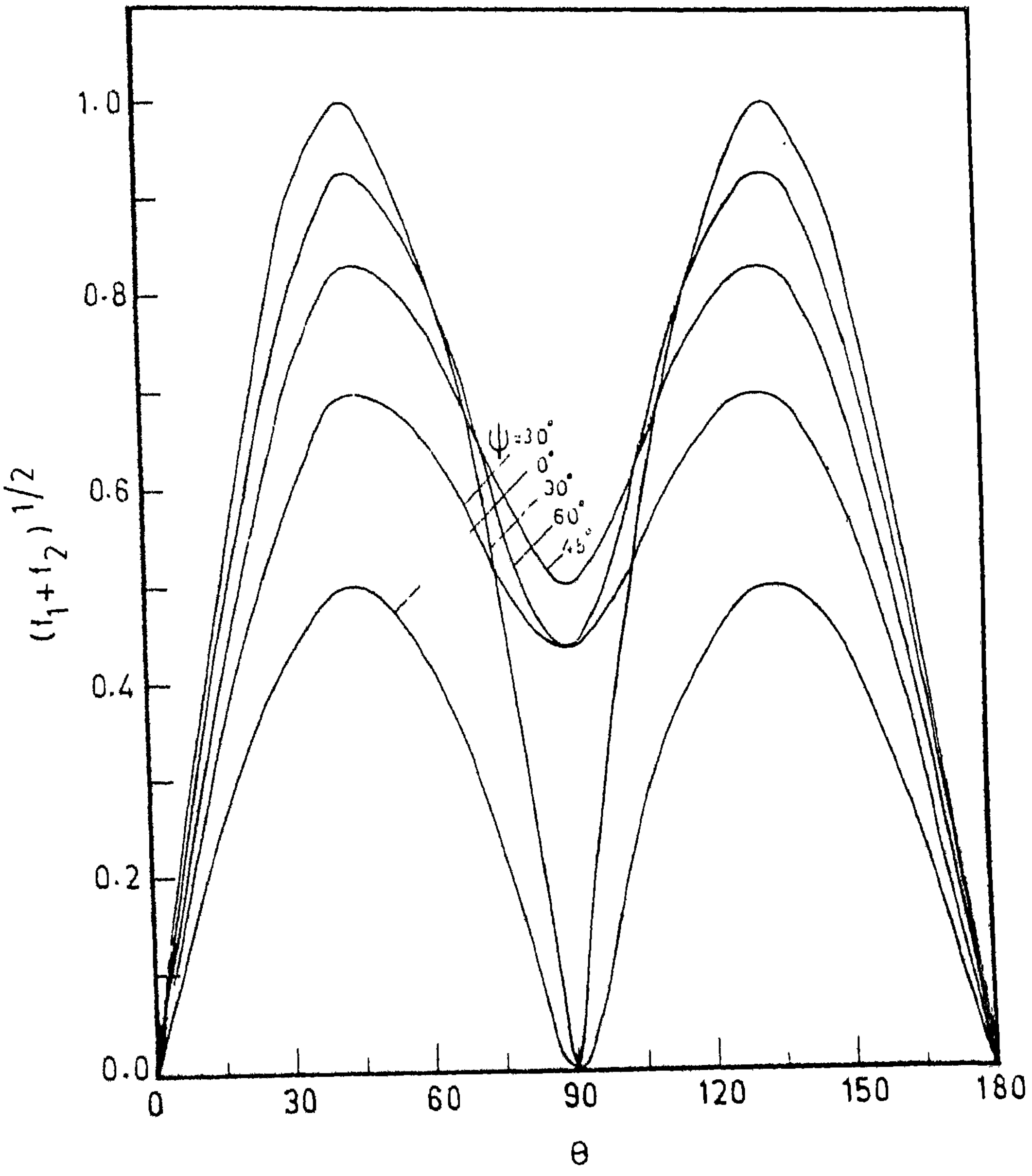

Figure 16. The variation of $\Delta\lambda$ with orientation of ϕ and ψ. The ordinate is the function given by equation (166) while the absoissa and labels are as in figure. 15.

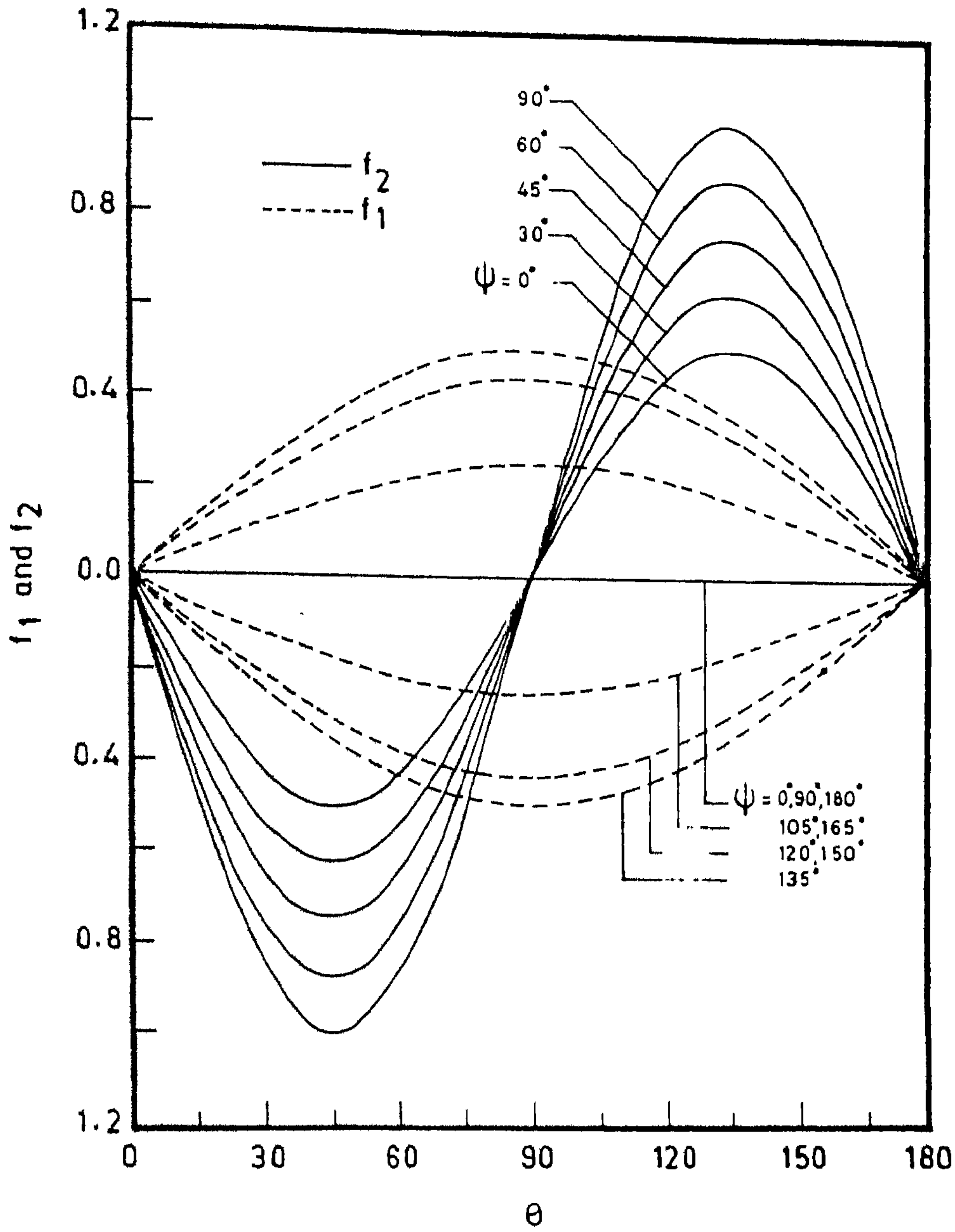

Figure 17. The variation of $\Delta\lambda_1$ and $\Delta\lambda_2$ with the orientation of ϕ and ψ. The curves depict the behaviour of the function f_1 and f_2 defined by equation (1.62). The abscissa and labels are as in figure 15.

CHAPTER 6
CONCLUSIONS

20 Summary

The tensor virial equations have been developed and used by Chandrasekhar and co-workers to study the equilibrium figures of rotating gas clouds (cf. Chandrasekhar 1969). In recent years these equations have been used to study the equilibrium form of ellipsoidal galaxies (Binney 1982). This technique has been extremely useful in explaining the recent observations regarding the shapes and velocity anisotropies in elliptical galaxies.

It has been suggested by Chandrasekhar and Elbert (1972) that the tensor virial equations can be used to study the dynamical evolution of stellar systems. They also outline some elementary applications of this technique. In this thesis we develop the necessary equations and apply them to other problems in stellar dyanmics.

The usual procedure adopted to obtain the virial equations is the following (Binney 1982): One first obtains the Jeans (or hydrodynamical) equations by integrating the Boltzmann's equation. The virial equations are then obtained by taking the moments of these equations. The equations thus derived are valid only for systems that satisfy the familiar Boltzmann's equation, namely, isolated systems observed from an inertial frame of reference. When other dynamical problems are considered one will have to repeat the analysis starting from first principles.

In chapter 2 of this work we suggest an alternative approach for obtaining the virial equations from purely kinematical considerations. We begin by assuming that the system is collisionless and hence treat the phase-space distribution for the system as a continuous fluid described by a distribution function $f(x, u, t)$. The various tensors required are then defined in terms of the distribution function. The virial equations are obtained by taking the moments of the six dimensional equation of continuity that the distribution function must satisfy. The virials in these expressions are expressed in terms of the acceleration field g. Being purely kinematical these relations are valid for any (collisionless) stellar system, isolated or non-isolated, and as observed from an inertial or a non-inertial frame of reference. When different dynamical problems are considered we can obtain the required relations by merely substituting for the acceleration, the forces that act on the system.

In the same chapter we also develop a relation for the rate of change of the kinetic energy tensor. This relation is the tensor analogue of the law of energy conservation, and is of use when the tensor virial equations are employed to study the dynamical evolution of stellar systems. In the final section of this chapter we use the results of Roberts (1962) to obtain expressions for the various tensors for a heterogeneous ellipsoid.

In an appendix to chapter 2 we also obtain by the same technique the Jeans (or hydrodynamical) equations of various orders. Besides the familiar equations of continuity and motion we also obtain an equation which is related to the heat transfer equation of statistical mechanics. The various macroscopic equations discussed above are shown to be moments of these equations. In particular the equation for the rate of change of the kinetic energy tensor is obtained by taking the zeroth order moment of equation of heat transfer. In later chapters, use is made of the virial equations of the order two only. However in another appendix to chapter 2, the virial equation of orders 3 and 4 are obtained for completeness.

In chapter 3 we reexamine the problem considered by Chandrasekhar & Elbert (1972) of a homogeneous spheroidal cluster with velocity distributions that always remain isotropic. such systems are not in equilibrium. We show that the equations of Chandrsekhar & Elbert can be used only when the energy is positive. We derive the equations and solve them for various negative values of the total energy and various eccentricities. We show that such systems execute finite amplitude oscillations both in size and between prolate and oblate shapes. We then extend this analysis to heterogeneous clusters and show that the period of the oscillation is independent of the central concentration and depends only on the total energy of the system. As expected heterogeneous systems with positive energies expand and are eventually dispersed. The expansion is accompanied by a decrease in the asphericity of the system.

In chapter 4 we relax the assumption that the velocity distribution always remains isotropic and consider systems in which this is true only initially. The equations derived reduce, in the case when the initial kinetic energy to zero, to those of Lin et al (1965) who study the collapse of a pressureless gas cloud. In this case the collapse of the system is accompanied by an increase in the prolateness or oblateness. When the initial kinetic energy is non-zero, the initial pressure is nonzero and this pressure increases as the system collapses. This stops the collapse and as a result the system executes finite amplitude oscillations both in size and eccentricity just as in chapter 3. However a reversal from oblate to prolate or prolate to oblate shapes as seen in chapter 3 is not observed. Systems with positive energies expand and are eventually dispersed. The initial expansion is accompanied by a sharp decrease in the eccentricity which then tends towards a finite value.

In chapter 5 we extend the work of Spitzer (1958) and employ the tensor virial equations to study the changes in the energy of a spheroidal cluster as a result of an impulsive interaction with a passing perturber. Besides the familiar dependence of the energy change on the impact parameter and velocity of the perturber (cf. Spitzer 1958) we show that it also depends on the orientation of the encounter, being a maximum when the orbit of the perturber is perpendicular to the equatorial plans of the test cluster. The energy change also increases with increasing prolateness, being a minimum for a disc shaped system and a maximum for a spindle shaped system.

When the test cluster is spherical there is no change in the angular momentum of the system as a result of an impulsive encounter. In chapter 5 we show however that such a change could take place in the case of a spheroidal cluster and this change increases with increasing asphericity of the system. Such a change in angular momentum could perhaps explain warps found in disc galaxies.

In the next section we will briefly survey certain other problems that can be studied using this technique.

21. Other applications of the virial equations:

The equations derived in chapter 2 apply to any stellar system. We have restricted ourselves to simple dynamical problems in this work. Several other problems could be studied using this technique.

It would be of interest to study systems in which rotation is important. The simplest case would be spheroidal systems rotating about the a_3 axis. In this case equation (123) would still apply and could be used to study how the presence of a nonzero λ_3 would effect the dynamical evolution of the system.

The study of triaxial systems would also be of interest. In this case the required equations can be obtained using equation (39) and (42).

Yet another interesting problem that can be studied using this technique is the dynamical evolution of a star cluster moving about the centre of the galaxy. The expressions for the acceleration of a star in the cluster can be obtained from Chandrasekhar (1967). Equations (35) and (42) can then be employed to study the evolution of such a cluster. Evolution of binary galaxies can also be studied in a similar fashion.

Icke (1973) using a technique similar to that of Lin et al (1965) has studied the collapse of a spheroidal gas cloud in an expanding universe. As was pointed out in chapter 4, Lin et al assumed that the pressure in the system is zero. The virial technique permits a study of systems with nonzero pressure and can be used to study the formation and collapse of galaxies in an expanding universe when the pressure is non zero.

The equations of chapter 2 apply only to collisionless systems, and are therefore inappropriate to the study of small clusters wherein collisions play a major role. Similar equations can be set up starting from the Lioville's equation in 6N-dimensional phase-space and can be used to study the evolution of star-clusters in which collisions are important.

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